131253

УДК 51

YU ISSN 0352—0900

УНИВЕРЗИТЕТ У НОВОМ САДУ — UNIVERSITY OF NOVI SAD

ЗБОРНИК РАДОВА
ПРИРОДНО-МАТЕМАТИЧКОГ ФАКУЛТЕТА
СЕРИЈА ЗА МАТЕМАТИКУ

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

Vol 20

No. 1

Vol. 21
No I
1991

(GK)
131253

131253

НОВИ САД — NOVI SAD
1990.

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

INSTITUTE OF MATHEMATICS
TRG DOSITEJA OBRADOVIĆA 4
21000 NOVI SAD
YUGOSLAVIA

EDITORIAL BOARD

Editor in Chief: **O. Hadžić**, Novi Sad
Associate Editor in Chief: **D. Herceg**, Novi Sad
Editorial Secretary: **R. Vulanović**, Novi Sad

Editors:
E. Allgower, Fort Collins, CO
K.Gilezan, Novi Sad
A. Ivić, Belgrade
Z. Ivković, Belgrade
A. Kamiński, Katowice
V. Marić, Novi Sad
G. Milovanović, Niš
E. Pap, Novi Sad
S. Pilipović, Novi Sad
M. Prvanović, Novi Sad
B. Stanković, Novi Sad
Z. Stojaković, Novi Sad
B. Šešelja, Novi Sad
R. Tošić, Novi Sad
J. Ušan, Novi Sad
V. Vladimirov, Moscow

The journal publishes original research papers from all fields of pure and applied mathematics. When preparing the manuscript please consider "Instructions to Authors" at the end of each number. Manuscripts and inquiries should be directed to the associate editor in chief. Twenty offprints of each paper are supplied free of charge. Additional copies may be ordered at cost price at time when proofs are returned.

Annual subscription rate for foreign countries: US $ 60 (one volume of two numbers).

Orders, exchange journals, manuscripts and all corespondence concering offprints should be sent to the associate editor in chief:

D. Herceg
Institute of Mathematics
University of Novi Sad
Trg Dositeja Obradovića 4
21000 Novi Sad, Yugoslavia

Published by the Institute of Mathematics, Novi Sad, Yugoslavia
Printed by FTN, Štamparija za grafičku delatnost, Novi Sad
Circulation: 500

CS
A 4

REVIEW OF RESEARCH

FACULTY OF SCIENCE, MATHEMATICS SERIES
Volume 20, Number 1, 1990

131253

Contents

D. Herceg: A note on iterative process for solving linear systems of equations 1
D. Herceg, Lj. Cvetković : On an iterative method for a system of equations 11
D. Herceg, K. Surla, Lj. Cvetković : A family of exponential spline difference schemes 17
M. A. H. Veron : An acceleration procedure of the Whittaker method by means of convexity 27
D. M. Acketa, L. Novak : An algorithm for the generation of the matroid associated to a given matrix 39
Z. Stojaković : Dihedral n-quasigroups 53
A. Ivić : On the k-th prime factor of an integer 63
E. Pap : Distance preserving maps on Abelian lattice ordered groups 75
J. Nikić : Conformal diffeomorphism between two f-manifolds 83
O. Hadžić : A common fixed point theorem for a family of mappings in convex metric spaces 89
O. Hadžić : A fixed point theorem for nonexpansive mappings in random normed spaces 97
H. Howard, V. Marić, Z. Radašin : Asymptotics of nonoscillatory solutions of a second order linear differential equations 107
D. Nikolić - Despotović, S. Pilipović : Quasiasymptotic expansions and the Stieltjes transformation 117
Z. Ivković : Representation for a class of non-linear functionals of Gaussian martingales 129
E. Pap : An integral generated by a decomposable measure 135
A. Száz : Continuities of multi-preseminorms 145
A. Takači : On Sebastio e Silva's order of growth of distributions ... 155
J. Ewert, T. Lipski : On S-quasi-continuous multivalued maps 167
V. Popa : Weak continuity and almost continuity for multifunctions 185
M. Prvanović : A note on conformally quasi-recurent manifolds 195

Abstracted / Indexed in:
Mathematical Reviews
Zentralblatt für Mathematik
Referativnyĭ Zhurnal
Current Mathematical Publications

(GK)
131253

Instructions to Authors

Manuscripts written in English (preferably), German, French or Russian should be sent to the Associate Editor in Chief.Three clearly legible copies are required. Authors should also send two copies of any of their unpublished papers to which crucial reference is made in the manuscript. The paper submitted must not be previously published in, nor have been submitted to another journal. Author receive 20 free offprints of their articles.

Manuscripts should be typed double spaced, on one side of the sheet only. Below the title, the name (s) and full postal address (es) of author (s) should be given, followed by a short selfcontained abstract in English (together with the English translation of the title for non-English papers), AMS Mathematics Subject Classification (1990) and key words and phrases. If possible, at the end of the paper a summary in Serbian should be given together with the Serbian translation of the title.

Each section, including the introduction should be numbered, with formulas numbered in the left margin. Nonstandard notations should be explained in the text. Footnotes should be avoided. Drawings should be prepared for direct photographic reproduction at printed size. References should be listed alphabetically and numbered consecutively at the end of manuscript, following the model given below:

[1] O Reilly, M.J.: A uniform scheme for the singularly perturbed Riccati equation. Numer. Math. 50(1987), 483-501.

[2] Ortega, J.M., Rheinboldt, W.C.: Iterative solution of nonlinear equations in several variables. New York and London: Academic Press 1970.

[3] Bohl, E.: Discrete versus continuous models for dissipative systems. In: Numerical models for bifurcation problems. (T. Krüpper, H.D. Mittelmann, H. Weber, eds.), pp. 68-78. Basel, Boston, Stuttgart: Birkhäuser Verlag 1984.

References should be indicated in the text by corresponding arabic numbers in square brackets.

Manuscripts will be retyped and the author will receive a copy of galley proofs. Authors are requested to correct typographical errors only. Alternations exceeding 10% of the typesetting will be charged to the author.

The Review is produced using LATEX. Authors who are able to do so are invited to prepare their manuscripts using LATEX. Hard copy only should be submitted for assesment, but if paper is accepted the author will be asked to send the text on a 5.25″ IBM PC compatible diskette, or to send it by e-mail to following adress:

e-mail: review%unsim@yubgef51.bitnet

UDK 51

YU ISSN 0352—0900

УНИВЕРЗИТЕТ У НОВОМ САДУ — UNIVERSITY OF NOVI SAD

ЗБОРНИК РАДОВА
ПРИРОДНО-МАТЕМАТИЧКОГ ФАКУЛТЕТА
СЕРИЈА ЗА МАТЕМАТИКУ

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

20

No. 2

НОВИ САД — NOVI SAD

1990.

Samoupravna interesna zajednica za naučni rad SAP Vojvodine delimično je učestvovala u štampanju ove publikacije.

Na osnovu mišljenja pokrajinskog sekretarijata za obrazovanje, nauku i kulturu br. 413-58/73. od 20. februara 1973. godine, ova knjiga je oslobođena plaćanja posebnog poreza na promet i usluga u prometu.

Izdaje: Prirodno-matematički fakultet u Novom Sadu,
21000 Novi Sad, Dr Ilije Đuričića 4, Jugoslavija

Štampa: FTN OOUR Štamparija za grafičku delatnost
Novi Sad, V. Vlahovića 3. tel: 54-245
Tiraž: 500 primeraka

REVIEW OF RESEARCH

FACULTY OF SCIENCE, MATHEMATICS SERIES

Volume 20, Number 2, 1990

Contents

D. Herceg: On the numerical solution of a singularly perturbed nonlocal problem 1
Z. Stojaković, Some classes of dihedral n-quasigroups 11
B. Šešelja, A. Tepavčević, Special elements of the lattice and lattice identities 21
G. Crombez, Group theoretical methods in gray-scale mathematical morphology 31
M. Lepović, Some results on the reduced energy of graphs, I 43
A.S. Mashour, M.H. Ghanim, F.I. Sidky, Normal fuzzy subgroups 53
B. Šešelja, A. Tepavčević, Weak congruences and homomorphisms . 61
B. Šešelja, A. Tepavčević, On a construction of codes by P-fuzzy sets 71
B. Fisher, L. Zhishen, The product of the distributions x_+^λ and $x_-^{-\lambda-p}$ defined by a distribution vector 81
Z.A. Ivković, Arbitrariness of multiplicity for a linear non-anticipative transformation of the Wiener process 91
M. Stojaković, R^n-valued fuzzy random variable 95
G. Isac, Remark on the paper of Chitra and Subrahmanyam about the section theorem of Ky Fan 105
B. Fisher, On powers of Heaviside's function 109
E. Pap, Regular Borel t-decomposable measure 113
E. Pap, Extensions of the continuous t-conorm decomposable measure 121
E. Pap, Ch. Swartz, On summable boundedness in convergence linear spaces 131
D. Nikolić-Despotović, S. Pilipović, On the behaviour of the Stieltjes transformation at the origin 145
O. Hadžić, A common fixed point theorem for two sequences of mappings in convex metric spaces 153
O. Hadžić, On multivalued contractions in probabilistic metric spaces 161

Abstracted / Indexed in:
Mathematical Reviews
Zentralblatt für Mathematik
Referativnyĭ Zhurnal
Current Mathematical Publications

Instructions to Authors

Manuscripts written in English (preferably), German, French or Russian should be sent to the Associate Editor in Chief.Three clearly legible copies are required. Authors should also send two copies of any of their unpublished papers to which crucial reference is made in the manuscript. The paper submitted must not be previously published in, nor have been submitted to another journal. Author receive 20 free offprints of their articles.

Manuscripts should be typed double spaced, on one side of the sheet only. Below the title, the name (s) and full postal address (es) of author (s) should be given, followed by a short selfcontained abstract in English (together with the English translation of the title for non-English papers), AMS Mathematics Subject Classification (1990) and key words and phrases. If possible, at the end of the paper a summary in Serbian should be given together with the Serbian translation of the title.

Each section, including the introduction should be numbered, with formulas numbered in the left margin. Nonstandard notations should be explained in the text. Footnotes should be avoided. Drawings should be prepared for direct photographic reproduction at printed size. References should be listed alphabetically and numbered consecutively at the end of manuscript, following the model given below:

[1] O Reilly, M.J.: A uniform scheme for the singularly perturbed Riccati equation, Numer. Math. 50(1987), 483-501.

[2] Ortega, J.M., Rheinboldt, W.C.: Iterative solution of nonlinear equations in several variables, New York and London: Academic Press 1970.

[3] Bohl, E.: Discrete versus continuous models for dissipative systems. In: Numerical models for bifurcation problems, (T. Krüpper, H.D. Mittelmann, H. Weber, eds.), pp. 68-78. Basel, Boston, Stuttgart: Birkhäuser Verlag 1984.

References should be indicated in the text by corresponding arabic numbers in square brackets.

Manuscripts will be retyped and the author will receive a copy of galley proofs. Authors are requested to correct typographical errors only. Alternations exceeding 10% of the typesetting will be charged to the author.

The Review is produced using LATEX. Authors who are able to do so are invited to prepare their manuscripts using LATEX. Hard copy only should be submitted for assesment, but if paper is accepted the author will be asked to send the text on a 5.25″ IBM PC compatible diskette, or to send it by e-mail to following adress:

e-mail: review%unsim@yubgef51.bitnet

UDK 51 YU ISSN 0352 – 0900

УНИВЕРЗИТЕТ У НОВОМ САДУ — UNIVERSITY OF NOVI SAD

ЗБОРНИК РАДОВА
ПРИРОДНО-МАТЕМАТИЧКОГ ФАКУЛТЕТА
СЕРИЈА ЗА МАТЕМАТИКУ

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

21

No. 1

НОВИ САД — NOVI SAD

1991.

Samoupravna interesna zajednica za naučni rad SAP Vojvodine delimično je učestvovala u štampanju ove publikacije.

Na osnovu mišljenja pokrajinskog sekretarijata za obrazovanje, nauku i kulturu br. 413-58/73. od 20. februara 1973. godine, ova knjiga je oslobođena plaćanja posebnog poreza na promet i usluga u prometu.

REVIEW OF RESEARCH

FACULTY OF SCIENCE, MATHEMATICS SERIES

Volume 21, Number 1, 1991

Contents

D. Herceg: On a modification of nonlinear overrelaxation methods 1

D. Herceg, Lj. Cvetković, On a method for solving an almost linear system of equations 15

H.A.S. Abujabal, M.S. Khan, Some commutativity conditions for rings 23

H.A.S. Abujabal, M.S. Khan, Some commutativity theorems for rings via iteration 33

L. Novak, Ž. Karadžić, D.M. Acketa, A space optimal algorithm for enumerating spanning trees of a connected graph 39

M. V. Petrić, Connectedness of the generalized direct product of regular digraphs 57

E. Pap, Many dimensional Lyapunov convexity theorem for an E_λ - decomposable measure 65

E. Pap, The Brooks - Jewett theorem for non - additive set functions . 75

V. Popa, Fixed points on two complete metric spaces 83

F.M. Khan, M.A. Khan, The sequence space $Ces(p,s)$ and related matrix transformations 95

Sh. Chang, Uniform convexity and the fixed point property 105

L. Kaniok, On sequential measures of noncompactness in topological vector spaces 117

S. Hahn, S. Juckelandt, Zur Approximation von Fixpunkten bei mengenwertigen oberhalbstetigen kompakten Abbildungen 127

B. Fisher, L.C. Kuan, A non - commutative neutrix product of distributions on R^m 135

B. Fisher, Common fixed points of commuting set - valued mappings 147

J. Hejduk, Convergence with respect to some σ - ideals 157

M. Budinčević, Oscillations and the asymptotic behaviour of certain second order neutral difference equations 165

M.I. Kamenskiĭ, V.V. Obukhovskiĭ, On periodic solutions of differential inclusions with unbounded operators in Banach spaces 193

O. Hadžić, On coincidence points in probabilistic metric spaces with a convex structure 193

O. Hadžić, Continuous dependence of the fixed points on parameters in random normed spaces 203

Abstracted / Indexed in:
Mathematical Reviews
Zentralblatt für Mathematik
Referativnyĭ Zhurnal
Current Mathematical Publications

Instructions to Authors

Manuscripts written in English (preferably), German, French or Russian should be sent to the Associate Editor in Chief.Three clearly legible copies are required. Authors should also send two copies of any of their unpublished papers to which crucial reference is made in the manuscript. The paper submitted must not be previously published in, nor have been submitted to another journal. Author receive 20 free offprints of their articles.

Manuscripts should be typed double spaced, on one side of the sheet only. Below the title, the name (s) and full postal address (es) of author (s) should be given, followed by a short selfcontained abstract in English (together with the English translation of the title for non-English papers), AMS Mathematics Subject Classification (1990) and key words and phrases. If possible, at the end of the paper a summary in Serbian should be given together with the Serbian translation of the title.

Each section, including the introduction should be numbered, with formulas numbered in the left margin. Nonstandard notations should be explained in the text. Footnotes should be avoided. Drawings should be prepared for direct photographic reproduction at printed size. References should be listed alphabetically and numbered consecutively at the end of manuscript, following the model given below:

[1] O Reilly, M.J.: A uniform scheme for the singularly perturbed Riccati equation, Numer. Math. 50(1987), 483-501.

[2] Ortega, J.M., Rheinboldt, W.C.: Iterative solution of nonlinear equations in several variables, New York and London: Academic Press 1970.

[3] Bohl, E.: Discrete versus continuous models for dissipative systems. In: Numerical models for bifurcation problems, (T. Krüpper, H.D. Mittelmann, H. Weber, eds.), pp. 68-78. Basel, Boston, Stuttgart: Birkhäuser Verlag 1984.

References should be indicated in the text by corresponding arabic numbers in square brackets.

Manuscripts will be retyped and the author will receive a copy of galley proofs. Authors are requested to correct typographical errors only. Alternations exceeding 10% of the typesetting will be charged to the author.

The Review is produced using LATEX. Authors who are able to do so are invited to prepare their manuscripts using LATEX. Hard copy only should be submitted for assesment, but if paper is accepted the author will be asked to send the text on a 5.25″ IBM PC compatible diskette, or to send it by e-mail to following adress:
e-mail: review%unsim@yubgef51.bitnet

UDK 51 YU ISSN 0352–0900

УНИВЕРЗИТЕТ У НОВОМ САДУ — UNIVERSITY OF NOVI SAD

ЗБОРНИК РАДОВА
ПРИРОДНО-МАТЕМАТИЧКОГ ФАКУЛТЕТА
СЕРИЈА ЗА МАТЕМАТИКУ

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

20

No. 1

НОВИ САД — NOVI SAD

1990.

Samoupravna interesna zajednica za naučni rad SAP Vojvodine delimično je učestvovala u štampanju ove publikacije.

Na osnovu mišljenja pokrajinskog sekretarijata za obrazovanje, nauku i kulturu br. 413-58/73. od 20. februara 1973. godine, ova knjiga je oslobođena plaćanja posebnog poreza na promet i usluga u prometu.

Izdaje: Prirodno-matematički fakultet u Novom Sadu,
21000 Novi Sad, Dr Ilije Đuričića 4, Jugoslavija

Štampa: FTN OOUR Štamparija za grafičku delatnost
Novi Sad, V. Vlahovića 3. tel: 54-245
Tiraž: 500 primeraka

Univ. u Novom Sadu
Zb. Rad. Prirod.-Mat. Fak.
Ser. Mat. 20, 1 (1990), 1-9

Review of Research
Faculty of Science
Mathematics Series

A NOTE ON ITERATIVE PROCESS FOR SOLVING LINEAR SYSTEMS OF EQUATIONS

Dragoslav Herceg[1]
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

A simple modification of an algorithm from [3] for numerical solution of linear system of equations is considered.The numerical examples show that introducing a parameter the basic method can be accelerated.

AMS Mathematics Subject Classification (1980): 65F10
*Key words and phrases:*Linear system, iterative method, extrapolation.

1. Introduction

Research in the area of economic models is concentrated on determining the convergence conditions for some solution tehniques applied to systems of equations which have none of the usual features of physical systems, such as symmetry, diagonal dominance, or nonnegativity, Fisher, Hallet, [1]. In Kudrinskii, Ostapchuk, [3], an algorithm for the iterative solution of the system Ax=b, assuming only that matrix A is nonsingular, is given. This algorithm is nonstationary and can be applied without the user's interventions. Some numerical experiments show that by introducing a parameter, the basic method can be accelerated.

In this paper we shall prove the convergence of our modification of the basic method assuming that the introduced parameter satisfies a condition.

---

[1] This research was partly supported by NSF and SMCI for Research of SAP Vojvodina through funds made available to the U.S.-Yugoslav Joint Board on Scientific and Technological Cooperation (Grant JF 799).

2. Preliminaries

We shall consider a system of linear equations

$$Ax = b$$

where $A = [a_{ij}] \in \mathbf{R}^n$ is a nonsingular matrix and $b = [b_1, \ldots, b_n]^T$ is a known vector.

Before describing our modification of the basic method from Kudrinskii, Ostapchuk, [3], we shall give some notations.

Let x^T be the transpose of vector $x \in \mathbf{R}^n$ and let $(x, y) = x^T y$ be the inner product of $x, y \in \mathbf{R}^n$. We denote by $\|x\|$ the Euclidean norm: $\|x\| = \sqrt{(x^T, x)}$ and by $\|x\|_\infty$ and $\|A\|_\infty$ the maximum vector norm $(\|x\|_\infty = max\{| x_i |: i = 1, 2, \ldots, n\})$ and the corresponding matrix norm. Let $a_i = [a_{i1}, \ldots, a_{in}], i = 1, 2, \ldots, n$ be row vectors.

We shall use the following notations (for $A \in \mathbf{R}^{n,n}$ and $x \in \mathbf{R}^n$):

$$N = \{1, 2, \ldots, n\}.$$

$$r(x^k) = b - Ax^k = [r_1^k, r_2^k, \ldots, r_n^k]^T \in \mathbf{R}^n.$$

Now, we are going to describe our method. With arbitrary $x^0 \in \mathbf{R}^n$ we form the iterations by

(1) $$x^{k+1} = Gx^k, \quad k = 0, 1, \ldots,$$

where

(2) $$Gx = x + \frac{\sigma}{\|a_i^T\|^2} r_i a_i^T,$$

(3) $$| r_i | = \|r(x)\|_\infty \quad \text{and}$$

(4) $$0 < \sigma < 2.$$

3. Convergence theorem

The proof of the following theorem is based on Kudrinskii, Ostapchuk, [3], and More [4].

Theorem 1. *Let A be a nonsingular matrix. Then for any $x^0 \in \mathbf{R}^n$ the iterations (1)-(4) converge to the unique solution x^* of the system $Ax = b$.*

Proof. The nonsingularity of A ensures that the sequence $\{x^k\}$ is well defined and that the unique solution x^* of the system $Ax = b$ exists. It is easy to see that is continuous for $x \in \mathbf{R}^n$. Let us define

$$\sigma_k = \frac{\sigma}{\|a_{i_k}^T\|^2}.$$

From (1)-(3) we obtain

$$x^{k+1} = x^k + \frac{\sigma}{\|a_{i_k}^T\|^2}\, r_{i_k}^k\, a_{i_k}^T, \quad |\, r_{i_k}^k \,| = \|r(x^k)\|_\infty,$$

and

$$x^* - x^{k+1} = x^* - x^k - \sigma_k\, r_{i_k}^k\, a_{i_k}^T,$$

$$\|x^* - x^{k+1}\|^2 = (x^* - x^k - \sigma_k\, r_{i_k}^k\, a_{i_k}^T)^T (x^* - x^k - \sigma_k\, r_{i_k}^k\, a_{i_k}^T)$$

$$= \|x^* - x^k\|^2 - 2\sigma_k\, r_{i_k}^k\, a_{i_k}(x^* - x^k) + (\sigma_k\, r_{i_k}^k\, \|a_{i_k}^T\|)^2.$$

From the definition of $r(x^k)$ we have $r(x^k) = b - Ax^k = Ax^* - Ax^k$ and, thus

$$r_{i_k}^k = a_{i_k}\,(x^* - x^k).$$

So, we now have

$$\|x^* - x^{k+1}\|^2 = \|x^* - x^k\|^2 - \sigma_k\,(r_{i_k}^k)^2\,(2 - \sigma_k\|\, a_{i_k}^T \,\|^2).$$

If $r^k_{i_k} = 0$, then, (3), $\|Ax^k - b\|_\infty = 0$ and follows immediately that x^k is the solution of the system $Ax = b$, and the Theorem is proved.
If $r^k_{i_k} \neq 0$, then because of (4) : $0 < \sigma_k,\ 2 - \sigma_k \|a^T_{i_k}\|^2 > 0$, we obtain

$$\|x^* - x^{k+1}\| < \|x^* - x^k\|.$$

So, $\varepsilon_k = \|x^* - x^k\|$ is a decreasing sequence of nonnegative numbers and hence convergent. Thus $\{x^k\}$ is bounded, and if $\{x^{k_i}\}$ is any convergent subsequence such that

$$\lim_{i\to\infty} x^{k_i} = y^* \neq x^*,$$

then

$$\lim_{i\to\infty} \varepsilon_{k_i+1} = \|Gy^* - x^*\| < \|y^* - x^*\| = \lim_{i\to\infty} \varepsilon_{k_i},$$

which is in contradiction with the fact that $\{\varepsilon_k\}$ is convergent. Therefore,

$$\lim_{i\to\infty} x^{k_i} = x^*,$$

and consequently, $\lim_{k\to\infty} x^k = x^*$. □

In Kudrinskii, Ostapchuk, [3], $\sigma = 1$ in each iteration. Now, we are going to consider the following modification of method (1)-(4):

(5) $$x^{k+1} = G_k\, x^k, \quad k = 0, 1, \ldots,$$

(6) $$G_k x = x + \frac{f(k)}{\|a^T_i\|^2}\, r_i a^T_i,$$

(7) $$|\, r_i \,| = \|r(x)\|_\infty \quad \text{and}$$

(8) $$f(k) \in (0,2),\ \lim_{k\to\infty} f(k) = \sigma \in (0,2).$$

Theorem 2. *Let A be a nonsingular matrix. Then for any $x^0 \in \mathbf{R}$ the iterations (5)-(8) converge to the unique solution x^* of the system $Ax = b$.*

Proof. Let us denote $\sigma_k = \frac{f(k)}{\|a_{i_k}^T\|^2}$. From (5)-(7) we obtain

$$x^{k+1} = x^k + \frac{f(k)}{\|a_{i_k}^T\|^2}\, r_{i_k}^k\, a_{i_k}^T\,,\ \mid r_{i_k}^k \mid = \|r(x^k)\|_\infty .$$

As in the proof of the Theorem 1 we obtain

$$\|x^* - x^{k+1}\| < \|x^* - x^k\|.$$

So, $\varepsilon_k = \|x^* - x^k\|$ is a decreasing sequence of nonnegative numbers and hence convergent. Thus $\{x^k\}$ is bounded and, since

$$G_k\, x^k - Gx^k = (f(k) - \sigma)\, \frac{1}{\|a_{i_k}^T\|^2}\, r_{i_k}^k\, a_{i_k}^T\,,$$

we have for arbitrary small $\varepsilon > 0$

$$\text{(9)} \qquad \|G_k\, x^k - Gx^k\| \leq \mid f(k) - \sigma \mid \frac{\|r(x^k)\|_\infty}{\|a_{i_k}^T\|} < \varepsilon \text{ for } k > K(\varepsilon).$$

If $\{x^{k_i}\}$ is any convergent subsequence such that

$$\lim_{i\to\infty} x^{k_i} = y^* \neq x^*,$$

then for arbitrary $\varepsilon > 0$ from (9) and the continuity of G it follows that

$$\|G_{k_i}\, x^{k_i} - Gy^*\| \leq \|G_{k_i}\, x^{k_i} - Gx^{k_i}\| + \|Gx^{k_i} - Gy^*\| < \varepsilon \text{ for } i > I(\varepsilon),$$

i.e.

$$\lim_{i\to\infty} G_{k_i}\, x^{k_i} = Gy^*.$$

Now we have

$$\lim_{i\to\infty} \varepsilon_{k_i+1} = \|G_{k_i}\, x^{k_i} - x^*\| = \|Gy^* - x^*\| < \|y^* - x^*\| = \lim_{i\to\infty} \varepsilon_{k_i},$$

which is in contradiction with the fact that $\{\varepsilon_k\}$ is convergent. Therefore

$$\lim_{i\to\infty} x^{k_i} = x^*,$$

and consequently, $\lim_{i\to\infty} x^k = x^*$. □

4. Numerical examples

In this section we shall describe the results of numerical experiments which were designed to illustrate the effectiveness of the method discussed in the previous section. To do this we solve the test problem $Ax = b$, where

$$A = \begin{bmatrix} 3 & -1 & & & & \\ -1 & 3 & -1 & & & \\ & -1 & 3-1 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 3 & -1 \\ & & & & -1 & 3 \end{bmatrix} \in \mathbf{R}^{10,10}, \quad b = \begin{bmatrix} 2 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 2 \end{bmatrix} \in \mathbf{R}^{10}.$$

For this problem the solution $\bar{x}$ is known and is unity everywhere. We define the iteration error vector by

$$\varepsilon^k = x^k - \bar{x},$$

and always let E_T denote the true error measure

$$E_T = \frac{\|\varepsilon^k\|}{\|\bar{x}\|}.$$

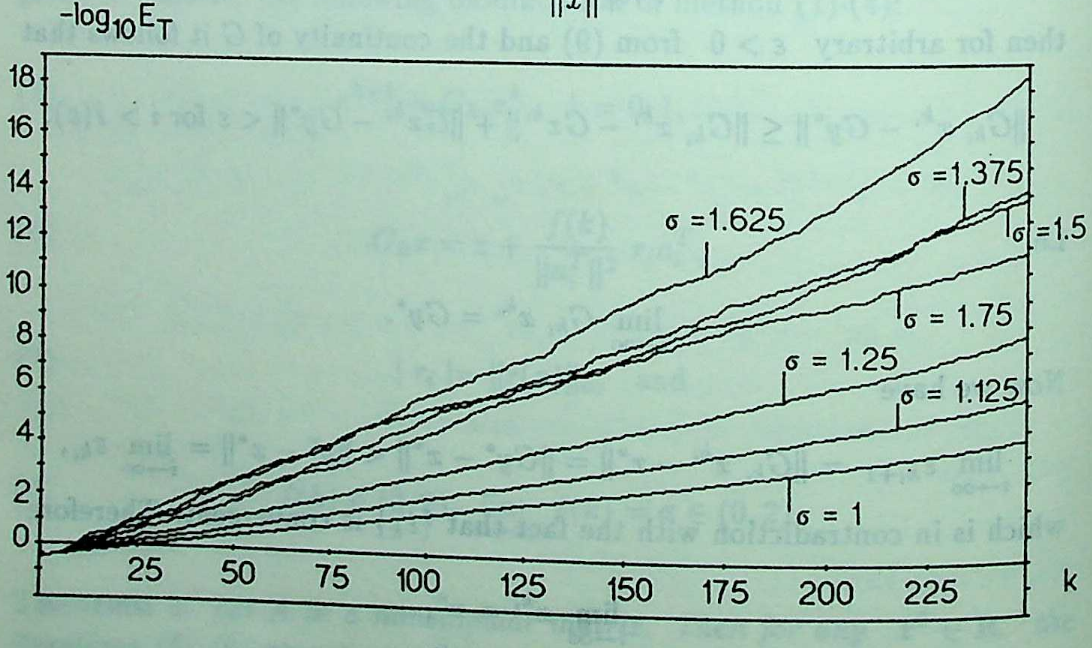


Figure 1.

In Fig. 1 graphs of $-\log_{10} E_T$ versus k are given for different values of σ in solving the problem $Ax = b$. We used the null vector as the initial guess for $\bar{x}$.

Table 1 gives for each σ the number of iterations needed to reduce the norm of the error vector by a factor 10^{-3}, as compared with the norm of the initial error vector, i.e., when $\|\varepsilon^k\|/\|\varepsilon^0\| \leq 10^{-3}$.

In order to illustrate the behaviour of the adaptive procedure (1)-(3) with $x^0 = 0$,

$$(10)\quad f(1) = 1.999,\ f(k) = 2 - \omega + \frac{\omega}{ln(1+k)},\ \omega \in (0,2),\ k = 2,3,\ldots,$$

as a function of ω, we are giving again for each ω the number of iterations needed to have $\|\varepsilon^k\|/\|\varepsilon^0\| \leq 10^{-3}$. As a start value we use $f(0) = 1.999$ for each ω.

Table 1.

σ	k
1.000	293
1.125	226
1.250	170
1.375	112
1.500	104
1.625	94
1.750	99
1.875	192

Table 2.

ω	k
0.2500	141
0.3125	97
0.3750	93
0.4375	86
0.5000	73
0.5625	80
0.6250	83
0.6875	81

Numerical examples show that, very often, our methods (1)-(3) with parametar $\sigma \in (0,2)$, and its adaptive version given by (5)-(8), converge faster than the basic method from [3], i.e. method (1)-(3) with $\sigma = 1$.

Acknowledgement

A part of these investigations was carried out while the author was visiting the Departamento de Mathematica, Universidade de Coimbra, Portugal. The author wishes to express his gratitude to Departamento de Mathematica for its support during his visit to Coimbra.

References

[1] Fisher, P.G., Hughes Hallet, A.J.: Nonstationary model sulution techniques and USA algorithm. Forthcoming.

[2] Hageman, L.A., Young, D.M.: Applied iterative method. New York, London, Toronto, Sydney, San Francisco : Academic Press 1981.

[3] Kudrinskii, V.Yu., Ostapchuk, V.S.: Ob odnom iteracionnom metode resheniya sistemy lineinyinj algebraicheskih uravnenii, Vychisl. i prikl. matematika 55, Kiev, 1985, 64-68.

[4] More, J.J.: Nonlinear generalizations of matrix diagonal dominance with application to Gauss-Seidel iterations. SIAM J. Numer. Anal. 9(1972), 357-378.

[5] Ortega, J.M., Rheinboldt, W.C.: Iterative solution of nonlinear equations in several variables. New York and London: Academic Press 1970.

[6] Varah, J.M.: A lower bound for the smallest singular value of a matrix. Lin. Alg. Apll. 11 (1975), 3-5.

REZIME

NOTA NA ITERATIVNI PROCES ZA REŠAVANJE LINEARNOG SISTEMA JEDNAČINA

Posmatra se jedna modifikacija poznatog postupka za rešavanje sistema linearnih jednačina. Numerički primeri pokazuju da se uvodjenjem parametra osnovni postupak može ubrzati.

Received by the editors February 14, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 11-15

Review of Research
Faculty of Science
Mathematics Series

ON AN ITERATIVE METHOD FOR A SYSTEM OF EQUATIONS

Dragoslav Herceg and **Ljiljana Cvetković**
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In order to solve systems of linear equations a modification of the Accelerated Overrelaxation (AOR) method is used, introduced in [8]. By using a technique similar to Sassenfeld's criteria, criteria of convergence are obtained which do not demand additional properties of the matrix of the linear system.

AMS *Mathematics Subject Classification* (1980): 65F10.
Key words and phrases: Linear system, iterative method, relaxation method.

1. Introduction

There are a lot of iterative methods for solving systems of linear equations. Among them, the so-called relaxation methods play a very important role. The iteration matrices of these methods depend on several real parameters, so it is possible to accelerate the rate of convergence by using a proper choice of these parameters. For example, some well known methods of such a type are the Jacobi, Gauss-Seidel, SOR and AOR method.

In this paper we shall consider a more general method, introduced by James, [8] . As we shall see, it is a special case of the MAOR (Modified Accelerated Overrelaxation) method, which we have introduced for the nonlinear case in paper [2] and investigated in [3],[4].

We shal obtain criteria of convergence, which do not demand additional properties of the matrix of our linear system (such as SDD or something else).

2. Preliminaries

In order to solve a system of linear equations

$$(1) \qquad Ax = b,$$

where $A = [a_{ij}] \in C^{n,n}$, $b \in C^n$, $a_{ii} \neq 0$, $i \in N := \{1, 2, ..., n\}$, we shall use the following iterative method

$$(2) \qquad x^{k+1} = M(\sigma, \Omega)x^k + d, \;\; k = 0, 1, ...,$$

where

$$M(\sigma, \Omega) = (E - \sigma\Omega L)^{-1}((E - \Omega) + (1 - \sigma)\Omega L + \Omega U),$$

$$d = (E - \sigma\Omega L)^{-1}\Omega D^{-1}b,$$

$A = D(E - L - U)$ is the standard splitting, σ is a real parameter and $\Omega = diag(\omega_1, \omega_2, ..., \omega_n)$ is a real regular diagonal relaxation matrix.

If we choose $\omega_1 = \omega_2 = ... = \omega_n = \omega$, we obtain the AOR method with the parameters $\omega\sigma$ and ω, and if, in addition, $\sigma = 1$, the method reduced to the SOR method with the parameter ω.

Also, we can compare this method to the MAOR one. It is obvious that

$$M(\sigma, \Omega) = H(\Omega^{-1}; \sigma, 1),$$

where H denotes the MAOR iteration matrix (cf.[2], [3], [4]).

3. An upper bound for the maximum norm of iteration matrix

Theorem 1. *Let*

$$p_i = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{ii}|}(|1 - \sigma\omega_j| + |\sigma||\omega_j|p_j) + \sum_{j=i+1}^{n} \frac{|a_{ij}|}{|a_{ii}|}, \; i \in N.$$

Then

$$\|M(\sigma, \Omega)\|_\infty \leq \max_{i \in N}\{|1 - \omega_i| + |\omega_i|p_i\}.$$

Proof. Let y be a vector with the following property:

$$\|y\|_\infty = 1, \ \ \|M(\sigma,\Omega)\|_\infty = \|M(\sigma,\Omega)y\|_\infty.$$

If we denote $M(\sigma,\Omega)y = z$, we shall obtain

$$(E - \sigma\Omega L)z = ((E-\Omega) + (1-\sigma)\Omega L + \Omega U)\, y,$$

or, written by components

$$z_i + \sigma\omega_i \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} z_j = y_i - \omega_i y_i - (1-\sigma)\omega_i \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} y_j - \omega_i \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} y_j.$$

From the last equality we obtain

$$(3) \quad z_i - (1-\omega_i)y_i = \ -\omega_i \left[\sigma \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} z_j + (1-\sigma) \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} y_j + \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} y_j \right],$$

from which, by using mathematical induction, we can prove that

$$(4) \qquad |z_i - (1-\omega_i)y_i| \le |\omega_i| p_i, \ i \in N.$$

Indeed, for $i = 1$ it is obvious that

$$|z_1 - (1-\omega_1)y_1| \le |\omega_1| p_1.$$

Now, we suppose that

$$(5) \qquad |z_k - (1-\omega_k)y_k| \le |\omega_k| p_k, \ \ k = 1,2,...,i-1.$$

From (3) we obtain

$$|z_i - (1-\omega_i)y_i| \le |\omega_i| \left[\sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{ii}|} |\sigma z_j + (1-\sigma)y_j| + \sum_{j=i+1}^{n} \frac{|a_{ij}|}{|a_{ii}|} \right],$$

and it is sufficient to show that

$$|\sigma z_j + (1-\sigma)y_j| \le |\sigma||\omega_j| p_j + |1 - \sigma\omega_j|, \ \ j = 1,2,...,i-1.$$

Because of (5), it follows that for $j = 1,2,...,i-1$

$$|\sigma z_j + (1-\sigma)y_j| = |\sigma\,(z_j - (1-\omega_j)y_j) + (1-\sigma\omega_j)y_j|$$

$$\leq |\sigma||z_j - (1-\omega_j)y_j| + |1-\sigma\omega_j|$$

$$\leq |\sigma||\omega_j|p_j + |1-\sigma\omega_j|.$$

Hence, relation (4) is proved. Now, it is obvious that

$$|z_i| \leq |1-\omega_i| + |\omega_i|p_i, \; i \in N,$$

which completes the proof.

□

The result of Theorem 1 is a generalization of the Sassenfeld's criteria. One can see that Theorem 1 gives a sufficient condition for the convergence of method (2) without extra properties of matrix A. It is sufficient to choose a parameter σ and a matrix Ω, so that

$$\max_{i \in N}\{|1-\omega_i| + |\omega_i|p_i\} < 1.$$

References

[1] Cvetković,Lj., Herceg,D., Some sufficient conditions for convergence of AOR-method. Numerical Methods and Approximation Theory, Milovanović, G.V., ed., Faculty of Electronic Engineering, Niš, 1984, 13–18.

[2] Cvetković, Lj., Herceg, D., On a generalized vSOR-Newton's method. IV Conference on Applied Mathematics, Vrdoljak,B., ed., Faculty of Civil Engineering, Split, 1985, 99–102.

[3] Cvetković, Lj., On a local convergence of the vAORN method. Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 13(1983), 203–209.

[4] Cvetković, Lj., Herceg, D., Über die Konvergenz des VAOR-Verfahrens. Z. angew. Math. Mech. Bd. 66(1986), 405–406.

[5] Cvetković, Lj., Herceg, D., Some results on M- and H-matrices. Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 16,2(1986), 121–129.

[6] Cvetković, Lj., Herceg, D., Convergence theory for AOR method. Journal of Computational Mathematics. 8(1990), 128–134.

[7] Cvetković, Lj., Herceg, D., An improvement for the area of convergence of the AOR method. Anal. Numer. Theor. Approx. 16(1987), 109–115.

[8] James, K.R., Convergence of matrix iterations subject to diagonal dominance. SIAM J. Numer. Anal. 10(1973), 478–484.

REZIME

O JEDNOM ITERATIVNOM POSTUPKU ZA SISTEM JEDNAČINA

Za rešavanje sistema linearnih jednačina koristi se jedna modifikacija AOR postupka, definisana u [8]. Korišćenjem tehnike slične Sassenfeldovom kriterijumu, dobijeno je jedno gornje ograničenje za maksimum normu iterativne matrice i kriterijum za konvergenciju, koji ne zahteva dodatne pretpostavke o matrici posmatranog linearnog sistema (SDD ili nešto drugo).

Received by the editors March 27, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 17-26

Review of Research
Faculty of Science
Mathematics Series

A FAMILY OF EXPONENTIAL SPLINE DIFFERENCE SCHEMES

Katarina Surla, Dragoslav Herceg and **Ljiljana Cvetković**
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

Using a spline in tension for the problem: $-\varepsilon y" + p(x)y = f(x),\ 0 < x < 1,\ y(0) = \alpha_0,\ y(1) = \alpha_1,\ 0 < \varepsilon << 1,$ a family of difference schemes is derived. The schemes have a second order of uniform convergence. Some of them converge with respect to ε.

AMS Mathematics Subject Classification (1980): 65L10
Key words and phrases : Singular perturbation, small parameter, uniform convergence, spline in tension, difference scheme.

1. Introduction

We shall consider the problem

$$(1)\qquad \begin{cases} -\varepsilon y" + p(x)y = f(x),\ 0 < x < 1, \\ y(0) = \alpha_0,\ \ y(1) = \alpha_1, \end{cases}$$

where $0 < x < 1$, $p(x)$ and $f(x)$ are smooth functions and $p(x) \geq p > 0$.

It is known that problem (1) has a unique solution y, which in general displays boundary layers at $x = 0$ and $x = 1$. The follownig lemma describes some properties of the exact solution $y = y(x)$.

This research was supported partly by NSF and SCI for Scientific Research of SAP Vojvodina through funds made available to the U.S.-Yugoslav Joint Board on Scientific and Technological Cooperation (Grant JF 799).

Lemma 1 ([1]). *Let* $y(x) \in C^4[0,1]$ *and* $p'(0) = p'1) = 0$. *The the solution of (1) can be written in the form*

$$y(x) = u(x) + w(x) + g(x),$$

where

$$\begin{aligned} u(x) &= q_0 \exp(-x(p(0)/\varepsilon)^{1/2}) \\ w(x) &= q_1 \exp(-(1-x)(p(1)/\varepsilon)^{1/2}) \end{aligned}$$

q_0, q_1 *are bounded functions of* ε *independent of* x *and*

$$| q^{(i)}(x) | \le M(1 + \varepsilon^{1-(1/2)i}), \;\; i = 0(1)4,$$

M *is a constant independent of* ε. □

According to Lemma 1 we have

$$y(x) = \begin{cases} u(x) + g_1(x), & 0 < x \le 1/2, \\ w(x) + g_2(x), & 1/2 < x \le 1, \end{cases}$$

where

$$| g_j^{(i)}(x) | \le M(1 + \varepsilon^{1-(1/2)i}), \;\; i = 0(1)4, \;\; j = 1,2.$$

Taking into account this in [3] the collocation method via the spline in tension for problem (1) id derived. A boundary layer functions $u(x)$ and $w(x)$ are introduced into the base of the spline. The uniform convergence of the second order is achived. In [4] the corresponding difference scheme is analysed. The optimal order of the convergence in the sense of [1] is proved. In this paper we formed differnt difference schemes via a spline in tension which satisfies different collocation conditions. They are consequence of a different approxiamation of functions $p(x)$ and $f(x)$ (piecewise constant and piecewise linear). Some of them heve the optimal error estimate in the sense of [1], i.e.

$$(2) \qquad | y_i - v_i | \le M h \min(h, \sqrt{\varepsilon}),$$

where $y_i = y(x_i)$, v_i is the approximate value for y_i, $x_i = ih$, $i = 0(1)n+1$, $h = 1/n+1$, M is a constant independent of ε and h.

The corresponding results for the cubic spline are given in [6].

Throughout the paper M denotes the positive constants that may take different values in different formulas, but are always independent of ε and h. All the constants in the asymptotic equalities are independent of ε and h.

2. Derivation of the schemes

We seek the solution of problem (1) in the form of the spline in tension $e(x)$, on the mesh $x_i = ih$, $i = 0(1)n+1$, $h = 1/n+1$. On each interval $[x_j, x_{j+1}]$, $e(x)$ has the form

$$(3) \qquad e(x) = e_j(x) = v_{j+1}t + v_j(1-t) + \frac{d_{j+1}}{\rho_j^2}\left(\frac{sh\mu_j t}{sh\mu_j} - t\right) +$$

$$\frac{d_j}{\rho_j^2}\left(\frac{sh\mu_j(1-t)}{sh\mu_j} - (1-t)\right),$$

where $\mu_j = \rho_j h$, $t = \frac{x - x_j}{h}$, $x \in [x_j, x_{j+1}]$.

The function $e(x)$ belong to $C^2[0,1]$ and $e_j(x) \in$ span $\{1, x, \exp(-\rho_j x), \exp(\rho_j x)\}$.
The values ρ_j are tension paramenters wich will be detemined. The unknown coefficients d_j and d_{j+1} will be determined so that the function $e(x)$ satisfies the "comparison" problem:

$$(4) \qquad \begin{cases} -\varepsilon e''(x) + \tilde{p}(x)e(x) = \tilde{f}(x), \\ e(0) = \alpha_0, \ \ e(1) = \alpha_1, \end{cases}$$

at the grid points. Here $\tilde{p}(x)$ a $\tilde{f}(x)$ are piecewise polinomial approximations to $p(x)$ and $f(x)$.

Let index j be fixed. Starting with piecewise constant approxiamtions we denote the approximate value for $p(x)$ by p^+ when $x \in [x_j, x_{j+1}]$ and by p^- when $x \in [x_j, x_{j-1}]$. Analogously we denote the approximation for $f(x)$. According to (3) and (4) we put

$$d_{j+1} : \; = d_j^+ = -\frac{f^+ - p^+ v_{j+1}}{\varepsilon} \quad \text{for} \quad x \in [x_j, x_{j+1}] \text{ and}$$

$$d_{:} \; = \; d_j^- = -\frac{f^- - p^- \; v_{j+1}}{\varepsilon} \quad \text{for} \quad x \in [x_{j-1}, x_j].$$

Using $\rho_j = \rho^- = \sqrt{p^-/\varepsilon}$ for $x \in [x_{j-1}, x_j]$ and $\rho_j = \rho^+ = \sqrt{p^+/\varepsilon}$ for $x \in [x_j, x_{j+1}]$, from equation

$$e'_{j-1}(x_j) = e'_j(x_j) \tag{5}$$

and the boundary conditions, we obtain the scheme

$$\tilde{R} v_j = \tilde{Q} f_j, \quad j = 1(1)n, \tag{6}$$

where

$$\begin{aligned}
\tilde{R} v_j &= R^- v_{j-1} + R^c v_j + R^+ v_{j+1}, \\
\tilde{Q} f_j &= Q^- f^- + Q^+ f^+, \\
R^- &= \frac{1}{h} + \frac{p^-}{\varepsilon(\rho^-)^2}\left(-\frac{\rho^-}{sh\mu^-} + \frac{1}{h}\right) \\
R^+ &= \frac{1}{h} + \frac{p^+}{\varepsilon(\rho^+)^2}\left(-\frac{\rho^+}{sh\mu^+} + \frac{1}{h}\right)
\end{aligned}$$

$$R^c = \frac{p^-}{\varepsilon(\rho^-)^2}\left(\rho^- cth\mu^- - \frac{1}{h}\right) + \frac{p^+}{\varepsilon(\rho^+)^2}\left(\rho^+ cth\mu^+ - \frac{1}{h}\right) + \frac{2}{h}$$

$$\begin{aligned}
Q^- &= \frac{1}{\varepsilon(\rho^-)^2}\left(\rho^- cth\mu^- - \frac{\rho^-}{sh\mu^-}\right), \\
Q^+ &= \frac{1}{\varepsilon(\rho^+)^2}\left(\rho^+ cth\mu^+ - \frac{\rho^+}{sh\mu^+}\right), \\
v_0 &= \alpha_0, \quad v_{n+1} = \alpha_1, \quad \mu^- = \rho^- h, \quad \mu^+ = \rho^+ h.
\end{aligned}$$

The chose of approximation to $p(x)$ and $f(x)$ determines the particular scheme.

Let $p^+ = p^- = p(x_j)$ and $f^- = f^+ = f(x_j) = f_j$. Then scheme (6) obtains the form

131253

$$r^{-}v_{j-1} + r^{c}v_j + r^{+}v_{j-1} = q^{c}f_j, \tag{7}$$

where

$$r^{-} = r^{+} = -\frac{\rho_j}{sh\,\mu_j},\ \ r^{c} = 2\,\rho_j\,cth\,\mu_j,\ \ \rho_j = \sqrt{p_j/\varepsilon},\ \ \mu_j = \rho_j h,$$

$$q^{c} = \frac{1}{\sqrt{\varepsilon p_j}}(2\,cth\,\mu_j - \frac{2}{sh\,\mu_j}).$$

If we consider ρ_j as a tension parameter independent of collocation conditions and if $\rho_j \to 0$, the scheme (7) gets the form

$$-\,\varepsilon\,h^{-2}(v_{j-1} - 2v_j + v_{j-1}) = f(x_j). \tag{8}$$

Then the spline in tension becomes a cubic one and scheme (8) is derived in [6] via the cubic spline.

Let $f^{-} = f_{j-1}$, $f^{+} = f_{j+1}$, $p^{-} = p_{j-1}$, and $p^{+} = p_{j+1}$. Now scheme (6) has the form

$$r^{-}v_{j-1} + r^{c}v_j + r^{+}v_{j-1} = q^{-}f_{j-1} + q^{+}f_{j+1}, \tag{9}$$

$$\begin{aligned}
r^{-} &= -\frac{\sqrt{p_{j-1}/\varepsilon}}{sh\,\mu_{j-1}},\ \ r^{+} = -\frac{\sqrt{p_{j+1}/\varepsilon}}{sh\,\mu_{j+1}},\\
r^{c} &= \sqrt{p_{j-1}/\varepsilon}\,cth\,\mu_{j-1} + \sqrt{p_{j+1}/\varepsilon}\,cth\,\mu_{j+1},\\
q^{-} &= \frac{1}{p_{j-1}}(\sqrt{p_{j-1}/\varepsilon}\,cth\,\mu_{j-1} - \frac{\sqrt{p_{j-1}/\varepsilon}}{sh\,\mu_{j-1}}),\\
q^{+} &= \frac{1}{p_{j+1}}(\sqrt{p_{j+1}/\varepsilon}\,cth\,\mu_{j+1} - \frac{\sqrt{p_{j+1}/\varepsilon}}{sh\,\mu_{j+1}}).
\end{aligned}$$

In the case $f^{+} = (f_{j+1} + f_j)/2$, $f^{-} = (f_{j-1} + f_j)/2$, $p^{+} = (p_{j+1} + p_j)/2$, $p^{-} = (p_{j-1} + p_j)/2$, the correspondong scheme is the derived in [2]:

$$(10)\qquad r^- v_{j-1} + r^c v_j + r^+ v_{j-1} = q^- f_{j-1} + q^c f_j + q^+ f_{j+1},$$

$$r^- = -\frac{\rho_{j-1}}{sh\,\mu_{j-1}},\ r^+ = -\frac{\rho_j}{sh\,\mu_j},\ r^c = \rho_{j-1} cth\,\mu_{j-1} + \rho_j cth\,\mu_j,$$

$$\begin{aligned}
\rho_j &= \sqrt{(p_{j+1}+p_j)/2\varepsilon},\ \mu_j = h\,\rho_j,\\
q^- &= \frac{1}{2\varepsilon\,\rho_{j-1}}\,(cth\,\mu_{j-1} - \frac{1}{sh\,\mu_{j-1}}),\\
q^+ &= \frac{1}{2\varepsilon\,\rho_{j+1}}\,(cth\,\mu_{j+1} - \frac{1}{sh\,\mu_{j+1}}),\ q^c = q^- + q^+.
\end{aligned}$$

In [2] it is proved that scheme (10) has a second order of the uniform convergence.

When $\tilde{p}(x) = \frac{p_j - p_{j-1}}{h}\,(x - x_{j-1}) + p_{j-1}$ for $x \in [x_{j-1}, x_j]$ equation (5) leads to the scheme derived in [5]:

$$(11)\qquad r^- v_{j-1} + r^c v_j + r^+ v_{j-1} = q^- f_{j-1} + q^c f_j + q^+ f_{j+1},$$

$$\begin{aligned}
r^- &= -\frac{\mu_{j-1}}{sh\,\mu_{j-1}},\ r^+ = -\frac{\mu_{j+1}}{sh\,\mu_{j-1}},\ r^c = 2\mu_j\ cth\,\mu_j,\\
q^- &= -\frac{1}{p_{j-1}}\,(1 - \frac{\mu_{j-1}}{sh\,\mu_{j-1}}),\ q^+ = -\frac{1}{p_{j+1}}\,(1 - \frac{\mu_{j+1}}{sh\,\mu_{j+1}}),\\
q^c &= -\frac{2}{p_j}\,(-1 + \mu_j\ cth\mu_j),\ \mu_j = h\,\sqrt{p_j/\varepsilon}.
\end{aligned}$$

For scheme (11) estimate (2) is valid ([5]).

When ε is fixed and $h \to 0$, the mentioned schemes become within $0(h^4)$ the schemes derived in [6] via the spline without the fitting factors.

3. Convergence of the schemes

The following theorem gives sufficient conditions for the convergence of schemes (6).

Theorem 1 *Let in (1)* $y(x) \in c^4[0,1]$ *and* $p'(0) = p'(1) = 0$. *Let* v_j *be the approxiamtion to* $y(x_j)$ *obtained using scheme (6) where*

$$\begin{aligned} p^- &= p(x_j - \alpha h) + 0(h^2), \ p^+ = p(x_j + \alpha h) + 0(h^2), \\ f^- &= f(x_j - \alpha h) + 0(h^2), \ f^+ = f(x_j + \alpha h) + 0(h^2), 0 \le \alpha \le 1. \end{aligned}$$

Then, the estimate

$$(12) \qquad | \, y(x_j) - v_j \, | \le Mh^2$$

holds.

Proof. Let A be a matrix corresponding to scheme (6). Then

$$(13) \qquad | \, y(x_j) - v_j \, | \le \| A^{-1} \| \max_j | \, \tilde{\tau}_j(y) \, |$$

where

$$\begin{aligned} \tilde{\tau}_j(y) &= \tilde{R}y_j - \tilde{R}v_j = \tilde{R}y_j - \tilde{Q}f_j = \tilde{R}y_j - Qf_j + Qf_j - \tilde{Q}f_j, \\ Qf_j &= q^+(-\varepsilon y"_j + p^+ y_j) + q^-(-\varepsilon y"_j + p^- y_j). \end{aligned}$$

After some Taylor developments, we obtain that

$$(14) \qquad | \, Qf_j - \tilde{Q}f_j \, | \le \begin{cases} Mh^3/\varepsilon & \text{for } h^2 \le \varepsilon \\ Mh^2/\sqrt{\varepsilon} & \text{for } \varepsilon \le h^2. \end{cases}$$

According to Lemma 1, we have

$$(13) \qquad \tau_j(y) = \tilde{R}y_j - Qf_j = \tau_j(u) + \tau_j(w) + \tau_j(g),$$

Now, we shall estimate separately the contribution to the error from functions u, w and g. Let $\varepsilon \le h^2$. Then

$$\tau_j(g) = T_0 g_j + T_1 g_{j'} + T_2 g_{j''} + T_3 g_{j'''} + R^- g^{iv}(\sigma_1) + R^+ g^{iv}(\sigma_2),$$

where $$x_{j-1} < \sigma_1 < x_j < \sigma_2 < x_{j+1},$$

$$\begin{aligned} T_0 &= R^- + R^c + R^+ - Q^- p^- - Q^+ p^+ = 0, \ T_1 = -hR^- + hR^+, \\ T_2 &= h^2/2(R^- + R^+) + \varepsilon(Q^- + Q^+), \ T_3 = h^3/3! \, (-R^- + R^+). \end{aligned}$$

By Taylor developments and Lemma 1 we obtain

$$|\tau_j(g)| \le Mh^3/\varepsilon. \tag{14}$$

One can check that $\tau_j(u) = 0$ for $p(x) = p(0) = \text{const}$. Thus, if denote

$$\tau_j(u) = \tau_j(u, p^-, p^+),$$

we have $\tau_j(u, p(0), p(0)) = 0$ and $\tau_j(u) = \tau_j(u, p^-, p^+) - \tau_j(u, p(0), p(0))$. Using Taylor expansions about $p(0)$ we obtain

$$|\tau_j(u)| \le Mh^3/\varepsilon. \tag{15}$$

Further, $\tau_j(w, p(1), p(1)) = 0$ where $\tau_j(w) = \tau_j(w, p^+, p^-)$. In a similar way as for $\tau_j(u)$, we can prove than

$$|\tau_j(w)| \le Mh^3/\varepsilon. \tag{16}$$

Since

$$\| A^{-1} \| \le \begin{cases} Mh^3/\varepsilon & \text{for } h^2 \le \varepsilon \\ Mh^2/\sqrt{\varepsilon} & \text{for } \varepsilon \le h^2, \end{cases} \tag{17}$$

from (11) - (17) we obtain the statement of Theorem 1 when $h^2 \le \varepsilon$.

Let $h^2 \ge \varepsilon$. Since $|R^-|, |R^+| \le M/h$, $|q^-|, |q^+| \le M/\sqrt{\varepsilon}$, from the expression

$$\tau_j(g) = T_0 g_j + T_1 g_{j'} + R^- h^2 g_{j''}(\xi_1) + R^+ h^2 g_{j''}(\xi_2) + \varepsilon(Q^- + Q^+) g_j'',$$

we obtain

$$|\tau_j(g)| \le Mh. \tag{18}$$

Further,

$$\tau_j(u) = \tau_j(u, p, p) - \tau_j(u, p(0), p(0)) = (R^- - R^-(p(0)))u_{j-1} +$$
$$+ (R^c - R^c(p(0), p(0)))u_j + (R^+ - R^+(p(0)))u_{j+1}$$

where $R^- = R^-(p^-)$, $R^+ = R^+(p^+)$, $R^c = R^c(p^-, p^+)$. Using the known properties of exponential functions [1], we have

$$|\tau_j(u)| \le M\varepsilon/h. \tag{19}$$

Similarly $|\tau_j(w)| \le M\varepsilon/h$ and from (13), (18), (19) and (20) we obtain the statement of Theorem 1.

Theorem 2 *Let the assumptions of Theorem 1 be fulfilled. Let v_j be the approximate value for $y(x_j)$ obtained using scheme (7). Then*

$$| y(x_j) - v_j | \leq M \ \min(h^2, \varepsilon), \quad j = 0(1)n+1.$$

Proof. In the case $h^2 \leq \varepsilon$ the statement follows from Theorem 1 ($\alpha = 0$ and the constant in the asymptotic member is equal to zero). For $\varepsilon \leq h^2$ we have that $Qf_j - \tilde{Q}f_j = 0$. Since $|r^-|, |r^+| \leq M\sqrt{\varepsilon}$ and $| \tau_j(g) | \leq M\sqrt{\varepsilon}$, following the proof of the Theorem 1 we obtain that Theorem 2 holds.

References

[1] Doolan, P.E., Miller, H.J.J., Schilders, Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole Press, Dublin, 1980.

[2] Hegarty, F.A., Miller, H.J.J., O'Riordan, E., Uniform second order difference schemes for singular perturbation problems, Boundary and Interior Layer-Computational and Asymptotic Methods (J.J.H. Miller, ed.), Boole Press, Dublin, 1980, 301-305.

[3] Surla, K., A collocation by spline in tension, Approx. Theory and its Appl. 6:2 (1990) 101-110.

[4] Surla, K., Herceg, D., Exponential spline difference scheme for singular perturbation problem, Proceedings of the Conference ISAM'89, Spline in Numerical Analysis, Weissig 1989, 171-180.

[5] Surla, K., Stojanović, M., Solving a singularly perturbed boundary value problem by spline in tension, J. Comput. Appl. Math. 24 (1988) 355-363.

[6] Uzelac, Z., Surla, K., A family of uniformly convergent spline difference schemes for self-adjoint problems, Proceedings of the VI-th Seminar on Applied Mathematics, Tara 1988, B.S. Jovanović, ed. 243-249.

REZIME

FAMILIJA EKSPONENCIJALNIH SPLAJN DIFERENCNIH ŠEMA

Koristeći splajn u tenziji za problem $-\varepsilon y'' + p(x)y = f(x),\ 0 < x < 1,\ y(0) = \alpha_0,\ y(1) = \alpha_1,\ 0 < \varepsilon << 1$, izvedena je familija diferencnih šema. Šeme imaju drugi red uniformne konvergencije. Neke od njih konvergiraju u odnosu na ε.

Received by the editors November 7, 1988.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 27-38

Review of Research
Faculty of Science
Mathematics Series

AN ACCELERATION PROCEDURE OF THE WHITTAKER METHOD BY MEANS OF CONVEXITY

Miguel A. Hernández Verón
Dpto de Matemática Aplicada
Colegio Universitario de La Rioja
C/ Obispo Bustamante No 3
26001 LOGROÑO (La Rioja), SPAIN.

Abstract

The paper studies the influence the convexity of a real function f has in the Whittaker method, in order to get the solution of $f(x)=0$, and obtains a method for accelerating this iterative process. Also, some new iterative processes are obtained.

AMS Mathematic Subject Classification (1980): 65H05
Key words and phrases: Nonlinear equations, iterative method, convexity.

1. Introduction

The aim in this paper is to study the influence of the convexity in the solution of $f(x) = 0, x \in [a,b] \subset \mathbf{R}$, by the iteration $x_n = x_{n-1} - \phi_f f(x_{n-1})$, where ϕ_f is a constant, [1]-[6], in order to get accelerations of this method by means of the convexity of f. Moreover, we shall obtain new iterative processes of the second and third order, and also give sufficient conditions for their convergence.

Assume that the function f satisfies the conditions

$$f(a) < 0 < f(b), f'(x) > 0, f''(x) \geq 0 \text{ for } x \in [a,b]$$

These conditions imply that $f \in C^{(m)}([a,b])$, $m \geq 2$, and it has one and only one root $s \in (a,b)$. Besides, we consider $x_0 \in [a,b]$ such that $f(x_0) > 0$.

First, we are going to introduce the log-degree of convexity, which will provide us with an index of convexity measure of a function at each point. This index is given by $U[f](x) = f''(x) \cdot [f'(x)]^{-2}$.

We shall prove that when the log-degree of convexity of $f, U[f](x)$, decreases, the sequence $\{x_n\}$ converges faster to s. Since the straight lines have a minimum log-degree of convexity, we take $g(x) = f(x) - [f''(s)/2!](x-s)^2$ as an approximation to the tangent line of f at s. Hence, $U[g](x)$ tends to zero as x tends to s; therefore, if we asssume $f'''(x) \geq 0$ in (a, b), there exists an interval $(\alpha, \beta) \subset [a,b], f(\alpha) < 0 < f(\beta)$, such that g is an increasing convex function and $U[g](x) < U[f](x)$ in (α, β), and it is obvious that s is the unique root of g in (a, b). Therefore, when we take accelerations, we shall suppose $m \geq 3$ and $f'''(x) \geq 0$ in $[a, b]$.

Now, we shall consider Whittaker's method for $\phi_f = [f'(b)]^{-1}$. Then,

$$f''(s)(x_{n-1} - s)^2 \sim f''(x_{n-1})(x_{n-1} - x_n)^2 \left[\frac{f'(b)}{f'(x_{n-1})}\right]^2$$

since

$$\lim_n \frac{f''(s)(x_{n-1} - s)^2}{f''(x_{n-1})(x_{n-1} - x_n)^2 [\frac{f'(b)}{f'(x_{n-1})}]^2} = 1,$$

and it follows that

$$g(x_{n-1}) = f(x_{n-1}) - \frac{f''(x_{n-1})f(x_{n-1})}{2![f'(x_{n-1})]^2}$$

Then, we obtain $y_n = x_{n-1} - \phi_g g(x_{n-1})$, and this sequence converges faster to s than does $\{x_n\}$, and this acceleration allows us to define new iterative processes.

2. An index of convexity measure

Let $h \in C^{(2)}(V)$ a convex function and V a neighbourhood of a suitable $x_0 \in [a,b]$. The curvature K, [3], is a measure of the convexity of a function

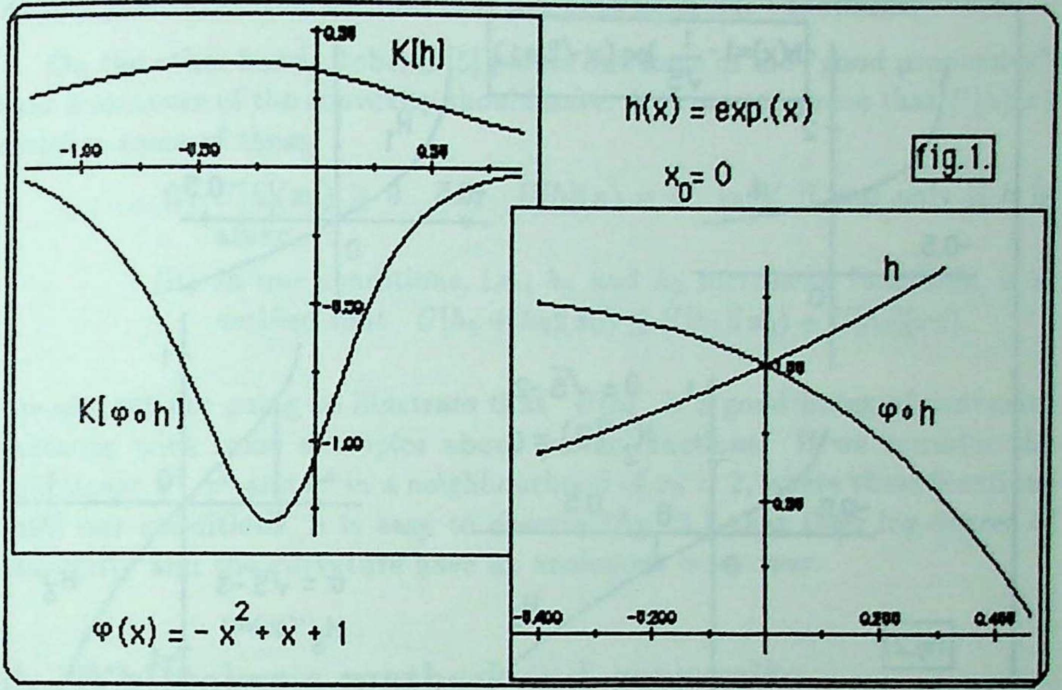


Figure 1:

at each point, note that if φ is a concave function in $C^{(2)}(U), h(x_0) \in U$, with $\varphi'(h(x_0)) = 1$, it is clear that $\varphi \circ h$ has a smaller curvature than h, fig.1., since that

$$K(\varphi \circ h)(x_0) = K(h)(x_0) + \frac{\varphi''(h(x_0))h'(x_0)^2}{[1 + h'(x_0)^2]^{3/2}}$$

Hence, by applying a concave operator to a convex function we obtain a function with a smaller curvature than the original convex function.

Taking this account, if we consider the logarithmic function and the convex function $T[h]$, with $T[h](x) = h(x) - h(x_0) + 1$, we are in the previous conditions. Now, if we apply the logarithmic operator to $T[h]$ sucessively, until we obtain a concave function, we can define an index of the convexity measure of a function at each point considering the number of times that we need to apply the logarithmic operator to get a concave

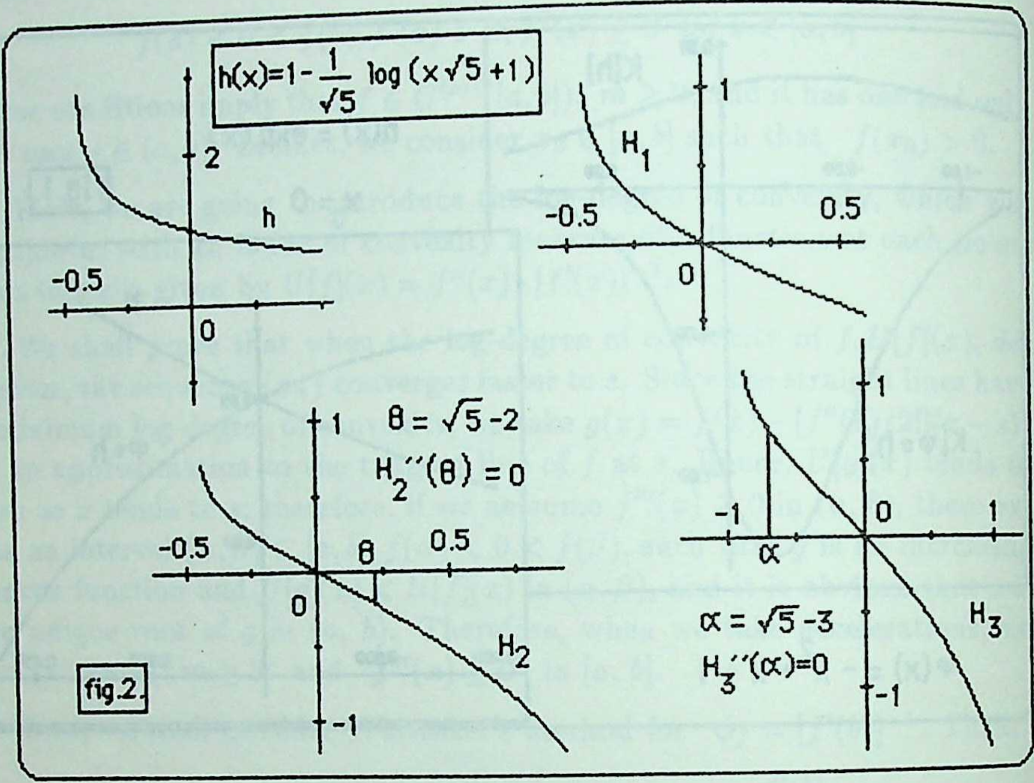


Figure 2:

function. So, if we define $H_n(x) = \log G_{n-1}(x)$ with $G_0(x) = T[f](x)$ and $G_n(x) = T[H_n](x)$ for $n \geq 1$, the sequence $\{H_n(x)\}$ will characterize the log-degree of convexity of h, since $K(h)(x_0) > K(H_1)(x_0) > \ldots > K(H_n)(x_0) \geq 0 > K(H_{n+1})(x_0) > \ldots$ (fig. 2.)

On the other hand, it is easy to prove by induction that $H_n''(x_0) = h''(x_0) - nh'(x_0)^2$ for all n, and therefore it follows that H_n is convex at x_0 if and only if $h''(x_0)[h'(x_0)]^{-2} \geq n$. When x_0 is a minimum of f, it follows that H_n is convex at x_0 for all n, and conversely. Then, the log-degree of convexity of h at x_0 is defined to be the positive real number given by

$$(1) \qquad U[h](x_0) = h''(x_0)[h'(x_0)]^{-2}$$

If x_0 is a minimum of h, we set $U[h](x_0) = +\infty$.

Note that there exists a similar behaviour between the curvature and the log-degree of convexity of a function, except in a neighbourhood of the critical points; a situation that it is not present in our conditions.

On the other hand, Roberts [5] points out some of the "good properties" that a measure of the convexity should have. It is easy to prove that $U[h](x)$ satisfies some of these.

(i) $U[h](x_0) \geq 0$ and $U[h](x) = 0$ in V if and only if h is affine.

(ii) In our conditions, i.e., h_1 and h_2 increasing functions, it is verified that $U[h_1 + h_2](x_0) \leq U[h_1](x_0) + U[h_2](x_0)$.

Finally, we are going to illustrate that $U[h]$ is a good index of convexity measure with some examples about power functions. If we consider the functions: x^2, x^3 and x^4 in a neighbourhood of $x_0 = 2$, where these functions fulfil our conditions, it is easy to observe (fig. 3.) that their log-degree of convexity and the curvature have an analogous behaviour.

3. Whittaker's method and convexity

Whittaker's method, [4], consists in applying the iterative process

$$x_n = F(x_{n-1}) \quad \text{with} \quad F(x) = x - \lambda f(x) \tag{2}$$

When the method is convergent, i.e. $0 < \lambda \leq [f'(b)]^{-1}$, we obtain a decreasing sequence $\{x_n\}$, such that $\lim_n x_n = s$, and this convergence is linear.

The following results point out the influence of convexity in the convergence of $\{x_n\}$.

Theorem 1. *Assume f and g as in the Introduction, $U[g](x) < U[f](x)$ in $[a,b]$ and let $h(x) = kg(x)$, where $k = \max\{1, f'(b)/g'(b)\}$. Then, the sequence $\{y_n\}$ defined by $y_n = y_{n-1} - \lambda h(y_{n-1})$, with $y_0 = x_0$ and $\lambda \leq \min\{[g'(b)]^{-1}, [f'(b)]^{-1}\}$, is converges to s and $y_n < x_n$ for all n.*

Proof. If we suppose that $f'(b) \leq g'(b)$, then the sequences $\{x_n\}$ defined in (2) and $\{y_n\}$ are two decreasing sequences to s when $\lambda \leq [g'(b)]^{-1}$. We shall prove that $y_n < x_n$, by using the induction.

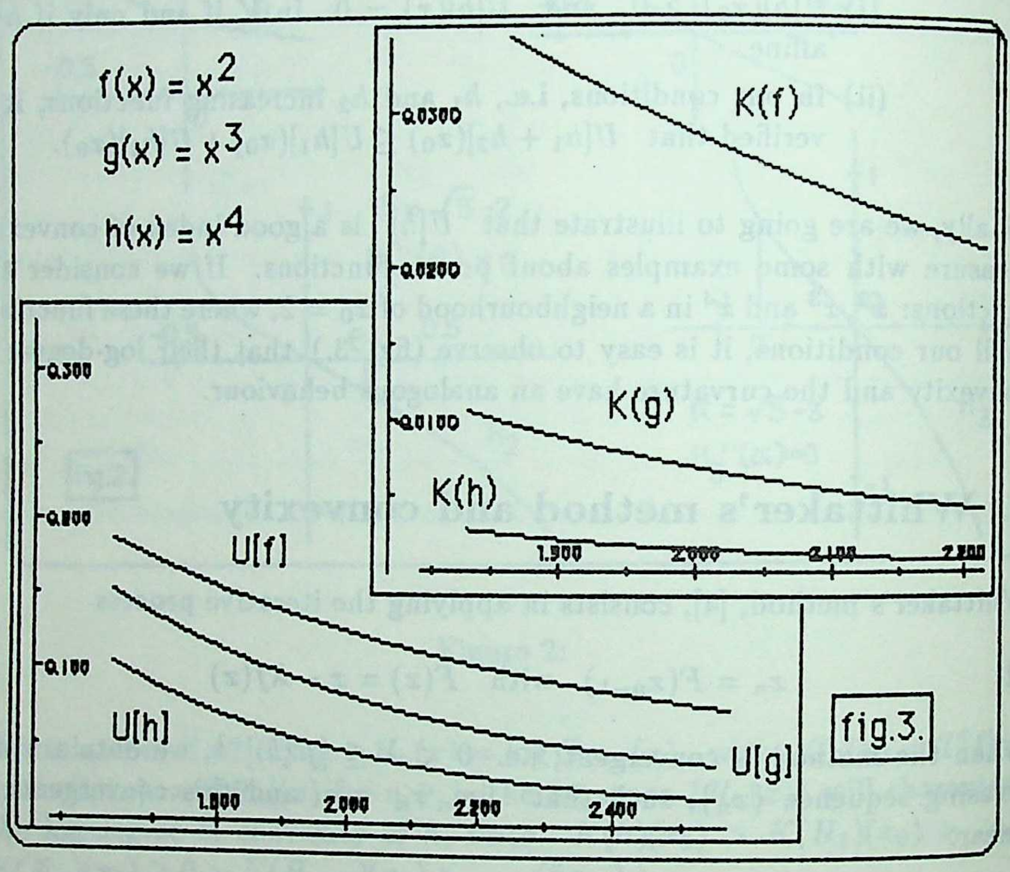

f(x) = x^2
g(x) = x^3
h(x) = x^4
K(f)
K(g)
K(h)
0.0300
0.0200
0.0100
1.900
2.000
2.100
2.200
U[f]
U[g]
U[h]
0.300
0.200
0.100
1.800
2.000
2.200
2.400
fig.3.

Figure 3:

Since $U[g](x) < U[f](x)$ in $[a,b]$, then $g'(x) > f'(x)$ in (a,b). Moreover, it follows that $g(x) > f(x)$ in (s,b), and therefore

$$x_1 - y_1 = \lambda[g(x_0) - f(x_0)] > 0$$

Now, assume $x_k > y_k$ for $k = 1, 2, \ldots, n-1$. Since F is an increasing function we have that $x_n - y_n \geq \lambda[g(y_{n-1}) - f(y_{n-1})]$. In a similar way to the case $n = 1$, we obtain that $x_n > y_n$.

If $g'(b) < f'(b)$, we consider $h(x) = f'(b)[g'(b)]^{-1}g(x)$. Then, $U[h](x) < U[f](x)$ in $[a,b]$ with $h'(b) = f'(b)$, and the result is proved by applying the above arguments to function h.

Since the best approximation in Whittaker's method is obtained when $\lambda = [f'(b)]^{-1}$, we are going to take an acceleration of Whittaker's method for this λ. By reasoning as in the Introduction, as $g'(b) \sim f'(s) \sim f'(x_{n-1})$, we get.

Corollary 1. *Let* $m \geq 3$ *with* $f'''(x) \geq 0$ *in [a, b]. Then, there exists* $n_0 \in N$ *such that the sequence*

$$y_n = x_{n-1} - \frac{f(x_{n-1})}{2f'(x_{n-1})}[2 - f(x_{n-1})U[f](x_{n-1})] \tag{3}$$

is an acceleration of $\{x_n\}$ *for* $n > n_0$.

Proof. Denote $L_f(x) = f(x) \cdot U[f](x)$ and let $F(x) = x - \frac{f(x)}{2f'(x)}[2 - L_f(x)]$. Since $F'(x) = \frac{L_f(x)}{2}[4 - 3L_f(x)] + \frac{f(x)^2}{2f'(x)^3}f'''(x)$, then F is an increasing function in a neghbourhood of s, and so $\{y_n\}$ is a decreasing sequence to s.

Note that as

$$x_n - y_n = f(x_{n-1})[\frac{1}{f'(x_{n-1})}(1 - \frac{f(x_{n-1})U[f](x_{n-1})}{2}) - \frac{1}{f'(b)}],$$

then there exists $n_0 \in N$ such that $x_n > y_n$.

If x_0 is a good approximation to s, the previous acceleration is effective from the first iteration. This is the convex acceleration of Whittaker's method and it enables us to define a new iterative process, and to give a theorem of convergence.

Theorem 2. *Let f be as before with $m \geq 3$. If $U[f](x) < 1/f(b)$ and $U[f'](x) \geq (-1)/f'(a)$ in $[a,b]$, then the iterative process*

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{2f'(x_{n-1})}[2 - f(x_{n-1})U[f](x_{n-1})] \tag{4}$$

yields a decreasing sequence which converges to s.

Proof. If F is as in the Corollary, then

$$F'(x) = \frac{L_f(x)}{2}[(4 - L_f(x)) + L_f(x)(f'(x)U[f'](x) - 2)]$$

Since $L_f(x) < 1$ in (s,b), then $F'(x) \geq 0$ if $f'(x)U[f'](x) \geq -1$, and therefore F is an increasing function in (s,b).

As $x_1 - s = F(x_0) - F(s) = F'(\xi_0)(x_0 - s)$ for $\xi_0 \in (s, x_0)$, it follows that $x_1 \geq s$. Repeating the reasoning we obtain that $x_n \geq s$ for all n. Moreover, it is obvious that $\{x_n\}$ is a decreasing sequence and therefore there exists $\lim_n x_n = u$, and since $f(u)U[f](u) < 1$, one can conclude that $u = s$ taking the limits in (4).

If $m \geq 4$, it is easy to pove that $F(s) = s$, $F'(s) = 0$ and $F''(s) \neq 0$, from which it follows that the iteration (4) is of the second order.

Notice that we have used the condition $f'(x)U[f'](x) \geq -1$, which is weaker than $f'''(x) \geq 0$.

Below we are going to study the convex acceleration of this new iterative process. Let f, g be as before, and defiene $C(x) = g(x)[f(x)]^{-1}$ if $x \neq s$ and $C(s) = g'(s)[f'(s)]^{-1}$. Denote $M = \max\{C(x)/x \in [a,b]\}$.

Theorem 3. *Let $x_n = F(x_{n-1})$ and $y_n = G(y_{n-1})$, where $F(x) = x - \frac{f(x)}{f'(x)}H(L_f(x))$, $G(x) = x - \frac{g(x)}{g'(x)}H(L_g(x))$ and $H(x) = 1 - \frac{x}{2}$. Consider f and g as in Theorem 2, with $U[f](x) > MU[g](x)$ and $x_0 = y_0 \in [a,b]$, Then $y_n < x_n$ for all n.*

Proof. As $L_f(x) > L_g(x)$ in (s,b), then $f(x)/f'(x) < g(x)/g'(x)$. Hence, $F(x) > G(x)$ in (s,b) and by using the induction, $x_n > y_n$ holds for all n.

Now, proceeding as in the Introduction, we obtain that the convex acceleration of $\{x_n\}$ in (4) is

$$(5)\, y_n = x_{n-1} - \frac{f(x_{n-1})}{4f'(x_{n-1})}[2 - L_f(x_{n-1}) + \frac{4+2L_f(x_{n-1})}{2-L_f(x_{n-1})(2-L_f(x_{n-1}))}]$$

This acceleration provides us with a new iterative process. It is known [2], that if we have an iterative process $x_n = F(x_{n-1})$ with $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}H(L_f(x_{n-1}))$ and $H(0) = 1, H'(0) = \frac{1}{2}$ and $|H''(0)| < +\infty$, it has a cubic convergence. Then, if $m \geq 4$, it is easy to prove that the iterative process given by the analogous to (5) is of the third order.

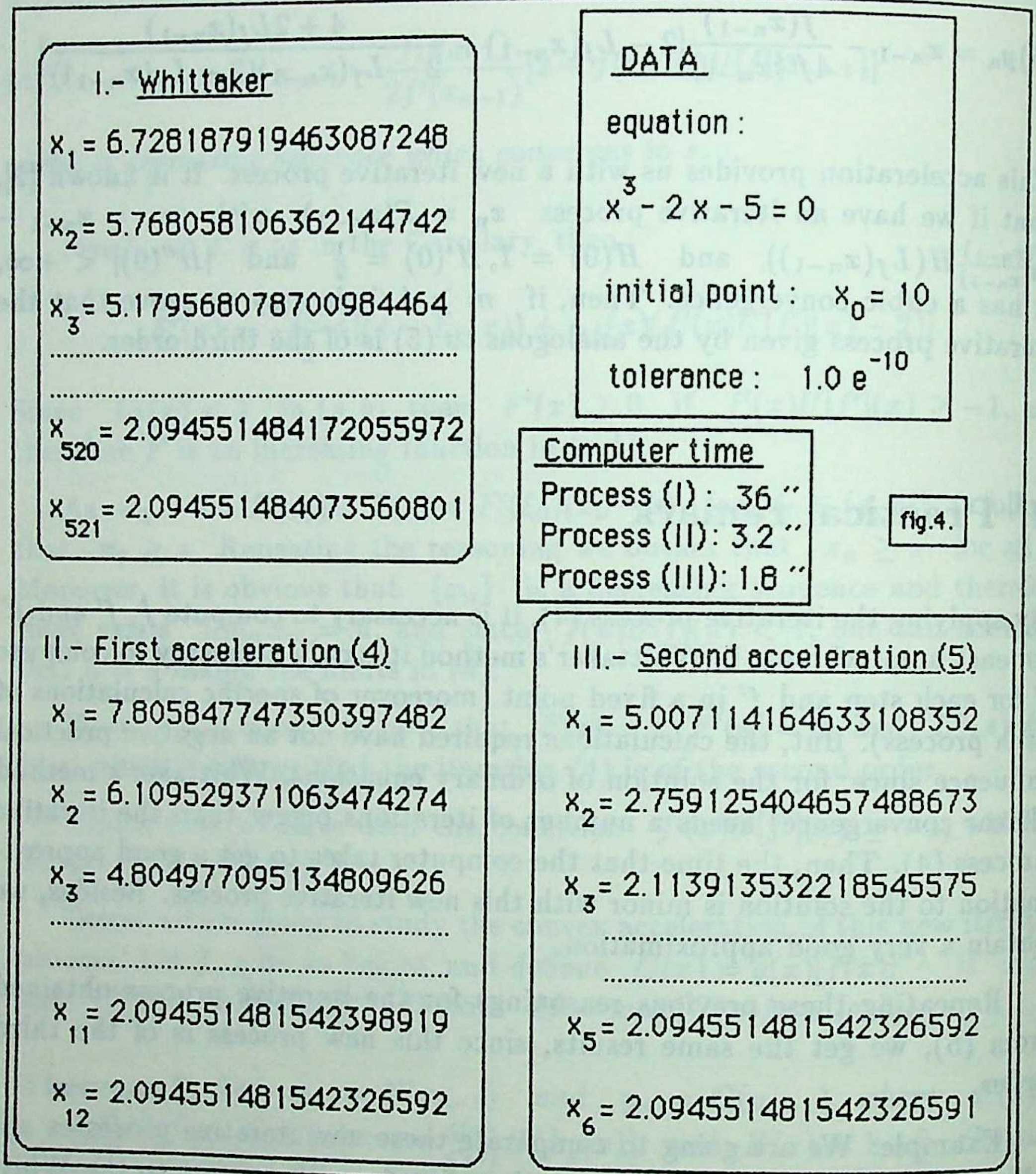
4. Practical remark

By applying the iterative process (4), it is necessary to compute f, f' and f'' for each step, whereas in Whittaker's method it is only necessary to compute f for each step and f' in a fixed point (moreover of specific calculations of each process). But, the calculations required have not an negative practical influence since, for the solution of ordinary equations, Whittaker's method (linear convergence) needs a number of iterations bigger than the iterative process (4). Then, the time that the computer takes to get a good approximation to the solution is minor with this new iterative process. Besides, we obtain a very good approximation.

Repeating these previous reasonings for the iterative process obtained from (5), we get the same results, since this new process is of the third order.

Example: We are going to comparate these new iterative processes applied to the Wallis polynomial equation, fig. 4., with respect to the Whittaker method.

The computer time of each process is shorter than in the Whittaker method, fig.4, and it provides us a better approximation to the solution of the equation (the solution to 18 decimal places is $\gamma = 2.094551481542326591$, [4]). If we consider another tolerance between the iterations, we can obtain a better approximation to the solution with the Whittaker method, but the computer time increases.



```
I.- Whittaker

x1 = 6.728187919463087248
x2 = 5.768058106362144742
x3 = 5.179568078700984464
.......................................
.......................................
x520 = 2.094551484172055972
x521 = 2.094551484073560801

DATA

equation :

x3 - 2 x - 5 = 0

initial point :   x0 = 10

tolerance :   1.0 e-10

Computer time
Process (I) :  36 ''
Process (II): 3.2 ''
Process (III): 1.8 ''

fig.4.

II.- First acceleration (4)
x1 = 7.805847747350397482
x2 = 6.109529371063474274
x3 = 4.804977095134809626
.......................................
.......................................
x11 = 2.094551481542398919
x12 = 2.094551481542326592

III.- Second acceleration (5)
x1 = 5.007114164633108352
x2 = 2.759125404657488673
x3 = 2.113913532218545575
.......................................
.......................................
x5 = 2.094551481542326592
x6 = 2.094551481542326591
```

Figure 4:

5. General remarks

If function f is decreasing, all the previous results turn out to be valid by changing slightly the used reasoning.

The condition $f(x_0) > 0$ does not affect the results obtained. If $f(x_0) < 0$, then the only change is that the sequence $\{x_n\}$ is increasing.

If function f is concave, then we have to consider the corresponding exp-degree of concavity and, by applying the previously studied reasoning, we obtain analogous results.

References

[1] Ehrmann, H., Konstruktion und durchduhrung von iterationsverfaharen hoherer ordnung, *Arch. Rational Mech. Anal.* 4 (1959) 65-88.

[2] Gander, W., On Halley's iteration method, *Amer. Math. Monthly Vol.* 92 No 2 (1985) 131-134

[3] O'Neill, B., Elementary differential geometry, *Academic Press, New York,* 1966.

[4] Ostrowski, A.M., Solution of equations and systems of equations, *Academic Press, New York,* 1973.

[5] Roberts, A.W. and Varberg, D.E., Convex functions, *Academic Press, New York,* 1973

[6] Traub, J.F., Iterative methods for solution of equations, *Prentice-Hall, Englewood Cliffs*, 1964.

REZIME

JEDNA UBRZANA PROCEDURA ZA WHITTAKEROV METOD POMOĆU KONVEKSNOSTI

U radu se posmatra uticaj konveksnosti realne funkcije f pri primeni Whittakerove metode na rešavanje jednačine $f(x) = 0$ i dobijen je metod za ubrzanje ovog iterativnog procesa. Takodje su dobijeni neki novi iterativni procesi.

Received by the editors August 13, 1988.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 39-51

Review of Research
Faculty of Science
Mathematics Series

AN ALGORITHM FOR THE GENERATION OF THE MATROID ASSOCIATED TO A GIVEN MATRIX

Dragan M. Acketa
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Ladislav Novak
Faculty of Technical Sciences, University of Novi Sad
Trg Dositeja Obradovića 6, 21000 Novi Sad, Yugoslavia

Abstract

An algorithm is sketched, which determines, up to an isomorphism, the matroid associated to a given matrix. A lemma which considerably shortens the corresponding computation time is used. The algorithm is given in detail for the matrices having at most 7 columns. Matroids on small ground-sets are represented by the following numerical parameters: cardinality of the ground-set, rank, number of bases and the vector of the sorted frequencies of elements w.r.t. the family of bases. It turns out that such a representation describes completely the matroids on at most 7 elements, while it is not sufficient on larger ground-sets.

AMS Mathematics Subject Classification (1980): 05B35
Key words and phrases: "Small" matroids, matrix representations, matroid invariants, matroid recognition, catalogue of matroids

1. Definitions and the problem statement

Matroid M on a finite set *(ground-set)* E is ([4]) an ordered pair (E, B), where B is a family of subsets of E, which satisfies the following "exchange" axiom :

If B_1 and B_2 belong to B, and x is an element of $B_1 - B_2$, then there exists an element y in $B_2 - B_1$, such that $(B_1 - x) + y$ belongs to B.

The sets in family B are called *bases* of M. All the bases of M have the same cardinality, which is called the *rank* of M.

Given a matroid M on E and a subset X of E, *rank* of X (w.r.t. M) is the maximal cardinality of the intersection of X with a base of M.

It is well-known that the complements (w.r.t. E) of bases of a rank r matroid M are the bases of so-called *dual* matroid M^*, the rank of which is equal to $\mid E \mid - rank(M)$.

Two matroids are *isomorphic* if there exists a bijection between their ground-sets, which maps the bases of one matroid onto the bases of the other.

Matroid M is *representable* over a field F if and only if there exists a matrix A (the *matrix representation* of M) with entries in F, such that there exists a rank-preserving bijection bij of the set E onto the set C of columns of A (thus, if X is a rank r subset of E w.r.t. M, then $bij(X)$ is a rank r submatrix of A -the number of rows in the submatrix is equal to the number of rows in A. Given a representable matroid M with a matrix representation A, we have that rank(M) =rank(A) and bases of M correspond to the maximal linearly independent sets of columns of A.

Given a family F of subsets of a finite set E and an element e from E, the *appearance frequency* of e w.r.t. F is the number of subsets of F, which contain e.

$C(n, r)$ denotes the binomial coefficient "n over R".

We shall give an algorithm for solving the following problem: Given a real-valued $m \times n$ matrix M, determine its associated matroid $f(M)$ and identify (recognize) it in a catalogue of non-isomorphic matroids.

We distinguish two stages in calculating $f(M)$:

(1) Extraction of matroid bases from the input matrix M

(2) Recognition of the generated matroid by comparision with matroids in catalogue [1]

2. The algorithm

2.1 Stage (1)

Stage (1) can be sketched as follows:

a) Determine rank r of M and extract an $r \times r$ submatrix S of rank r

b) Let Q denote the $r \times n$ submatrix of M, consisting of exactly those rows, which intersect S.

c) Extract the bases of $f(M)$ from Q in the following way: An $r-$tuple R of columns of M corresponds to a base of $f(M)$ if and only if the $r \times r$ submatrix consisting of exactly those fields of M, which belong to both Q and R – is of rank r.

When the rank calculation is considered, we use the efficient algorithm described in [2]. This algorithm is based on the reduction of the input matrix M to the so-called *row echelon* form. Such a reduction is a generalization of the Gaussian elimination to the rectangular case. The rank of a matrix is equal to the number of non-zero rows in its row echelon form. The columns and rows of the rank r submatrix S are easily determined from the reduction algorithm: the columns of S are exactly those, which contain the first non-zero elements in the first r rows of the row echelon form; the rows of S can be reconstructed by memorizing the row interchanges performed during the reduction process.

After the submatrix S is extracted, we should further calculate only $C(n,r)$ determinants corresponding to the $r \times r$ submatrices of Q (more precisely, we should just distinguish between zero- and non-zero- determinants among them). The reduction of search for bases of $f(M)$ to the columns of the "horizontal" band Q (instead of considering the whole columns of M) – is justified by the following Lemma:

Lemma 1. *Let an $m \times n$ matrix M of rank r be given and let K and L be two $r \times n$ submatrices of M, such that $rank(K) = rank(L) = r$. Let J denote an arbitrary $m \times r$ submatrix of M. Finally, let A and B be the $r \times r$ submatrices of M, which are composed of fields belonging both to K and J, respectively to both L and J. Then*

$$rank(A) = r \iff rank(B) = r$$

Proof. ([3]) Assume that $\text{rank}(A) = r$ and $\text{rank}(B) < r$ (the opposite assumption is treated in the same way). Since $\text{rank}(L) = r$, there exists an $r \times r$ submatrix C in the "horizontal band" L, such that $\text{rank}(C) = r$.

Consider a row R of M, which belongs to $L - K$. The assumption $\text{rank}(B) < r$ implies the linear dependence of rows of B. In particular, the subrow $R \bigcap B$ may be expressed as a linear combination of the other rows of B. It follows that there exists a linear transformation T of M, which consists of multiplying the rows of $L - R$ by the appropriate coefficients and adding these products to R, and which makes the fields of $B \bigcap R$ equal to zero. It is obvious that at least one field F of $C \bigcap R$ is not equal to zero after applying T (otherwise we would have a contradiction with $\text{rank}(C) = r$).

Now, consider the submatrix W of size $(r+1) \times (r+1)$ of the matrix $T(M)$, which consists of the matrix A and the additional row and column, which intersect in F (this row and column are extracted from M and all their elements, apart from F, lie in those columns, respectively rows, which intersect A). Since all the fields in the additional row, apart from F, are equal to zero, it follows that

$$\det(W) = \det(A) \cdot \det(F) \neq 0.$$

This implies that $\text{rank}(M) = \text{rank}(T(M)) \geq r + 1$, a contradiction. □

Consequence of Lemma. Arbitrary rank r $r-$tuple of rows is sufficient to determine the matroid bases. Namely, according to the Lemma, an $r-$tuple J of columns corresponds (or does not correspond) to a matroid base, without regard to the rank r $r-$tuple of rows, which is used.

The easiest way to determine matroid bases is to start from the first submatrix S of rank r (we assume that such a submatrix is detected in the process of rank calculating). Then the matroid bases associated to M are the same as the matroid bases associated to $H(S)$, where $H(S)$ denotes the "horizontal band" associated to S, i.e., the submatrix of M, which consists of exactly those rows, which intersect S.

2.2 Stage (2)

The comparison between families F_1 and F_2 of bases, which respectively correspond to the matroid $M_1 = f(M)$ and a matroid M_2 from the catalogue, is preceded by comparison of the following four types of numerical parameters in turn:

a) cardinalities of the ground-sets of M_1 and M_2 (this cardinality should be equal to n with the matroid M_2; note that there may be elements (loops) which do not belong to any base – therefore such a cardinality should be given in a catalogue independently from the family of bases). If the ground-sets of M_1 and M_2 are equicardinal, then they may be considered to coincide.

b) ranks of M_1 and M_2

c) the numbers of bases in F_1 and F_2

d) sorted (non-increasing) sequences of appearance frequencies of elements of the common ground-set w.r.t. the families F_1 and F_2 respectively.

Given a matrix M, the parameter a) (=the number of matrix columns) is known in advance, the parameters b) and c) are obtained in Stage (1), while the parameter d) is easily derived from the family of bases. On the other hand, when the family F_2 is considered, the parameters a), b) and c) are given with each matroid in catalogue [1]. The values of the parameter d) were derived from that catalogue, in order to be applied in Section 4.

Our hierarchy of numerical features associated to a matroid can be incirporated in a nested sequence of IF-statements in the following way:

```
IF the paramaters   a)   coincide THEN
  IF the paramaters   b)   coincide THEN
    IF we are not sure that $M_1 = M_2$ THEN
      IF the paramaters   c)   coincide THEN
        IF we are not sure that $M_1 = M_2$ THEN
          IF the paramaters   d)   coincide THEN
            IF we are not sure that $M_1 = M_2$ THEN
              Compare directly   $F_1$   and   $F_2$.
```

The three lines "IF we are not sure that $M_1 = M_2$ THEN" are inserted because we may stop searching for a matroid isomorphic to M_1 in the catalogue, in the cases when there is a unique matroid in the catalogue with the given numerical parameters.

The first stopping possibility works only for the matroids of rank 0 and n (on the ground-set of cardinality n). The second one works for matroids of ranks 1 and $n-1$, but also for some matroids of "interior" ranks as well. Finally, an inspection of catalogue [1] assures us that the third stopping possibility will work with all the matroids on ≤ 7 elements; thus we need not compare the families themselves for $n \leq 7$. This is not the case for higher values of n, as is shown by the counterexample in the next section.

3. The auxiliary list of matroids

We shall proceed with a list which enables the calculation of each matroid on ≤ 7 elements, provided that the corresponding numbers n (cardinality of the ground-set), r (rank), b (number of bases), as well as the vector of sorted appearance frequencies w.r.t. the family of bases are given. Each triple (n, r, b) is followed by the symbol ":", each matroid is followed by the symbol ";". If there exists only one matroid (up to an isomorphism) with some given parameters n, r, b, then it is represented by "one" ; otherwise it is represented by the vector v.

There exists only one (empty) rank O matroid on n elements. There exist exactly n non-isomorphic rank 1 matroids, which respectively have i bases for each i between 1 and n. If $n > r/2$, then the matroids on ≤ 7 elements are uniquely represented by the sorted vectors of appearance frequencies, which are associated to their dual matroids. Therefore, we may conclude that it suffices to give a list for (n, r) belonging to the set $\{(4,2),(5,2),(6,2),(6,3),(7,2),(7,3)\}$.

3.1 List

$n = 4, \quad r = 2$

$b = 1:$ one ; $b = 2:$ one ; $b = 3:$ $(2,2,2,0)$; $(3,1,1,1)$;
$b = 4:$ one ; $b = 5:$ one ; $b = 6:$ one ;

$n = 5, \quad r = 2$

$b = 1:$ one ; $b = 2:$ one ; $b = 3:$ $(2,2,2,0,0)$; $(3,1,1,1,0)$;
$b = 4:$ $(2,2,2,2,0)$; $(4,1,1,1,1)$; $b = 5:$ one ;
$b = 6:$ $(3,3,3,3,0)$; $(3,3,2,2,2)$; $b = 7:$ one ;
$b = 8:$ one ; $b = 9:$ one ; $b = 10:$ one ;

$n = 6, \quad r = 2$

$b = 1:$ one ; $b = 2:$ one ;
$b = 3:$ $(2,2,2,0,0,0)$; $(3,1,1,1,0,0)$;
$b = 4:$ $(2,2,2,2,0,0)$; $(3,3,2,2,0,0)$;
$b = 5:$ $(3,3,2,2,0,0)$; $(5,1,1,1,1,1)$;
$b = 6:$ $(3,3,3,3,0,0)$; $(3,3,2,2,2,0)$; $b = 7:$ one ;
$b = 8:$ $(4,3,3,3,3,0)$; $(4,4,2,2,2,2)$;
$b = 9:$ $(4,4,4,3,3,0)$; $(3,3,3,3,3,3)$;
$b = 10:$ one ; $b = 11:$ one ; $b = 12:$ one ;
$b = 13:$ $(5,5,5,3,3,3)$; $(5,5,4,4,4,4)$;
$b = 14:$ one ; $b = 15:$ one ;

$n = 6, \quad r = 3$

$b = 1:$ one ; $b = 2:$ one ;
$b = 3:$ $(3,2,2,2,0,0)$; $(3,3,1,1,1,0)$;
$b = 4:$ $(3,3,3,3,0,0)$; $(4,2,2,2,2,0)$; $(4,4,1,1,1,1)$;
$b = 5:$ one ;
$b = 6:$ $(4,4,4,3,3,0)$; $(6,3,3,2,2,2)$; $(6,3,3,3,3,0)$;
$b = 7:$ $(5,5,5,3,3,0)$; $(7,4,4,2,2,2)$;
$b = 8:$ $(5,5,5,5,4,0)$; $(8,4,3,3,3,3)$; $(4,4,4,4,4,4)$;
$b = 9:$ $(6,6,5,5,5,0)$; $(6,6,6,3,3,3)$; $(9,4,4,4,3,3)$;
$b = 10:$ $(6,6,5,5,4,4)$; $(7,7,7,3,3,3)$; $(10,4,4,4,4,4)$;
$b = 12:$ $(7,7,7,7,4,4)$; $(8,8,5,5,5,5)$; $(6,6,6,6,6,6)$;
$b = 13:$ one ; $b = 14:$ one ;
$b = 15:$ $(9,8,8,8,6,6)$; $(9,9,7,7,7,6)$;
$b = 16:$ $(9,9,9,9,6,6)$; $(8,8,8,8,8,8)$; $(10,10,7,7,7,7)$;
$b = 17:$ one ; $b = 18:$ $(9,9,9,9,9,9)$; $(10,9,9,9,9,8)$;
$b = 19:$ one ; $b = 20:$ one ;

$n = 7, \quad r = 2$

$b = 1:$ one ; $b = 2:$ one ;
$b = 3:$ $(2,2,2,0,0,0,0)$; $(2,1,1,1,0,0,0)$;

$b=4:\quad (2,2,2,2,0,0,0)\,;\ (4,1,1,1,1,0,0)\,;$
$b=5:\quad (3,3,2,2,0,0,0)\,;\ (5,1,1,1,1,1,0)\,;$
$b=6:\quad (3,3,2,2,2,0,0)\,;\ (6,1,1,1,1,1,1)\,;\quad b=7:\ \text{one}\,;$
$b=\ 8:\quad (4,3,3,3,3,0,0)\,;\ (4,4,2,2,2,2,0)\,;$
$b=\ 9:\quad (4,4,4,3,3,0,0)\,;\ (3,3,3,3,3,3,0)\,;\ (5,5,2,2,2,2,0)\,;$
$b=10:\quad (4,4,4,4,4,0,0)\,;\ (5,5,2,2,2,2,2)\,;$
$b=11:\quad (5,4,4,3,3,3,0)\,;\ (6,6,2,2,2,2,2)\,;$
$b=12:\quad (4,4,4,4,4,4,0)\,;\ (5,5,5,3,3,3,0)\,;\ (4,4,4,3,3,3,3)\,;$
$b=13:\quad \text{one}\,;\quad b=14:\ (5,5,5,5,4,4,0)\,;\ (6,5,5,3,3,3,3)\,;$
$b=15:\quad (5,5,5,5,5,5,0)\,;\ (6,4,4,4,4,4,4)\,;\ (6,6,6,3,3,3,3)\,;$
$b=16:\quad \text{one}\,;\quad b=17:\ \text{one}\,;$
$b=18:\quad (6,5,5,5,5,5,5)\,;\ (6,6,6,6,4,4,4)\,;$
$b=19:\quad \text{one}\,;\quad b=20:\ \text{one}\,;\quad b=21:\ \text{one}\,;$

$\underline{n=7,\quad r=3}$

$b=\ 1:\quad \text{one}\,;\quad b=2:\ \text{one}\,;$
$b=\ 3:\quad (3,2,2,2,0,0,0)\,;\ (3,3,1,1,1,0,0)\,;$
$b=\ 4:\quad (3,3,3,3,0,0,0)\,;\ (4,2,2,2,2,0,0)\,;\ (4,4,1,1,1,1,0)\,;$
$b=\ 5:\quad (5,3,3,2,2,0,0)\,;\ (5,5,1,1,1,1,1)\,;$
$b=\ 6:\quad (4,4,4,3,3,0,0)\,;\ (6,3,3,2,2,2,0)\,;\ (6,3,3,3,3,0,0)\,;$
$b=\ 7:\quad (5,5,5,3,3,0,0)\,;\ (7,4,4,2,2,2,0)\,;$
$b=\ 8:\quad (5,5,5,5,4,0,0)\,;\ (4,4,4,4,4,4,0)\,;\ (8,4,3,3,3,3,0)\,;$
$(8,4,4,2,2,2,2)\,;$
$b=\ 9:\quad (6,6,5,5,5,0,0)\,;\ (9,4,4,4,3,3,0)\,;\ (6,6,6,3,3,3,0)\,;$
$(9,5,5,2,2,2,2)\,;\ (9,3,3,3,3,3,3)\,;$
$b=10:\quad (10,4,4,4,4,4,0)\,;\ (7,7,7,3,3,3,0)\,;\ (6,6,6,6,6,0,0)\,;$
$(6,6,5,5,4,4,0)\,;\quad b=11:\ \text{one}\,;$
$b=12:\quad (6,6,6,6,6,6,0)\,;\ (8,8,5,5,5,5,0)\,;\ (7,7,7,7,4,4,0)\,;$
$(6,6,6,6,4,4,4)\,;\ (8,8,8,3,3,3,3)\,;\ (12,4,4,4,4,4,4)\,;$
$(12,5,5,5,3,3,3)\,;$
$b=13:\quad (8,8,7,6,5,5,0)\,;\ (9,9,9,3,3,3,3)\,;\ (13,5,5,4,4,4,4)\,;$
$b=14:\quad (9,9,7,7,5,5,0)\,;\ (8,8,7,7,4,4,4)\,;\ (14,5,5,5,5,4,4)\,;$
$b=15:\quad (9,8,8,8,6,6,0)\,;\ (9,9,7,7,7,6,0)\,;\ (9,9,6,6,5,5,5)\,;$
$(15,5,5,5,5,5,5)\,;$
$b=16:\quad (9,9,9,9,6,6,0)\,;\ (8,8,8,8,8,8,0)\,;\ (10,10,7,7,7,7)\,;$
$(8,8,8,6,6,6,6)\,;\ (9,9,9,9,4,4,4)\,;$
$b=17:\quad (9,9,9,8,8,8,0)\,;\ (11,11,7,7,5,5,5)\,;$
$b=18:\quad (9,9,9,9,9,9,0)\,;\ (10,9,9,9,9,8,0)\,;\ (9,9,8,8,8,6,6)\,;$

(9,9,9,9,6,6,6) ; (11,11,9,8,5,5,5) ;
$b = 19$: (10,10,10,9,9,9,0) ; (11,11,9,7,7,6,6) ; (12,12,9,9,5,5,5) ;
$b = 20$: (10,10,10,10,10,10,0) ; (12,12,8,7,7,7,7) ;
(12,8,8,8,8,8,8) ; (10,10,8,8,8,8,8) ;
$b = 21$: (11,11,9,8,8,8,8) ; (12,11,11,11,6,6,6) ;
(13,13,9,7,7,7,7) ; (12,12,9,9,9,6,6) ;
$b = 22$: (12,12,12,12,6,6,6) ; (13,13,9,9,8,7,7) ;
$b = 23$: (13,11,11,10,10,8,8) ; (14,14,9,9,9,7,7) ;
$b = 24$: (12,12,12,9,9,9,9) ; (12,11,11,11,11,8,8) ;
(12,11,11,11,9,9,9) ; (14,14,9,9,9,9,8) ;
$b = 25$: (13,13,13,9,9,9,9) ; (13,12,12,11,11,8,8) ;
(15,15,9,9,9,9,9) ;
$b = 26$: (13,13,12,11,11,9,9) ; (14,11,11,11,11,10,10) ;
(14,14,11,11,11,8,8) ;
$b = 27$: (13,13,13,12,12,9,9) ; (14,13,13,12,11,9,9) ;
(12,12,12,12,12,12,9) ;
$b = 28$: (13,13,13,13,12,10,10) ; (14,14,14,12,12,9,9) ;
(13,13,13,12,11,11,11) ; (12,12,12,12,12,12,12) ;
$b = 29$: (14,14,13,13,13,10,10) ; (13,13,13,12,12,12,12) ;
(14,14,13,12,12,11,11) ;
$b = 30$: (14,13,13,13,13,12,12) ; (14,14,14,14,14,10,10) ;
(13,13,13,13,13,13,12) ; (15,14,14,12,12,12,11) ;
(14,14,14,12,12,12,12) ;
$b = 31$: (15,13,13,13,13,13,13) ; (14,14,13,13,13,13,13) ;
(14,14,14,13,13,13,12) ; (15,15,15,12,12,12,12) ;
$b = 32$: (15,14,14,14,13,13,13) ; (14,14,14,14,14,13,13) ;
(14,14,14,14,14,14,12) ;
$b = 33$: (15,14,14,14,14,14,14) ; (15,15,14,14,14,14,13) ;
$b = 34$: one ; $b = 35$: one ;

3.2 A counterexample

The following counterexample (also derived from catalogue [1]) shows that the parameters n, r, b and the vector of sorted appearance frequencies are not sufficient to determine matroids on 8 elements up to an isomorphism:

The non-isomorphic rank 4 matroids M_1 and M_2 on 8 elements have 63 bases each, which are all the 4-subsets of the ground-set $\{A, B, C, D, E, F, G, H\}$, except for the subsets in the families

$\{ABCD, ABEF, ABGH, ACEG, BDFH, CDEF, CDGH\}$ and
$\{AEFG, ABCD, BEFH, ACGH, BDEG, CDFH, ADEH\}$
respectively.

The corresponding vectors of appearance frequencies of elements A, B, C, D, E, F, G, H (w.r.t. the families of bases of the matroids M_1 and M_2) are $(31, 31, 31, 31, 32, 32, 32, 32)$ and $(31, 32, 32, 31, 31, 32, 32, 31)$ respectively. After sorting, these two vectors become equal. This means that there exist two non-isomorphic matroids on 8 elements, with which the parameters b), c) and d) are coincident.

A further elaboration of this counterexample leads to a more fascinating conclusion that there exist even 22 non-isomorphic matroids on 8 elements with coincident a), b), c), d), although it was not possible to find only two matroids with all the four coinciding parameters on 7 elements. We shall give some further relevant details:

There exist 71 non-isomorphic rank 4 matroids on 8 elements with exactly 63 bases each ([1]). The distribution of these matroids with respect to the vectors of sorted appearance frequencies looks as follows:

$n = 8, \ r = 4, \ b = 63$
$(32, 32, 32, 32, 31, 31, 31, 31)$ – 22 matroids
$(32, 32, 32, 32, 32, 31, 31, 30)$ – 8 matroids
$(33, 32, 32, 32, 31, 31, 31, 30)$ – 8 matroids
$(33, 32, 32, 31, 31, 31, 31, 31)$ – 8 matroids
$(33, 33, 31, 31, 31, 31, 31, 31)$ – 3 matroids
$(32, 32, 32, 32, 32, 32, 30, 30)$ – 3 matroids

The following 19 vectors of sorted appearance frequencies have just one corresponding matroid each:

$(32, 32, 32, 32, 32, 32, 32, 28)$, $(33, 33, 33, 31, 31, 31, 31, 29)$,
$(33, 32, 32, 32, 32, 31, 31, 29)$, $(32, 32, 32, 32, 32, 32, 31, 29)$,
$(34, 32, 32, 32, 32, 30, 30, 30)$, $(33, 33, 32, 32, 31, 31, 30, 30)$,
$(34, 32, 32, 31, 31, 31, 31, 30)$, $(33, 33, 32, 31, 31, 31, 31, 30)$,
$(35, 31, 31, 31, 31, 31, 31, 31)$, $(34, 32, 31, 31, 31, 31, 31, 31)$,
$(33, 32, 32, 32, 32, 31, 30, 30)$, $(33, 33, 33, 32, 32, 30, 30, 29)$,
$(33, 33, 33, 33, 32, 30, 30, 28)$, $(34, 33, 33, 32, 32, 30, 29, 29)$,
$(33, 33, 33, 33, 32, 30, 29, 29)$, $(34, 33, 33, 31, 31, 30, 30, 30)$,

(34, 34, 33, 31, 31, 30, 30, 29) , (34, 34, 33, 31, 30, 30, 30, 30) ,
(35, 33, 33, 31, 30, 30, 30, 30)

4. An illustrative example

As an illustration, we shall give the results of the application of our algorithm, for $m = 2$ and $n = 4$, to a class $M(x, y)$ of 2×4 matrices, the entries of which are defined by using two independent real variables x and y in the following way:

$$\begin{array}{ll} a[1,1] = y - x - 3\ ; & a[1,2] = 0\ ; \\ \multicolumn{2}{l}{a[1,3] = 0 \text{ if abs(sqr}(x) + \text{ sqr}(y) - 49) < 7\ ,} \\ a[1,3] = 1 \text{ otherwise}\ ; & a[1,4] = x - 2\ ; \\ a[2,1] = 0\ ; & a[2,2] := y + 2 \cdot x + 5\ ; \\ a[2,3] := x \cdot (y - 3)\ ; & a[2,4] := -2 + \text{abs}(y - 3)\ ; \end{array}$$

Let the seven non-isomorphic matroids for $(n, r) = (4, 2)$, listed in the above order, be denoted by 1, 2, 3, 4, 5, 6, 7 respectively. Then the list of the values of $f(M(x, y))$ for $-10 \le x, y \le 10$ looks as in Table 1. below.

It can be easily verified (by computer) that all the considered 441 matroids are of rank 2. All the seven rank 2 matroids on 4 elements can be found among them. The seven "lines" corresponding to the matrix fields $a[1,1]$, $a[1,4]$, $a[2,2]$, $a[2,3]$, $a[2,4]$ (two "lines" in each of the last two cases), as well the "circle" associated to the matrix field $a[1,3]$ can be recognized in the table below.

References

[1] Acketa, D., M., The catalogue of all the non-isomorphic matroids on at most 8 elements, Inst. of Math. Novi Sad, spec. issue, 1983, XVIII + 157.

[2] Chua, L. O., Lin, P. - M., Computer – aided Analysis of Electronic Circuits (Algorithms and Computational Techniques), Prentice-Hall, 1975.

[3] Stojaković, Z., private comunication

[4] Welsh, D. J. A., Matroid Theory, Academic Press, 1973.

$y\backslash x$	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10	7	7	7	7	7	7	7	7	7	7	6	7	6	7	7	7	7	3	7	7	7
9	7	7	7	3	7	7	7	7	7	7	6	7	6	7	7	7	3	7	7	7	7
8	7	7	7	7	7	7	7	7	7	7	6	7	6	7	7	3	7	7	7	7	7
7	7	7	7	7	3	7	7	7	6	6	3	6	4	7	3	7	7	7	7	7	7
6	7	7	7	7	7	7	6	6	7	7	6	7	6	2	6	7	7	7	7	7	7
5	6	6	6	6	6	2	6	6	6	6	4	6	1	6	6	5	6	6	6	6	6
4	7	7	7	7	6	7	7	7	7	7	6	2	6	7	7	7	6	7	7	7	7
3	6	6	6	6	3	6	2	6	6	6	3	6	5	6	6	6	3	6	6	6	6
2	7	7	7	6	7	7	7	7	7	3	6	7	6	7	7	7	7	6	7	7	7
1	6	6	6	5	6	6	6	2	3	6	4	6	3	6	6	6	6	5	6	6	6
0	7	7	7	6	7	7	7	3	7	7	6	7	6	7	7	7	7	6	7	7	7
-1	7	7	7	6	7	7	3	7	3	7	6	7	6	7	7	7	7	6	7	7	7
-2	7	7	7	6	7	3	7	7	7	7	6	7	6	7	7	7	7	6	7	7	7
-3	7	7	7	7	2	7	7	7	7	3	6	7	6	7	7	7	6	7	7	7	7
-4	7	7	7	3	6	7	7	7	7	7	6	7	6	7	7	7	6	7	7	7	7
-5	7	7	3	7	7	6	7	7	7	7	2	7	6	7	7	6	7	7	7	7	7
-6	7	3	7	7	7	7	6	6	7	7	6	7	6	6	6	7	7	7	7	7	7
-7	3	7	7	7	7	7	7	7	6	6	3	3	4	7	7	7	7	7	7	7	7
-8	7	7	7	7	7	7	7	7	7	7	6	7	6	7	7	7	7	7	7	7	7
-9	7	7	7	7	7	7	7	7	7	7	6	7	3	7	7	7	7	7	7	7	7
-10	7	7	7	7	7	7	7	7	7	7	6	7	6	7	7	7	7	7	7	7	7

Table 1.

REZIME

JEDAN ALGORITAM ZA GENERISANJE MATROIDA PRIDRUŽENOG DATOJ MATRICI

U radu je opisan jedan algoritam, koji sa tačnošću do na izomorfizam odredjuje matroid pridružen datoj matrici. Algoritam je detaljno opisan za matrice, koje nemaju više od 7 kolona. Matroidi na malim nosačima su reprezentovani pomoću sledećih parametara: kardinalnost nosača, rang, broj baza, i vektor sortiranih učestanosti javljanja pojedinih elemenata nosača u familiji baza. Ovi parametri su dovoljni da opišu matroid sa najviše 7 elemenata, a nisu dovoljni da opišu matroide na većim nosačima.

Received by the editors October 23, 1989.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 53-61

Review of Research
Faculty of Science
Mathematics Series

DIHEDRAL n-QUASIGROUPS

Zoran Stojaković
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

An n-quasigroup (Q, f) is called dihedral iff $f(x_1, ..., x_n) = x_{n+1} \Leftrightarrow f(x_{\sigma(1)}, ..., x_{\sigma(n)}) = x_{\sigma(n+1)}$ for every permutation $\sigma \in D_{n+1}$, where D_{n+1} is the dihedral subgroup of the symmetric group S_{n+1} of degree $n+1$. Dihedral n-quasigroups (D-n-quasigroups) represent a generalization of totally symmetric binary quasigroups. Several equivalent definitions and some examples of D-n-quasigroups are given. It is proved that some retracts of D-n-quasigroups are also D-n-quasigroups.Autotopisms and regular permutations of D-n-quasigroups are cosidered and some of their properties determined.

AMS Mathematics Subject Classification (1980): 20N15
Key words and phrases: n-quasigroup, n-group, parastrophy, autotopism, regular permutation

1. Definitions and Notations

First we give some basic definitions and notations. Other notions from the theory of n-quasigroups can be found in [1].

The sequence $x_m, x_{m+1}, \ldots, x_n$ we shall denote by x_m^n or $\{x_i\}_{i=m}^n$. If $m > n$, then x_m^n will be considered empty. The sequence $x, x, \ldots, x$ (m times) will be denoted by $\overset{m}{x}$. If $m \leq 0$, then $\overset{m}{x}$ will be considered empty.

An n-ary groupoid (n-groupoid) (Q, f) is called an n-quasigroup iff the equation $f(a_1^{i-1}, x, a_{i+1}^n) = b$ has a unique solution x for every $a_1^n, b \in Q$ and every $i \in \{1, \ldots, n\} = N_n$.

An n-quasigroup (Q, f) is isotopic to an n-quasigroup (Q, g) iff there exists a sequence $T = (\alpha_1^{n+1})$ of permutations of Q such that the following identity

$$g(x_1^n) = \alpha_{n+1}^{-1} f(\{\alpha_i x_i\}_{i=1}^n)$$

holds. T is called an isotopism, g is an isotope of f and by $f^T = g$ we denote that f is isotopic to g by T. T^{-1} is defined by $T^{-1} = (\{\alpha_i^{-1}\}_{i=1}^{n+1})$. If T is an isotopism of (Q, f) to itself, that is, $f^T = f$, then T is called an autotopism of f. The set of all autotopisms of f under the composition of autotopisms is a group which we denote by $A(f)$.

By S_n we denote the symmetric group of degree n, by A_n its alternating subgroup, and by D_n its dihedral subgroup.

If (Q, f) is an n-quasigroup and $\sigma \in S_{n+1}$, then the n-quasigroup f^σ defined by

$$f^\sigma(\{x_{\sigma(i)}\}_{i=1}^n) = x_{\sigma(n+1)} \iff f(x_1^n) = x_{n+1}$$

is called a σ-parastrophe (or simply parastrophe) of f. If $f = f^\sigma$, then σ is called an autoparastrophism of f. The set of all autoparastrophisms of f is a subgroup of S_{n+1} which is denoted by $\Pi(f)$.

An n-quasigroup (Q, f) is called

a) totally symmetric (TS) if $f = f^\sigma$ for all $\sigma \in S_{n+1}$,

b) alternating symmetric (AS) iff $f = f^\sigma$ for all $\sigma \in A_{n+1}$ ([8]),

c) cyclic iff $f = f^\sigma$ for all $\sigma \in C$, where C is a subgroup of S_{n+1} generated by the cycle $(12 \ldots n+1)$ ([7],[10]).

If Q is a set, by ϵ we denote the identity mapping of Q.

2. Dihedral n-quasigroups

Definition 1. *An n-groupoid* (Q, f) *is called dihedral iff for every permutation* $\sigma \in D_{n+1}$ *and every* $x_1^{n+1} \in Q$

$$f(x_1^n) = x_{n+1} \iff f(\{x_{\sigma(i)}\}_{i=1}^n) = x_{\sigma(n+1)}.$$

It is not difficult to see that from the preceding definition it follows that every dihedral n-groupoid is necessarily an n-quasigroup, which we call a dihedral n-quasigroup (D-n-quasigroup).

Since every D-n-groupoid is a D-n-quasigroup, Definition 1 can be given in another form.

Definition 1'. *An n-quasigroup (Q, f) such that $f = f^\sigma$ for every $\sigma \in D_{n+1}$ is a D-n-quasigroup.*

In the symmetric group S_{n+1} the subgroup D_{n+1} is generated by permutations $\phi = (12\ldots n{+}1), \psi = \prod_{2\leq i<n+3-i}(i \quad n{+}3{-}i)$, which implies the following theorem.

In the sequel ϕ and ψ will always denote these two permutations.

Theorem 1. *An n-quasigroup (Q, f) is a D-n-quasigroup iff $f = f^\phi = f^\psi$.*

D-n-quasigroups can also be defined as n-quasigroups satisfying a system of identities. Since every equality of type $f = f^\sigma$, where $\sigma \in S_{n+1}$, is equivalent to an identity ([5]), from Theorem 1 we get the next theorem.

Theorem 2. *An n-groupoid (Q, f) is a D-n-quasigroup iff for all $x_1^{n+1} \in Q$*

$$\begin{cases} f(x_2^n, f(x_1^n)) = x_1, \\ f(x_1, f(x_1^n), x_n, x_{n-1}, \ldots, x_3) = x_2. \end{cases}$$

Since D-n-quasigroups can be defined by a system of identities, the class of all D-n-quasigroups is a variety, which implies that this class is closed under the formation of direct products, subalgebras and quotient algebras.

For $n = 2$ a D-n-quasigroup (Q, f) is a binary quasigroup $(Q, \cdot)$ satisfying the identities

$$y(xy) = x, \quad x(xy) = y$$

which implies that $(Q, \cdot)$ is a TS binary quasigroup. Hence D-n-quasigroups represent a generalization of TS binary quasigroups, different from other generalizations such as TS-n-quasigroup or AS-n-quasigroups.

For $n = 3$ a class of generalized idempotent D-3-quasigroups (see [9]) is equivalent to a new class of quadruple systems called dihedral quadruple systems, which lies between Steiner and Mendelsohn quadruple systems.

Since $C_{n+1} \subset D_{n+1} \subset S_{n+1}$, every TS-n-quasigroup is dihedral and every D-n-quasigroup is cyclic. If we denote the class of all TS-n-quasigroup by $\mathcal{A}(TS)$, the class of all D-n-quasigroups by $\mathcal{A}(D)$ and the class of all cyclic n-quasigroups by $\mathcal{A}(C)$, it follows

$$\mathcal{A}(TS) \subset \mathcal{A}(D) \subset \mathcal{A}(C).$$

That the class $\mathcal{A}(D)$ is different from both $\mathcal{A}(TS)$ and $\mathcal{A}(C)$ follows from [2], where it was proved that for every subgroup $G \subset S_{n+1}$ and every $m > n$, $p \geq 2$, there exists an n-quasigroup (Q, f) of order mp such that $\Pi(f) = G$.

If $n \equiv 0 \pmod 4$, then $D_{n+1} \subset A_{n+1}$, hence $\mathcal{A}(AS) \subset \mathcal{A}(D)$, where $\mathcal{A}(AS)$ denotes the class of all AS-n-quasigroups. So, in this case we have

$$\mathcal{A}(TS) \subset \mathcal{A}(AS) \subset \mathcal{A}(D) \subset \mathcal{A}(C).$$

For other values of n D_{n+1} is not contained in A_{n+1}.

Now we shall give some examples of D-n-quasigroups.

1. If $(G, +)$ is a commutative group, θ an automorphism of $(G, +)$ such that $\theta^2 = \epsilon$, b an element of G such that $\theta b = b$, n odd, then by

$$f(x_1^n) = \theta x_1 - x_2 + \theta x_3 - x_4 + \ldots + \theta x_n + b$$

a D-n-quasigroup (G, f) is defined.

2. Let the Euclidean plane be given and let A, B and C be arbitrary points (not necessarily distinct) in the plane. If D is the fourth vertex of the parallelogram $ABCD$ determined by vertices A, B and C (which can be degenerated if some of the points A, B and C coincide), then we define a ternary mapping f by

$$f(A, B, C) = D.$$

If S denotes the set of all points in the plane, then $f : S^3 \to S$ and it is not difficult to see that (S, f) is a D-3-quasigroup of infinite order which is not totally symmetric.

In the preceding example an infinite uncountable D-3-quasigroup was constructed. An analogous construction on the set of all points in a plane with integer coordinates gives an example of an infinite countable D-3-quasigroup.

3. As we have mentioned before, in [2] a method which enables a construction of D-n-quasigroups of order mp for every $m > n$, $p \geq 2$, such that $\Pi(f) = D_{n+1}$, was given.

3. Retracts of D-n-quasigroups

We shall prove that every D-n-quasigroup, where n is odd, defines a family of D-$(\frac{n+1}{2})$-quasigroups.

Theorem 3. *If (Q,f) is an D-n-quasigroup, where $n = 2k+1$, $k \in \{2,3,\ldots\}$, and a is an arbitrary element from Q, then by*

$$g(x_1^k) = x_{k+1} \iff f(a,x_1,a,x_2,\ldots,a) = x_{k+1}$$

a D-k-quasigroup (Q,g) is defined.

Proof. g is a k-quasigroup since it is a retract of an n-quasigroup. From the cyclicity of f it follows

$$f(a,x_1,a,x_2,\ldots,a) = x_{k+1} \iff f(a,x_{k+1},a,x_1,\ldots,a) = x_k,$$

and

$$g(x_1^k) = x_{k+1} \iff g(x_{k+1},x_1^{k-1}) = x_k,$$

which means that g is also cyclic.

Since f is dihedral $f = f^{\psi}$, where $\psi = \prod_{2\le i<n+3-i}(i \quad n+3-i)$, hence

$$f(a,x_1,a,x_2,\ldots,a) = x_{k+1} \iff f(a,x_{k+1},a,x_k,\ldots,a) = x_1$$

and

$$g(x_1^k) = x_{k+1} \iff g(x_{k+1},x_k,\ldots,x_2) = x_1.$$

Since g is cyclic, we get

$$g(x_1^k) = x_{k+1} \iff g(x_1,x_{k+1},x_k,\ldots,x_3) = x_2,$$

We have proved that $g = g^{\psi'}$, where $\psi' = \prod_{2\le i<k+3-i}(i \quad k+3-i)$, so by Theorem 1 g is dihedral.

4. Autotopisms of D-n-quasigroups

Lemma 1. *If $T = (\alpha_1^{n+1})$ is an autotopism of a D-n-quasigroup (Q,f), then for every $\sigma \in D_{n+1}$, $T^{\sigma} = (\{\alpha_{\sigma(i)}\}_{i=1}^{n+1})$ is also an autotopism of f.*

Proof. From $f^T = f^\sigma = f$, it follows that $f = (f^T)^\sigma = (f^\sigma)^{T^\sigma} = f^{T^\sigma}$. Hence T^σ is an autotopism of f.

In the theory of binary, as well as n-ary quasigroups, there exists a close relation between nuclei and groups of regular permutations. There exist several different generalizations of nuclei and regular permutations to n-ary case, here we shall consider regular permutations as defined in [6],[1],[3],[4].

Let (Q, f) be an n-quasigroup, $i \in N_n$. A permutation α of Q is said to be i-inverse regular for f iff $(\overset{i-1}{\epsilon}, \alpha, \overset{n-i}{\epsilon}, \alpha^{-1}) \in A(f)$. A permutation which is i-inverse regular for every $i \in N_n$ is called inverse regular for f. The set of all inverse regular permutations for f will be denoted by $V(f)$.

Theorem 4. *If (Q, f) is a D-n-quasigroup, n even, $T = (\overset{i-1}{\epsilon}, \alpha, \overset{j-i-1}{\epsilon}, \beta, \overset{n-j+1}{\epsilon}) \in A(f)$ for some $i, j \in N_n$, then $\beta = \alpha^{-1}$.*

Proof. a) $j - i$ odd.

Then there exists $t \in N_n$ such that T^{ϕ^t} or $T^{\phi^{-t}}$ has the following form: $S = (\overset{k-1}{\epsilon}, \alpha, \overset{m-k-1}{\epsilon}, \beta, \overset{n-m+1}{\epsilon})$, where $k + m = n + 3$. The last condition implies that $(k \quad m)$ is one of the transpositions from the product $\psi = \prod_{2 \leq i < n+3-i} (i \quad n+3-i)$. By Lemma 1 $S \in A(f)$, hence the following identity is valid

$$f(x_1^n) = f(x_1^{k-1}, \alpha x_k, x_{k+1}^{m-1}, \beta x_m, x_{m+1}^n). \tag{1}$$

Putting in (1) $x_1 = \ldots = x_n = x$, and since $f = f^\psi$, we get

$$f(\overset{k-1}{x}, \alpha^{-1}x, \overset{n-k}{x}) = f(\overset{m-1}{x}, \beta x, \overset{n-m}{x}) = f(\overset{k-1}{x}, \beta x, \overset{n-k}{x}).$$

Hence $\alpha^{-1}x = \beta x$, that is, $\beta = \alpha^{-1}$.

b) $j - i$ even.

In this case there exists $t_1 \in N_n$ such that $T^{\phi^{t_1}}$ or $T^{\phi^{-t_1}}$ is of the form $S_1 = (\overset{k-1}{\epsilon}, \beta, \overset{m-k-1}{\epsilon}, \alpha, \overset{n-m+1}{\epsilon})$, where $k + m = n + 3$, and the rest of the proof is analogous to the preceding case.

Theorem 5. *Let (Q, f) be a D-n-quasigroup, n odd. If $T = (\overset{i-1}{\epsilon}, \alpha, \overset{j-i-1}{\epsilon}, \beta, \overset{n-j+1}{\epsilon}) \in A(f)$ for some $i, j \in N_n$ such that $j - i$ is even, then $\beta = \alpha^{-1}$.*

Proof. Then there exists $t \in N_n$ such that T^{ϕ^t} or $T^{\phi^{-t}}$ has the form $S = (\overset{k-1}{\epsilon}, \alpha, \overset{m-k-1}{\epsilon}, \beta, \overset{n-m+1}{\epsilon})$ where $k + m = n + 3$. The rest of the proof is analogous to the proof of Theorem 4.

Theorem 6. *If* (Q, f) *is a D-n-quasigroup and for some* $i \in N_n$ $(\overset{i-1}{\epsilon}, \alpha, \alpha^{-1}, \overset{n-i}{\epsilon}) \in A(f)$, *then* $(\overset{k-1}{\epsilon}, \alpha, \overset{m-k-1}{\epsilon}, \alpha^{-1}, \overset{n-m+1}{\epsilon}) \in A(f)$ *for every* $k, m \in N_{n+1}$, $k < m$.

Proof. If we denote $T_i = (\overset{i-1}{\epsilon}, \alpha, \alpha^{-1}, \overset{n-i}{\epsilon})$, then $T_i^{\phi^{-p}} = T_{i+p} = (\overset{i+p-1}{\epsilon}, \alpha, \alpha^{-1}, \overset{n-i-p}{\epsilon}) \in A(f)$ for all $p = 1 - i, 2 - i, \ldots, n - i$. Hence $S_k = T_n T_{n-1} \ldots T_k = (\overset{k-1}{\epsilon}, \alpha, \overset{n-k}{\epsilon}, \alpha^{-1}) \in A(f)$ for every $k \in N_n$, and $S_k S_m^{-1} = (\overset{k-1}{\epsilon}, \alpha, \overset{m-k-1}{\epsilon}, \alpha^{-1}, \overset{n-m+1}{\epsilon}) \in A(f)$ for every $k, m \in N_{n+1}$, $k < m$.

From Theorems 4 and 6 we get the following corollary.

Corollary 1. *If* (Q, f) *is a D-n-quasigroup, n even, and for some* $i \in N_n$ $(\overset{i-1}{\epsilon}, \alpha, \beta, \overset{n-i}{\epsilon}) \in A(f)$, *then* $(\overset{k-1}{\epsilon}, \alpha, \overset{m-k-1}{\epsilon}, \alpha^{-1}, \overset{n-m+1}{\epsilon}) \in A(f)$ *for every* $k, m \in N_{n+1}$, $k < m$.

By $A_i(f)$ we shall denote the set of all i-th components of all autotopisms of the n-quasigroup (Q, f). $A_i(f)$ is obviously a group under the composition of permutations.

Theorem 7. *If* (Q, f) *is a D-n-quasigroup, then* $A_i(f) = A_j(f)$ *for all* $i, j \in N_{n+1}$.

Proof. Let $i, j \in N_{n+1}$ and let $T = (\alpha_1^{n+1})$ be an arbitrary autotopism of a D-n-quasigroup (Q, f). Then by Lemma 1 $T^{\phi^{j-i}}$ is also an autotopism of f such that its j-th component is α_i. So, $A_i(f) \subset A_j(f)$, and analogously it follows $A_i(f) \supset A_j(f)$, hence $A_i(f) = A_j(f)$.

The group of all components of all autotopisms of a D-n-quasigroup (Q, f) we shall denote by $A_0(f)$.

Theorem 8. *If* (Q, f) *is a D-n-quasigroup, n even, then* $V(f)$ *is a commutative group.*

Proof. If $\alpha,\beta \in V(f)$, then $T = (\alpha, \overset{n-1}{\epsilon}, \alpha^{-1}) \in A(f)$ and $S = (\beta, \overset{n-1}{\epsilon}, \beta^{-1}) \in A(f)$. Then $TS = (\alpha\beta, \overset{n-1}{\epsilon}, \alpha^{-1}\beta^{-1}) \in A(f)$ and $(TS)^{\phi^{-1}} = (\alpha^{-1}\beta^{-1}, \alpha\beta, \overset{n-1}{\epsilon}) \in A(f)$ hence by Theorem 4 $\alpha^{-1}\beta^{-1} = (\alpha\beta)^{-1}$, that is, $\alpha\beta = \beta\alpha$. Since $((TS)^{\phi^{-1}})^{-1} = (\alpha\beta, (\alpha\beta)^{-1}, \epsilon) \in A(f)$, by Theorem 6 it follows that $\alpha\beta \in V(f)$. From $T^{\phi^{-1}} = (\alpha^{-1}, \alpha, \overset{n-1}{\epsilon}) \in A(f)$ by Theorem 6 $\alpha^{-1} \in V(f)$.

An autotopism of an n-quasigroup (Q,f) is called inverse regular if all its components are inverse regular for f. The set of all inverse regular autotopisms of a D-n-quasigroup (Q,f), where n is even, is obviously a commutative group which will be denoted by $A'(f)$.

Theorem 9. *If (Q,f) is a D-n-quasigroup, n even, then $V(f)$ and $A'(f)$ are normal subgroups in groups $A_0(f)$ and $A(f)$ respectively.*

Proof. If $\alpha \in V(f)$, then $T = (\alpha, \overset{n-1}{\epsilon}, \alpha^{-1}) \in A(f)$ and if $\beta \in A_0(f)$, then there exists an autotopism of f of the form $S = (\alpha_1^n, \beta)$. Then $R = S^{-1}TS = (\alpha_1^{-1}\alpha\alpha_1, \overset{n-1}{\epsilon}, \beta^{-1}\alpha^{-1}\beta) \in A(f)$ and $R^{\phi^{-1}} = (\beta^{-1}, \alpha\beta, \alpha_1^{-1}\alpha\alpha_1, \overset{n-1}{\epsilon})$ which by Theorems 4 and 6, implies that $\beta^{-1}\alpha\beta \in V(f)$, hence $V(f)$ is a normal subgroup of $A_0(f)$.

From the preceding it follows that all components of an autotopism of the form $S^{-1}TS$, where $S \in A(f), T \in A'(f)$, are inverse regular permutations, hence $A'(f)$ is a normal subgroup of $A(f)$.

Theorem 10. *If $(\alpha, \beta, \gamma, \overset{n-2}{\epsilon})$ is an autotopism of a D-n-quasigroup (Q,f), n even, then $\alpha\gamma^{-1}$ is inverse regular for f.*

Proof. From $T = (\alpha, \beta, \gamma, \overset{n-2}{\epsilon}) \in A(f)$, it follows that $S = T^{\psi\phi^{-2}} = (\gamma, \beta, \alpha, \overset{n-2}{\epsilon})$. Then $TS^{-1} = (\alpha\gamma^{-1}, \epsilon, \gamma\alpha^{-1}, \overset{n-2}{\epsilon}) \in A(f)$, and $(TS^{-1})^{\phi^{-2}} = (\overset{2}{\epsilon}, \alpha\gamma^{-1}, \epsilon, \gamma\alpha^{-1}, \overset{n-4}{\epsilon}) \in A(f)$. Continuing this, we obtain that $R = (\alpha\gamma^{-1}, \overset{n-1}{\epsilon}, \gamma\alpha^{-1}) \in A(f)$ and $(R^{\phi^{-1}})^{-1} = (\alpha\gamma^{-1}, \gamma\alpha^{-1}, \overset{n-1}{\epsilon}) \in A(f)$, which by Theorem 6 implies that $\alpha\gamma^{-1}$ is inverse regular for f.

References

[1] Belousov, V.D.: n-arnye kvazigruppy. Shtiintsa, Kishinev, 1972.

[2] Hoffman, D.G.: On the spectrum of n-quasigroups with given conjugate invariant subgroup, J. Combin. Theory Ser. A 35(1983), 98-99.

[3] Murathudzhaev, S.: Gruppy regulyarnyh podstanovok v TS-n-kvazigruppe, Izv. Akad. Nauk Moldav. SSR Ser. Fiz. Tekhn. Nauk 2(1976), 27-32.

[4] Murathudzhaev, S.: Tip n-arnyh kvazigrupp, Mat. Issled. 43(1976), 129-137.

[5] Paunić Dj., Stojaković, Z.,: Parastrophy invariant n-quasigroups, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 13(1983), 251-257.

[6] Sandik, M., Sokolov, E.I.: Total'no-simmetrichnye n-kvazigruppy, Mat. Issled. T. III, 2(1968), 170-182.

[7] Stojaković, Z.: Cyclic n-quasigroups, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser.Mat. 12(1982), 407-415.

[8] Stojaković, Z.: Alternating symmetric n-quasigroups, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser.Mat. 13(1983), 259-272.

[9] Stojaković, Z.: Dihedral quadruple systems (to appear).

[10] Stojaković, Z., Paunić, Dj.: Self-orthogonal n-quasigroups, Aequationes Math. 30(1986), 252-257.

REZIME

DIEDARSKE n-KVAZIGRUPE

n-kvazigrupa (Q, f) se naziva diedarska ako je $f(x_1, ..., x_n) = x_{n+1} \Leftrightarrow f(x_{\sigma(1)}, ..., x_{\sigma(n)}) = x_{\sigma(n+1)}$ za svaku permutaciju $\sigma \in D_{n+1}$, gde je D_{n+1} diedarska podgrupa simetrične grupe S_{n+1} stepena $n + 1$. Diedarske n-kvazigrupe (D-n-kvazigrupe) predstavljaju generalizaciju totalno simetričnih binarnih kvazigrupa. Navedene su neke ekvivalentne definicije i dati primeri D-n-kvazigrupa. Dokazano je da su neki retrakti D-n-kvazigrupa takodje n-kvazigrupe. Razmatrane su autotopije i regularne permutacije D-n-kvazigrupa i odredjene neke njihove osobine.

Received by the editors June 18, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 63-73

Review of Research
Faculty of Science
Mathematics Series

ON THE k-TH PRIME FACTOR OF AN INTEGER

Aleksandar Ivić
Katedra Matematike RGF-a Univerziteta u Beogradu
Djušina 7, 11000 Beograd, Jugoslavija

Abstract

Let $P_k(n)$ be the k-th largest prime factor of an integer $n \geq 1$ if n has at least k prime factors, and let $P_k(n)$ be zero otherwise. An asymptotic formula for the sum $\sum_{n\leq x} P_k(n)$ is obtained.

AMS Mathematics Subject Classification (1980): 10H15, 10H25
Key words and phrases: k-th largest prime factor of an integer, asymptotic formulas for summatory functions.

Let $P_k(n)$ denote the k-th largest prime factor of an integer $n \geq 1$ if $\Omega(n) \geq k$, and let otherwise $P_k(n) = 0$, where $\Omega(n)$ is the number of all prime factors of n. In other words $P_k(1) = 0$, and if $P(n) = P_1(n)$ is the largest prime factor of $n \geq 2$, then $P_2(n) = P(n/P(n)), \ldots, P_k(n) = P(n/P_{k-1}(n))$ if $\Omega(n) \geq k$. In [1] Alladi and Erdös (see (1.21), p. 284) showed that, for $k \geq 1$ a fixed integer,

$$\sum_{n\leq x} P_k(n) = A_k \frac{x^{1+1/k}}{\log^k x} + O\left(\frac{x^{1+1/k}\log\log x}{\log^{k+1} x}\right). \tag{1}$$

Their proof was elementary, but quite complicated. Recently R. Balasubramanian [3] found a simple proof of (1) that works even if k is not fixed, but lies in a suitable interval depending on x. In addition, he evaluated

$A_k = k^{2k}\zeta(1+1/k)/(k+1)!$. In [4] J.-M. De Koninck and I proved, for any fixed integer $N \geq 1$,

$$\sum_{n\leq x} P_1(n) = A_{1,1}\frac{x^2}{\log x} + \ldots + A_{1,N}\frac{x^2}{\log^N x} + O(\frac{x^2}{\log^{N+1} x}), \tag{2}$$

where $A_{1,1} = \pi^2/12, \ldots, A_{1,N}$ are certain absolute constants which may be explicitly evaluated. The aim of this note is to generalize (2) as to include a sharper version of (1). The result is the following

Theorem. *Let* $k \geq 1$ *be a fixed integer, and* $N \geq 0$ *an arbitrary, but fixed integer. Then there exist constants* $A_{k,1} > 0, \ldots, A_{k,N+1}$ *which may be explicitly evaluated such that*

$$\sum_{n\leq x} P_k(n) = A_{k,1}\frac{x^{1+1/k}}{\log^k x} + \ldots + A_{k,N+1}\frac{x^{1+1/k}}{\log^{k+N} x} + O(\frac{x^{1+1/k}}{\log^{k+N+1} x}). \tag{3}$$

The proof of the asymptotic formula (3) will be also elementary. We shall make use of the formulas

$$\sum_{p\leq x}\frac{p^\alpha}{\log^\beta p} = c_1\frac{x^{\alpha+1}}{\log^{\beta+1} x} + \cdots + c_N\frac{x^{\alpha+1}}{\log^{\beta+N} x} + O(\frac{x^{\alpha+1}}{\log^{\beta+N+1} x}), \tag{4}$$

$$(\alpha > 0)$$

$$\sum_{y_1<p\leq y_2}\frac{1}{p^\alpha(\log\frac{x}{p})^\beta} = (\frac{d_1}{t^{\alpha-1}\log t(\log\frac{x}{t})^\beta} + \cdots \tag{5}$$

$$+\frac{d_N}{t^{\alpha-1}\log t(\log\frac{x}{t})^{\beta+N-1}})\Big|_{y_1}^{y_2} + O(\frac{1}{y_1^{\alpha-1}\log y_1(\log\frac{x}{y_1})^{\beta+N}}),$$

and

$$\sum_{n\leq x^\gamma}\frac{1}{n^\alpha(\log\frac{x}{n})^\beta} = \frac{e_1}{\log^\beta x} + \cdots + \frac{e_N}{\log^{\beta+N-1} x} + O(\frac{1}{\log^{\beta+N} x}), \tag{6}$$

where $0 < \gamma < 1$, and $\alpha > 1$ in (5) and (6). Here and in the sequel p (with or without indices) stands for primes, the constants $c_j = c_j(\alpha,\beta), d_j = d_j(\alpha,\beta), e_j = e_j(\alpha,\beta) (j = 1,\ldots,N)$ may be explicitly evaluated, $N \geq 1$ is an arbitrary fixed integer, and in (5) $y_1 < y_2 < x, y_1 \to \infty$. Moreover we shall use the standard notation $f = O(g)$ and $f << g$, which both mean that $|f| \leq Cg$ for some absolute $C > 0$. The proofs of the above formulas follow without difficulty from the prime number theorem and elementary analysis. For example, we have

$$\sum_{y_1<p\le y_2}\frac{1}{p^{\alpha}(\log\frac{x}{p})^{\beta}}=\int_{y_1+0}^{y_2-0}\frac{d\pi(t)}{t^{\alpha}(\log\frac{x}{t})^{\beta}}=\int_{y_1}^{y_2}\frac{dt}{t^{\alpha}(\log\frac{x}{t})^{\beta}\log t}$$

$$+O(y_1^{1-\alpha}(\log\frac{x}{y_1})^{-\beta-N}(\log y_1)^{-1})$$

by the prime number theorem in the standard form

$$\pi(x)=\sum_{p\le x}1=\int_2^x\frac{dt}{\log t}+O(xe^{-C\sqrt{\log x}})\quad (C>0),$$

and successive integrations by parts give (5).

We begin the proof of (3) by noting that

$$(7)\qquad \sum_{n\le x}P_k(n)=\sum_{mp_1p_2\ldots p_k\le x,P(m)\le p_1,p_1\le p_2\le\ldots\le p_k}p_1$$

$$=\sum_{p_1p_2\ldots p_k\le x,p_1\le p_2\le\ldots\le p_k}p_1\psi(\frac{x}{p_1p_2\ldots p_k},p_1),$$

where as usual $\psi(x,y)$ denotes the number of $n\le x$ all of whose prime factors are $\le y$. It will be shown now how the first equality in (7) may be used in a simple way to obtain lower and upper bounds for the sum of $P_k(n)$. Namely, let in (7) $m=1,x^{1/k}2^{-k}\le p_1\le x^{1/k}2^{1-k},\ldots,\frac{1}{2}x^{1/k}\le p_k\le x^{1/k}$. By the prime number theorem there are $>> x^{1/k}/\log x$ choices for each $p_j (j=1,\ldots,k)$. Therefore

$$(8)\qquad \sum_{n\le x}P_k(n)>>x^{1/k}(x^{1/k}/\log x)^k=x^{1+1/k}\log^{-k}x,$$

which represents a lower bound of the right order of magnitude. On the other hand, we have

$$x\ge mp_1p_2\ldots p_k\ge mp_1^k,$$

hence $p_1\le(x/m)^{1/k}$, which gives

$$(9)\qquad \sum_{n\le x}P_k(n)\le x^{1/k}\sum_{m\le x}m^{-1/k}\sum_{p_1p_2\ldots p_k\le x/m}1$$

$$<<x^{1+1/k}\sum_{m\le x}m^{-1-1/k}\frac{(\log\log x)^{k-1}}{\log x}$$

$$\leq \zeta(1+1/k)x^{1+1/k}(\log\log x)^{k-1}\log x$$
$$<< x^{1+1/k}(\log\log x)^{k-1}\log x.$$

Here we used the classical elementary estimate

$$\sum_{n\leq x,\omega(n)=k} 1 < \frac{Ax(\log\log x + B)^{k-1}}{\log x(k-1)!}$$
$$(A, B > 0; k \geq 1, x > 2)$$

of Hardy and Ramanujan (see [7], p. 265), where A, B are suitable absolute constants and $\omega(n)$ is the number of distinct prime factors of n. Thus the upper bound in (9) is only by a factor of $(\log\log x)^{k-1}$ smaller than the true order of the summatory function of $P_k(n)$.

The main obstacle in evaluating the last sum in (7) is the presence of the ψ-function. However, noting that $\psi(x,y) = [x]$ if $y \geq x$ ($[x]$ is the integer part of x) and using the trivial $\psi(x,y) \leq x$ for $y \leq x$, this difficulty may be overcome, and we may obtain essentially (7) without the condition $P_1(m) \leq p_1$. In this way the problem will be reduced to a rather technical one which involves a k-fold summation over primes. We have

$$(10) \qquad \sum_{p_1p_2\ldots p_k\leq x, p_1\leq p_2\leq\ldots\leq p_k} p_1\psi(\frac{x}{p_1p_2\ldots p_k}, p_1)$$

$$= \sum_{p_1p_2\ldots p_k\leq x, p_1\leq p_2\leq\ldots\leq p_k, p_1^2p_2\ldots p_k>x} p_1\psi(\frac{x}{p_1p_2\ldots p_k}, p_1)$$
$$+ \sum_{p_1^2p_2\ldots p_k\leq x, p_1\leq p_2\leq\ldots\leq p_k} p_1\psi(\frac{x}{p_1p_2\ldots p_k}, p_1)$$
$$= \sum_{p_1p_2\ldots p_k\leq x, p_1\leq p_2\leq\ldots\leq p_k, p_1^2p_2\ldots p_k>x} p_1[\frac{x}{p_1p_2\ldots p_k}]$$
$$+ \ O(\sum_{p_1^2p_2\ldots p_k\leq x, p_1\leq p_2\leq\ldots\leq p_k} \frac{xp_1}{p_1p_2\ldots p_k}).$$

In the sum appearing in the O-term the conditions $p_1 \leq p_2 \leq \ldots \leq p_k$ and $p_1^2p_2\ldots p_k \leq x$ imply that $p_1 \leq x^{1/(k+1)}$. Therefore this sum is

$$\leq \sum_{p_1^2p_2\ldots p_k\leq x} \frac{x^{1+1/(k+1)}}{p_1p_2\ldots p_k}$$

$$<< x^{1+1/(k+1)} \sum_{p_1 \le x} \frac{1}{p_1} \cdots \sum_{p_k \le x} \frac{1}{p_k} << x^{1+1/(k+1)} (\log\log x)^k,$$

on using the elementary estimate

$$\sum_{p \le x} \frac{1}{p} = \log\log x + O(1).$$

Thus we obtain

$$(11) \qquad \sum_{n \le x} P_k(n) = \sum_{p_1 p_2 \ldots p_k \le x, p_1 \le p_2 \le \ldots \le p_k} p_1 [\frac{x}{p_1 p_2 \ldots p_k}]$$

$$+O(x^{(k+2)/(k+1)} (\log\log x)^k),$$

$$= \sum_{n p_1 p_2 \ldots p_k \le x, p_1 \le p_2 \le \ldots \le p_k} p_1 + O(x^{k+2)/(k+1)} (\log\log x)^k),$$

where n stands for natural numbers, and the portion of the sum containing the greatest integer function for $p_1^2 p_2 \ldots p_k \le x$ is estimated trivially in exactly the same way as the O-term in (10). Two further simplifications may be made in the last sum in (11). Firstly, we may restrict n to the range $1 \le n \le x^{\varepsilon}$, where $0 < \varepsilon < 1$ is any fixed number. This follows from

$$\sum_{n p_1 p_2 \ldots p_k \le x, n > x^{\varepsilon}, p_1 \le p_2 \le \ldots \le p_k} p_1 << x^{1/k} \sum_{n > x^{\varepsilon}} n^{-1/k} \sum_{p_1 p_2 \ldots p_k \le x/n} 1$$

$$<< x^{1+1/k} \sum_{n > x^{\varepsilon}} n^{-1-1/k} << x^{1+1/k-\varepsilon/k},$$

since the last expression is of lower order of magnitude than any term on the right-hand side of (3). Secondly, the contribution from $p_k > (x/2n)^{1/(k-1)}$ in (11) ($k \ge 2$ may be henceforth assumed to hold in view of (2)) is also negligible. Namely, from $p_k > (x/2n)^{1/(k-1)}, p_1 \le \ldots \le p_k$ and $p_1 p_2 \ldots p_k \le x/n$ we have $p_1 << (x/n)^{(k-2)/(k-1)^2}$. Hence for $k > 2$

$$\sum_{n \le x^{\varepsilon}} \sum_{p_1 p_2 \ldots p_k \le x/n, p_1 \le p_2 \le \ldots \le p_k, p_k > (x/2n)^{1/(k-1)}} p_1$$

$$<< \quad x^{1+(k-2)/(k-1)^2} \sum_{n \le x^{\varepsilon}} n^{-1-(k-2)/(k-1)^2}$$

$$\le \quad x^{1+(k-2)/(k-1)^2} \zeta(1 + (k-2)/(k-1)^2) << x^{1+(k-2)/(k-1)^2},$$

on using the same argument as in (9), and for $k = 2$ we get the bound $x \log x$. Actually, even the contribution for $p_k > (x/n)^{1/k} (\log x)^D, D =$

$D(k) > 0$ sufficiently large, is also seen to be negligible by the above argument, but it will be enough to have $p_k \le (x/2n)^{1/(k-1)}$ in (11). We obtain

$$\sum_{n\le x} P_k(n) = \sum_{n\le x^{\varepsilon}} \sum_{p_1p_2\dots p_k\le x/n,\, p_k\le (x/2n)^{1/(k-1)},\, p_1\le\dots\le p_k} p_1 + O(x^{1+\Theta}),$$

where $\Theta < 1/2$ if $k = 2$ and for $k > 2$ we have

$$\Theta = \Theta(k,\varepsilon) = \max\left(\frac{1-\varepsilon}{k}, \frac{k-2}{(k-1)^2}\right) < \frac{1}{k}.$$

Moreover, the double sum above may be written as a multiple sum, so that in fact we have

(12) $$\sum_{n\le x} P_k(n) = O(x^{1+\Theta}) +$$

$$+ \sum_{n\le x^{\varepsilon}} \sum_{p_k\le (x/2n)^{1/(k-1)}} \sum_{p_{k-1}\le \min(p_{k'} \frac{x}{2^{k-2}np_k})} \dots$$

$$\dots \sum_{p_2\le \min(p_{3'} \frac{x}{2p_3\cdots p_k n})} \sum_{p_1\le \min(p_{2'} \frac{x}{p_2\cdots p_k n})} p_1.$$

But the conditions $p_2 \le p_3 \le \dots \le p_k$ and $p_k \le (x/2n)^{1/(k-1)}$ imply

$$\min(p_{k'} \frac{x}{2^{k-2}p_k n}) = p_k, \dots, \min(p_{3'} \frac{x}{2p_3\dots p_k n}) = p_{3'}$$

and finally one has

(13) $$\min(p_{2'} \frac{x}{p_2p_3\dots p_k n}) = \begin{cases} p_2 & \text{if } p_2 \le (x/(p_3\dots p_k n))^{1/2}, \\ \frac{x}{p_2p_3\dots p_k n} & \text{if } p_2 > (x/(p_3\dots p_k n))^{1/2}. \end{cases}$$

It follows that (12) becomes

(14) $$\sum_{n\le x} P_k(n) = O(x^{1+\Theta}) +$$

$$+ \sum_{n\le x^{\varepsilon}} \sum_{p_k\le (x/2n)^{1/(k-1)}} \sum_{p_{k-1}\le p_k} \dots$$

$$\dots \sum_{p_3\le p_4} \sum_{p_2\le p_3} \sum_{p_1\le \min(p_{2'} \frac{x}{p_2p_3\cdots p_k n})} p_1.$$

Now from (14) it becomes fairly clear why (3) must hold. We have k summations over primes $p_j (1 \le j \le k)$, each of which accounts for an additional $\log x$ factor in the denominators in (3), on using (4) and (5). The summation over n will "preserve" log-factors in view of (6), and the final exponent of x in all the terms must be the same, and consequently it will be $1+1/k$ in view of (8) and (9). To elucidate this in more detail suppose now that in (14) we have $p_k \le (x/n)^{1/k}$. Then in (13) we have the first case, since

$$p_2^2 p_3 \ldots p_k \le p_k^k \le x/n.$$

Therefore the corresponding portion of the multiple sum in (14) equals

$$\sum_{n\le x^\epsilon}\ \sum_{p_k\le(x/n)^{1/k}}\ \sum_{p_{k-1}\le p_k}\cdots\sum_{p_2\le p_3}\ \sum_{p_1\le p_2} p_1 \tag{15}$$

$$\begin{aligned}
&= \sum_{n\le x^\epsilon}\ \sum_{p_k\le(x/n)^{1/k}}\ \sum_{p_{k-1}\le p_k}\cdots\sum_{p_2\le p_3}\Big(\sum_{j=1}^{N+1}\frac{c_{1,j}p_2^2}{\log^j p_2}+O\Big(\frac{p_2^2}{\log^{N+2} p_2}\Big)\Big)\\
&= \ldots = \sum_{n\le x^\epsilon}\ \sum_{p_k\le(x/n)^{1/k}}\Big(\sum_{j=1}^{N+1}\frac{c_{k-1,j}p_k^k}{\log^{j+k-2} p_k}+O\Big(\frac{p_k^k}{\log^{N+k} p_k}\Big)\Big)\\
&= \sum_{n\le x^\epsilon}\Big(\sum_{j=1}^{N+1}\frac{c_{k,j}(x/n)^{(k+1)/k}}{\log^{j+k-1}(x/n)}+O\Big(\frac{(x/n)^{(k+1)/k}}{\log^{N+k+1} x}\Big)\Big)\\
&= B_{k,1}\frac{x^{1+1/k}}{\log^k x}+B_{k,2}\frac{x^{1+1/k}}{\log^{k+1} x}+\ldots+\\
&+ B_{k,N+1}\frac{x^{1+1/k}}{\log^{k+N} x}+O\Big(\frac{x^{1+1/k}}{\log^{k+N+1} x}\Big),
\end{aligned}$$

where $B_{k,j}$ are suitable constants $(B_{k,1} > 0)$, and where (4) and (6) was used.

The remaining portion of the multiple sum in (14), where now

$$(x/n)^{1/k} < p_k \le (x/2n)^{1/(k-1)}$$

is estimated in an analogous way, using (4)-(6). Consider the range $p_{k-1} \le p_k$ and split it into the ranges $p_{k-1} \le (x/np_k)^{1/(k-1)}$ and $(x/np_k)^{1/(k-1)} < p_{k-1} \le p_k$. If $p_{k-1} \le (x/np_k)^{1/(k-1)}$, then $p_2 \le (x/p_3 \ldots p_k n)^{1/2}$, and we obtain that the corresponding portion of the multiple sum equals

$$\sum_{n\le x^\epsilon}\ \sum_{(x/n)^{1/k}<p_k\le(x/2n)^{1/(k-1)}}\ \sum_{p_{k-1}\le(x/np_k)^{1/(k-1)}}\ \sum_{p_{k-2}\le p_{k-1}}\cdots$$

$$\ldots \sum_{p_2 \le p_3} \sum_{p_1 \le p_2} p_1$$

$$= C_{k,1}\frac{x^{1+1/k}}{\log^k x} + C_{k,2}\frac{x^{1+1/k}}{\log^{k+1} x} + \ldots + C_{k,N+1}\frac{x^{1+1/k}}{\log^{k+N} x} + O(\frac{x^{1+1/k}}{\log^{k+N+1} x}).$$

This processes is continued, and in the j-th step $(2 \le j \le k-1)$ the range of summation for p_{k-j+1} is split into

$$p_{k-j+1} \le (\frac{x}{np_kp_{k-1}\ldots p_{k-j+2}})^{1/(k-j+1)}$$

and

$$(\frac{x}{np_kp_{k-1}\ldots p_{k-j+2}})^{1/(k-j+1)} < p_{k-j+1} \le p_{k-j+2}.$$

In the first range we have $p_{k-j} \le p_{k-j+1}, \ldots, p_2 \le p_3, p_1 \le p_2$; and this portion yields analogously as before an expression of the type appearing on the right-hand side of (3). In the second range, we continue the process of splitting up the range of summation of the next variable. All the ensuing sums will be eventually of the same type, and the last one will be

$$\sum_{n \le x^\epsilon} \sum_{(x/n)^{1/k} < p_k < (x/2n)^{1/(k-1)}} \sum_{(x/np_k)^{1/(k-1)} < p_{k-1} \le p_k}$$

$$\ldots \sum_{(x/np_3\ldots p_k)^{\frac{1}{2}} < p_2 \le p_3} \sum_{p_1 \le \frac{x}{p_2p_3\cdots p_kn}} p_1$$

$$= \sum_{n \le x^\epsilon} \sum_{p_k} \sum_{p_{k-1}} \ldots \sum_{(x/np_3\ldots p_k)^{\frac{1}{2}} < p_2 \le p_3} \times$$

$$\times(\sum_{j=1}^{N+1} \frac{c_{1,j}x^2}{p_2^2\ldots p_k^2n^2\log^j(\frac{x}{p_2\ldots p_kn})} + O(\frac{x^2}{p_2^2\ldots p_k^2n^2\log^{N+2}(\frac{x}{p_2\ldots p_kn})})).$$

This will eventually reduce again to an expression analogous to the one on the right-hand side (3), since the terms coming from both limits of summation for $p_2, \ldots, p_{k-1}$ will at the end give rise to the same type of terms. Take, for example, the lower limits of summation. Using (5) it is seen that summation over $p_2 > (x/np_3\ldots p_k)^{1/2}$ (i.e. the lower limit) gives rise to terms of the type

$$x^{3/2}n^{-3/2}p_3^{-3/2}\ldots p_k^{-3/2}\log^{-j-1}(x^{3/2}\ldots p_k^{-3/2})\ (j = 1, \ldots, N+2),$$

and in general summation over $p_m > (x/(np_{m+1}\dots p_k))^{1/2}(2 \le m \le k-1)$ gives rise to terms of the type

$$x^{(m+1)/m} n^{-(m+1)/m} p_{m+1}^{-(m+1)/m} \dots$$

$$\dots p_k^{-(m+1)/m} \log^{-j-m+1}(x^{(m+1)/m} \dots p_k^{-(m+1)/m}).$$

Finally summation over p_k gives

$$(16) \quad \sum_{n \le x^{\epsilon}} x^{(k+1)/k} n^{-(k+1)/k} \Big(\sum_{j=1}^{N+1} D_{k,j} \log^{1-k1j}(x/n) + O(\log^{-k-N-1} x)\Big)$$

$$= x^{(k+1)/k} \Big(\sum_{j=1}^{N+1} E_{k,j} \log^{1-k-j} x + O(\log^{-k-N-1} x)\Big)$$

with suitable constants $D_{k,j}$ and $E_{k,j}(E_{k,1} > 0)$, since the upper limit of summation $(x/2n)^{1/(k-1)}$ for p_k gives by (5) an expression absorbed in the error term in (3) and (16). Likewise the upper limits of summation for $p_2, p_3, \dots, p_{k-1}$ will eventually yield again expression of the type (16), so collecting all such expressions it is seen that (3) is established. The constants $A_{k,j}(j = 1, \dots, N)$ in (3) can be certainly written down explicitly in closed form, although the above proof would lead to fairly cumbersome expressions for these constants. It may be also remarked that our theorem gives easily the asymptotic formula

$$(17) \quad \sum_{n \le x} (B(n) - P_1(n) - \dots - P_{k-1}(n))$$

$$= x^{1+1/k} \Big(\sum_{j=1}^{N+1} A_{k,j} \log^{1-j-k} x + O(\log^{-k-N-1} x)\Big),$$

where $k \ge 2$ and the hypotheses are the same as in (3). Here $B(n) = \sum_{p^\alpha \| n} \alpha p$ is the sum of all prime divisions of n. This formula, with the right-hand side as in (1), was given by Alladi and Erdös [2]. One obtains easily (17) from (3) when one observes that the sum in (17) is empty for $\Omega(n) \le k-1$, and that for $r = \Omega(n) \ge k$ one has

$$(18) \quad B(n) - P_1(n) - \dots - P_{k-1}(n) = P_k(n) + \dots + P_r(n)$$

$$= P_k(n) + O(P_{k+1}(n) \log n),$$

since trivially $\Omega(n) << \log n$. Hence summing (18) with the aid of (3) we obtain (17).

By the same method we sould evaluate asymptotically the sum $\sum_{n \le x}(P_k(n))^c$, where $c > 0$ is a fixed real number, and $k \ge 1$ is a fixed integer. The result would be an expression similar to the right-hand side of (3), with $x^{1+1/k}$ replaced by $x^{1+c/k}$, and $A_{k,j} = A_{k,j}(c)$. The case $c < 0$ requires new methods. For $c = -1$ (sum of reciprocals, where $n = 1$ is excluded from summation and n with $\Omega(n) < k$) this problem was solved in [5] and [6].

References

[1] K. Alladi and P. Erdös, On an additive arithmetic function, *Pacific J. Math.* 77 (1977), 275-294.

[2] K. Alladi and P. Erdös, On the asymptotic behavior of large prime factors of integers, *Pacific J. Math.* 82 (1979), 295-315.

[3] R. Balasubramanian, On the k-th largest factor of an integer, unpublished.

[4] J. -M. De Koninck and A. Ivić, The distribution of the average prime divisor of an integer, *Archiv Math.* 43 (1984), 37-43.

[5] P. Erdös and A. Ivić, On sums involving reciprocals of certain arithmetical functions, *Publs. Inst. Mathématique* 34(46) (1982), 49-56.

[6] P. Erdös, A. Ivić and C. Pomerance, On sums involving reciprocals of the largest prime factor of an integer, *Glasnik Matematički* 21(41) (1986), 283-300.

[7] S. Ramanujan, Collected Papers, Chelsea, New York, 1962.

REZIME

O k - TOM PROSTOM FAKTORU PRIRODNOG BROJA

Neka je $P_k(n)$ k-ti najveći prosti faktor prirodnog broja $n \geq 1$ ako n ima bar k prostih faktora, a neka je $P_k(n) = 0$ u suprotnom. Dobivena je asimptotska formula za sumu $\sum_{n \leq x} P_k(n)$.

Received by the editors November 11, 1988.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 75-82

Review of Research
Faculty of Science
Mathematics Series

DISTANCE PRESERVING MAPS ON ABELIAN LATTICE ORDERED GROUPS

E.Pap
Institute of Mathematics, University of Novi Sad,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

If f is a map from an Abelian lattice ordered group G_1 (endowed with a root function γ_2 of order two) onto an Archimedean Abelian lattice ordered group G_2 with $f(0) = 0$ and $|f(x) - f(y)|$ depends functionally on $|x - y|$, then f is additive.

AMS Mathematics Subject Classification (1980): 46A06, 06F20
Key words and phrases: Abelian lattice ordered group, isometry, Archimedean order.

Mazur and Ulam [7] have proved that every isometry of a normed real vector space onto a normed vector space is linear up to translation. This result was extended in many different directions. Vogt [12] has replaced isometries by the more general maps with the property that the distance between image points depends functionally on the distance between domain points.

Swamy [10],[11] has defined an isometry in an Abelian lattice ordered group G as a bijection $f : G \to G$ with the property

$$|f(x) - f(y)| = |x - y| \quad (x, y \in G)$$

(without bijectivity by Jakubik [10] under the name weak isometry). Swamy [10] has proved that every isometry on a lattice ordered group G is of the form $T(x)+a$ where a is a fixed element of G and T is an involutory isometric group automorphism of G.

We shall investigate, in this paper, maps from Abelian lattice ordered group G_1 into an Archimedean Abelian lattice ordered group G_2 which preserves the equlity of distance in the sense of autometrized spaces.We shall prove that the surjective distance preserving map f with $f(0) = 0$ is additive.

1. Let G be an Abelian group written additively with a neutral element 0. G is an Abelian lattice order group if G is also a lattice under a partial ordered relation $\leq$ with the property that $a \leq b$ implies $c+a \leq c+b$ for all $c \in G$. A partially ordered group G is said to be Archimedean if $a > 0$ and $b > 0$ than $na > b$ for a suitable $n \geq 1$, where $na = \underbrace{a + \cdots + a}_{n}$. We extract the following known result (see [2], footnote on page 12).

Proposition 1. *If G is an Archimedean Abelian lattice ordered group, then $na < b\ (n = 0, \pm 1, \pm 2, \ldots)$ implies $a = 0$.*

Let $G^+ = \{x : x \in G, x \geq 0\}$. The absolute $|a|$ of an element $a \in G$ is defined by $|a| = a \bigvee (-a)$. It has the following properties:

(i) $|a| \in G^+$, i.e. $|a| \geq 0$, for all $a \in G$ and the equality holds if and only if $a = 0$;

(ii) $|-a| = |a|$;

(iii) $|a+b| \leq |a| + |b|$;

(iv) $|na| = n|a|$.

Remark. The property (iii) holds only for Abelian groups. The properties (i), (ii) and (iv) hold also for the noncommutative case.

Let G_1 and G_2 be lattice ordered Abelian groups.

Definition 1. *A map $f : G_1 \to G_2, f(0) = 0$ preserves the equality of distance if there exists a function $p : G_1^+ \to G_2^+$ such that for each x and y from G*

$$|f(x) - f(y)| = p(|x-y|).$$

The function p is called the gauge function for f.

Theorem 1. *Let G_1 and G_2 be Abelian lattice ordered groups.If f is a map from G_1 into G_2 with $f(0) = 0$ then the following statements are equivalent:*
a) f preserves the equality of distance;
b) whenever x, y, z and u are in G_1 and $|x - y| = |z - u|$,then

$$|f(x) - f(y)| = |f(z) - f(u)|;$$

If $G_1 = G_2$ then the above assertions are equivalent to the following one:
c) there exists an integer n such that

$$| f(x) - f(y) | = n | x - y | \quad (x, y \in G_1).$$

Proof. a) implies b). Let x, y, z and u be from G_1 such that $|x - y| = |z - u|$. Then we have

$$|f(x) - f(y)| = p(|x - y|) = p(|z - u|) = |f(z) - f(u)|.$$

b) implies a). We define $p(z) = |f(z)|$ $(z \in G_1^+)$. Then p is a gauge function for f. We have for any $x, y \in G_1$ and $z = |x - y|$,

$$|x - y| = z = |z| = |z - 0|.$$

Therefore, by b) we obtain

$$p(|x - y|) = p(z) = |f(z)| = |f(x) - f(y)|.$$

Let $G_1 = G_2$. a) implies c). Let $d(x, y) = p(|x - y|)$. Then $d(x, y)$ is a translation invariant map, *i.e.* $d(a, b) = d(a + c, b + c) = d(c + a, c + b)$ and symmetric map, *i.e.*$d(a, b) = d(b, a)$. Hence d is an intrinsic metric in the sense of Holland [3]. By the corollary from [3], there exists an integer n such that

$$p(|x - y|) = d(x, y) = n|x - y|.$$

c) implies a). If a map f satisfies c) then a gauge function for f is $p(z) = nz$, $(z \in G_1^+)$.

Remark. By the preceding theorem the gauge function p for a map $f : G \to G$ which preserves the equality of distance has to be of the form $p(z) = nz$, for an integer n. We shall call $n|x - y|$ an intrinsic n- metric. Jakubik [1] and [2] has defined an n-isometry on a lattice ordered group as a function from G onto G which preserves the intrinsic n-metric.1-isometries are always n-isometries for every n. For Abelian lattice ordered group every n-isometry is also an 1-isometry.

2. Using the ideas from [1] and [12] we shall prove the following theorem.

Theorem 2. *Let G be an Archimedean Abelian lattice ordered group. Let A be a bounded subset of G, i.e. there exists an element b from G such that $|x| \leq b$ for all x from A. Suppose there exists an element $a \in A$, a surjective isometry (congruence in the sense of Swamy [10]) $g : A \to A$ and a natural number m such that for all $x \in A$*

$$(1) \qquad m|a-x| \leq |g(x)-x|.$$

Then, every surjective isometry $h : A \to A$ fixes a.

Proof . Since each isometry in an Abelian lattice ordered group G is an injection ($x \neq y$ implies $|x-y| \neq 0$, hence $|f(x)-f(y)| \neq 0$ and so $f(x) \neq f(y)$), h^{-1} and g^{-1} exist. We have that h, g, h^{-1} and g^{-1} are bijective isometries. Hence any finite composition of them is also a bijective isometry. We define a sequence $\{g_n\}$ of isometries on A in the following way:

$$g_1 = g,\ g_2 = hg_1h, \ldots,\ g_{n+1} = g_{n-1}g_n(g_{n-1})^{-1}.$$

We also define a sequence $\{a_n\}$ from G in the following way:

$$a_1 = a, a_2 = h(a), \cdots, a_{n+1} = g_{n-1}(a_n) \quad (n \geq 2).$$

Starting from (1) we obtain, by induction,

$$(2) \qquad m|a-x| \leq |g_n(x)-x| \quad (x \in A).$$

Taking $x = a_{n+1}$ in (2), we have

$$m|a_{n+1}-a_n| = m|a_n-a_{n+1}| \leq |g_n(a_{n+1})-a_{n+1}| = |a_{n+2}-a_{n+1}|.$$

Hence, by induction, we obtain

$$(3) \qquad m^n|a_2-a_1| \leq |a_{n+2}-a_{n+1}| \quad (n \in N).$$

Since A is ordered bouned, we have for all $n \in N$

$$|a_{n+2}-a_{n+1}| \leq |a_{n+2}| + |a_{n+1}| \leq 2b.$$

Hence by (3), we obtain

(4) $\quad n|a_2 - a_1| \leq m^n|a_2 - a_1| \leq 2b$ for each $n \in N$.

Since G is an Archimedean lattice ordered group, (4) implies by Proposition 1. $|a_2 - a_1| = 0$. Hence,

$$a = a_1 = a_2 = h(a).$$

This completes the proof.

A function $\gamma_n : G \to G$ is a root function of order $n \in N \cup \{0\}$ on an Abelian group G if

(a) $\gamma_n(x + y) = \gamma_n(x) + \gamma_n(y)$;

(b) $n\gamma_n(x) = x$ (see [8]);
hold.

Proposition 2. *Let G be an Abelian lattice ordered group. Then, for each non-negative integer n there exists at most one root function γ_n on G.*

Proof. Since every lattice ordered group is isolated (see E in 5.1 from [2]) it is also torsion-free. Hence, for any $n \in N \cup \{0\}$ there exists at most one root function γ_n of order n.

Now, we have the main result of this paper.

Theorem 3. *Let G_1 and G_2 be Abelian lattice ordered groups such that (G_1 has a root function γ_2) G_2 is Archimedean. Let $f : G_1 \to G_2$ with $f(0) = 0$ be a surjective map which preserves the equality of distance. Then, f is additive.*

Proof. Let x be an arbitrary but fixed element from G_1. We define

$$A = \{y : y \in G_2 \text{ and } \quad |y| = |2f(x) - y| \leq 2|f(x)|\}$$

and $g : A \to A$ by $g(y) = 2f(x) - y$. It is obvious that g is an isometry from A onto A. Taking $a = f(x)$, we have

$$2|a - y| = 2|f(x) - y| = |(2f(x) - y) - y| = |g(y) - y|.$$

Hence, g satisfies condition (1) for $m = 2$ from Theorem 2. Since the set A is bounded , we can apply Theorem 2. Therefore, we obtain that every surjective isometry of A fixes a. Now we can define an appropriate isometry h. Let $u = f^{-1}(2f(x))$. We define $h : A \to A$ by

$$h(y) = f(u - f^{-1}(y)).$$

Now, we can prove that h is well-defined. If $f(x_1) = f(x_2) = y$ then we have

$$|f(u - x_1) - f(u - x_2)| = p(|(u - x_1) - (u - x_2)|) = p(|x_2 - x_1|) =$$
$$= |f(x_2) - f(x_1)| = |y - y| = 0.$$

Hence $f(u - x_1) = f(u - x_2)$.

If $f(x_1) = y_1$ and $f(x_2) = y_2$, then

$$|h(y_1) - h(y_2)| = |f(u - x_1) - f(u - x_2)| =$$
$$p(|x_2 - x_1|) = |f(x_2) - f(x_1)| = |y_2 - y_1|,$$

i.e. h is an isometry. We shall prove that h is an isomerty from A into A. Namely, if $y_1 \in A$ and $y_1 = f(x_1)$, then

$$|2f(x) - h(y_1)| = |2f(x) - f(u - x_1)| = |f(u) - f(u - x_1)| =$$
$$p(|x_1 - 0|) = |f(x_1) - f(0)| = |y_1 - 0| = |y_1| = |2f(x) - y_1| =$$
$$|f(u) - f(x_1)| = p(|u - x_1|) = p(|(u - x_1) - 0|) =$$
$$|f(u - x_1) - f(0)| = |h(y_1) - 0| = |h(y_1)|,$$

i.e. $h(y_1) \in A$. Since h is its own inverse, h is a surjective isometry from from A onto A.

Now, we shall prove the following equality

(5) $f(2x) = 2f(x)$ $(x \in G_1)$. Since by Theorem 2 h fixes a and

$$f(x) = a = h(a) = f(u - f^{-1}(a)) = f(u - f^{-1}(f(x)))$$

(we shall use also that $u = f^{-1}(2f(x))$ we obtain

$$|f(2x) - 2f(x)| = |f(2x) - f(u)| = p(|2x - u|) =$$
$$p(|x - (u - x)|) = |f(x) - f(u - x)| = 0.$$

Hence (5).

We define for an arbitrary but fixed $y \in G_1$

$$F_y(x) = f(x+y) - f(y) \quad (x \in G_1).$$

It is obvious that $F_y(0) = 0$ and that F_y is surjective. Since

$$|F_y(x_1) - F_y(x_2)| = |f(x_1+y) - f(x_2+y)| =$$

$$p\left(|(x_1+y) - (x_2+y)|\right) = p\left(|x_1 - x_2|\right).$$

F_y preserves the equality of distance. By Proposition 2 there exists a root function γ_2 of order 2 on G_1. Hence, by (5) it holds that

$$F_y(z) = 2F_y(\gamma_2(z)) \ \ (z \in G_1).$$

This equality implies that for any x and y from G_1 it holds that

$$f((x-y)+y) - f(y) = F_y(x-y) = 2F_y(\gamma_2(x-y)) =$$

$$2[f(\gamma_2(x-y)+y) - f(y)].$$

Hence,

$$f(x) + f(y) = 2f(\gamma_2(x+y)) = f(x+y).$$

References

[1] Aronszajn,N., Caractérisation métrique de l'espace de Hilbert, des espaces vectoriels et de certains groupes metriques, Comptes Rendus Acad. Sci. Paris 201, 1935, 811–813.

[2] Fuchs,L., Partially ordered algebraic systems, Pergamon Press, 1963.

[3] Holland,W.C., Intrinsic metrics for lattice ordered groups, Algebra Univ. 19, 1984, 142–150.

[4] Jakubik, J., Isometries of lattice ordered groups, Czechoslovak Mat.J.30,1980,142–152.

[5] Jakubik, J., On isometries of non -Abelian lattice ordered group, Math.Slovaca 31,1981,171–175.

[6] Jakubik, J. Weak isometries of lattice ordered groups, Math.Slovaca 38, 1988, 133–138.

[7] Mazur,S. Ulam,S., Sur les transformations isométriques d'espaces vectoriels normés,Comptes Rendus Acad. Sci. Paris 194, 1932, 946-948.

[8] Pap, E., n-convex functions on a semigroup with a root function, Univ. u Novom Sadu Zb. Rad. Prirod. - Mat. Fak. Ser. Mat. 6, 1976, 7–13.

[9] Swamy, K.L.N., Autometrized lattice ordered groups I,Math. Annalen 154, 1964, 406-412.

[10] Swamy,K.L.N.,Isometies in autometrized lattice ordered groups, Algebra Univ. 8, 1978, 59-64.

[11] Swamy,K.L.N.,Isometries in autometrized lattice ordered groups II, Math. Seminar Notes, Kobe Univ. 5, 1977, 211–214.

[12] Vogt,A., Maps which preserve equality of distance, Studia Math. 45, 1973, 43–48.

REZIME

PRESLIKAVANJA KOJA OČUVAVAJU RASTOJANJE NAD ABELOVIM MREŽASTIM GRUPAMA

U radu je dokazano da je svako preslikavanje f sa komutativne mrežaste grupe G_1 na Arhimedovsku komutativnu mrežastu grupu G_2 sa osobinama da $|f(x) - f(y)|$ funkcionalno zavisi od $|x - y|$ i $f(0) = 0$ uvek aditivno.

Received by the editors June 19, 1989.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 83-87

Review of Research
Faculty of Science
Mathematics Series

CONFORMAL DIFFEOMORPHISM BETWEEN TWO f – MANIFOLDS

Jovanka Nikić
Faculty of Technical Sciences, University of Novi Sad
Trg Dositeja Obradovića 6, 21000 Novi Sad, Yugoslavia

Abstract

f-manifolds have been studied by various authors including Blair D. [1], Endo H. [2], Yano K. [5] and the integrability conditions of this structure [6]. Singh K. and Srivastava R. in [4] prove a number of theorems involving the fundamental tensor for almost Hermitian manifolds with torsion and study a torsion preserving conformal diffeomorphism.

The paper studies the conformal diffeomorphism between two Riemannian f-manifolds and the Nijenhuis tensor on such manifolds. The result is obtained: The structure f' on $\mathcal{M}'^n$ is integrable iff and only iff the structure f on $\mathcal{M}^n$ is integrable, where $\mathcal{M}^n$ and $\mathcal{M}'^n$ are conformal diffeomorphically Riemannian f-manifolds with the structure tensors f and f', respectively.

AMS Mathematics Subject Classification (1980): 53C10, 53C15, 53C40, 51H20
Key words and phrases: $f-$manifolds, conformal diffeomorphism, Nijenhuis tensor.

1. Introduction

Let $\mathcal{M}^n$ be a C^∞ real differentiable manifold, $\mathcal{R}(\mathcal{M}^n)$ the ring of real valued differentiable functions over $\mathcal{M}^n$, $\mathcal{H}(\mathcal{M}^n)$ the module of derivatives of

$\mathcal{R}(\mathcal{M}^n)$. Then $\mathcal{H}(\mathcal{M}^n)$ is a Lie algebra over the real numbers and elements of $\mathcal{H}(\mathcal{M}^n)$ are called vector fields.

A C^∞ $n-$dimensional differentiable manifold is called an $f-$manifold iff there exists a non-null tensor field f of type $(1,1)$, of constant rank r, $r \le n$, such that $f^3 + f = 0$. The $(1,1)$ tensor fields ℓ and m defined by

$$\ell = -f^2, \ m = I + f^2$$

are complementary projection operators, where I denotes the identity operator. It is easily seen that $\ell^2 = \ell, \ m^2 = m, \ \ell m = m\ell = 0, \ \ell + m = I$.

Let L and M be complementary distributions corresponding to ℓ and m respectively. $Dim\ L = r$, $dim\ M = n - r$. Then:

$$\begin{aligned} \ell f = f\ell = f, &\qquad fm = mf = 0, \\ f^2\ell = \ell f^2 = -\ell, &\quad f^2 m = mf^2 = 0. \end{aligned}$$

It can be seen from the above relations that f acts as an almost complex structure on L and as a null operator on M.

An $f-$manifold $\mathcal{M}^n$ always admits a positive definite Riemannian metric g such that

$$g(X,Y) = g(fX,fY) + g(mX,Y). \tag{1.1}$$

The above metric satisfies the following relations:

$$g(mX,Y) = g(X,mY), \ \ g(X,mY) = g(mX,mY),$$

$$g(fX,Y) = g(f^2X,fY), \ \ g(X,fY) = g(fX,f^2Y).$$

Let ∇ be the Riemannian connexion with respect to the metric g on $\mathcal{M}^n$. Thus,

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{1.2}$$

$$[X,Y] = \nabla_X Y - \nabla_Y X.$$

Let $\mathcal{N}$ denote the Nijenhuis tensor of f i.e.

$$\begin{aligned} [f,f](X,Y) \ &= \ \mathcal{N}(X,Y) = \\ = [fX,fY] - f[fX,Y] \ &- \ f[X,fY] + f^2[X,Y]. \end{aligned} \tag{1.3}$$

2. Conformal diffeomorphism between two f−manifolds

Let $\mathcal{M}^n$ and $\mathcal{M}'^n$ be two manifolds with structure tensors f and $f'(f^3+f=0,\ f'^3+f'=0)$, and admitting the Riemannian metric g and g' respectively as in (1.1).

Let $\phi : \mathcal{M}^n \to \mathcal{M}'^n$ be a diffeomorphism. For $X \in \mathcal{H}(\mathcal{M}^n)$, let $X' \equiv \phi_* X$ where ϕ_* is the Jacobian map or the differential of ϕ [3]. Then, ϕ is called a conformal diffeomorphism, if there exists some real valued function $\sigma \in \mathcal{R}(\mathcal{M}^n)$ such that

$$g'(X',Y') \circ \phi = e^{2\sigma} g(X,Y)\ ;\ X,Y \in \mathcal{H}(\mathcal{M}^n).$$

For a real valued function φ, $grad\varphi$ is defined by [4] in the following way

$$g(grad\varphi, X) = X(\varphi);\quad X \in \mathcal{H}(\mathcal{M}^n).$$

We shall be using the following Lemma [4].

Lemma 1. *Let $\phi : \mathcal{M}^n \to \mathcal{M}'^n$ be a conformal diffeomorphism on two C^∞ manifolds $\mathcal{M}^n$ and $\mathcal{M}'^n$ have ∇ and ∇' as Riemannian connexions. Then*

$$\nabla'_{X'}Y' = \{\nabla_X Y + X(\sigma)Y + Y(\sigma)X - g(X,Y)grad\sigma\}'. \tag{2.1}$$

Let us assume that ϕ preserves the f−structure i.e. $f'X' = (fX)'$.

Theorem 1.

$$\begin{aligned}(\nabla'_{X'}f')(Y') \;=\; &\{(\nabla_X f)(Y) + (fY)(\sigma)Y - Y(\sigma)(fX) + \\ &+g(fX,Y)grad\sigma + g(X,Y)f\ grad\sigma\}'.\end{aligned}$$

Proof. From (2.1), we have

$$(\nabla'_{X'}f')(Y') = \nabla'_{X'}(f'Y') - f'(\nabla'_{X'}Y') =$$

$$\begin{aligned}&= \{\nabla_X(fY) + X(\sigma)fY + fY(\sigma)X - g(X,fY)grad\sigma\}' - \\ &\quad - f'((\nabla_X Y) + X(\sigma)Y + Y(\sigma)X - g(X,Y)grad\sigma)' = \\ &= ((\nabla_X f)(Y) + fY(\sigma)X - Y(\sigma)fX + g(fX,Y)grad\sigma + g(X,Y)fgrad\sigma)'\end{aligned}$$

which proves the theorem. □

Theorem 2. *Let $\mathcal{N}$ and $\mathcal{N}'$ be Nijenhuis tensors, of f on $\mathcal{M}^n$ and of f' on $\mathcal{M}'^n$ respectively. Then we have*

$$(\mathcal{N}(X,Y))' = \mathcal{N}'(X',Y').$$

Proof. From (1.3) and the previous theorem we have:

$$\begin{aligned}
\mathcal{N}'(X',Y') &= [f'X',f'Y'] - f'[X',f'Y'] - f'[f'X',Y'] + f'^2[X',Y'] = \\
&= \nabla'_{(f'X')}(f'Y') - \nabla'_{(f'Y')}(f'X') - f'\nabla'_{X'}(f'Y') + f'\nabla'_{(f'Y')}X' - \\
&\quad - f'\nabla'_{(f'X')}Y' + f'\nabla'_{Y'}(f'X') + f'^2\nabla'_{X'}Y' - f'^2\nabla'_{Y'}X' = \\
&= (\nabla'_{f'X'}f')(Y') - (\nabla'_{f'Y'}f')(X') - f'(\nabla'_{X'}f')(Y') + f'(\nabla'_{Y'}f')(X') = \\
&= ((\nabla_{fX}f)(Y) + fY(\sigma)fX - Y(\sigma)f^2X + g(f^2X,Y)grad\sigma + \\
&\quad + g(fX,Y)fgrad\sigma - (\nabla_{fY}f)(X) - fX(\sigma)fY + X(\sigma)f^2Y - \\
&\quad - g(f^2Y,X)grad\sigma - g(fY,X)fgrad\sigma - f(\nabla_X f)(Y) - fY(\sigma)fX + \\
&\quad + Y(\sigma)f^2X - g(fX,Y)fgrad\sigma - g(X,Y)f^2grad\sigma + f(\nabla_Y f)(X) + \\
&\quad + fX(\sigma)fY - X(\sigma)f^2Y + g(fY,X)fgrad\sigma + g(Y,X)f^2grad\sigma)' = \\
&= (\mathcal{N}(X,Y))',
\end{aligned}$$

which proves the theorem. □

The following result [6] is very well known. The structure f is integrable iff and only iff $[f,f] = 0$.

The consequence of Theorem 2 is the following theorem:

Theorem 3. *Let $\phi : \mathcal{M}^n \to \mathcal{M}'^n$ be a structure preserving a conformal diffeomorphism between two $f-$manifolds $\mathcal{M}^n$ and $\mathcal{M}'^n$. The structure f' on $\mathcal{M}'^n$ is integrable iff and only iff the structure f on $\mathcal{M}^n$ is integrable.*

References

[1] Blair, D.E., Contact Manifolds in Riemannian Geometry, Lecture Notes in Mathematics 509, Springer Verlag 1976.

[2] Endo, H., Invariant Submanifolds in a Contact Riemannian Manifold, Tensor, N.S. Vol. 42, 1985, 86-89.

[3] Hicks, N.Y., Notes on Differential Geometry, D. Van Nostrand Company, INC., Princeton, New Jersey, 1965.

[4] Singh, K.D. and Srivastava, R.C., Almost Hermitian Manifolds with Torsion, Matematički vesnik, 8(23), 1971. 125-134.

[5] Yano, K., On structure defined by a tensor field f of type $(1,1)$ satisfying $f^3 + f = 0$, Tensor, N.S. 14, 1963, 99-109.

[6] Yano, K., Ishihare S.: On integrability conditions of a structure f satisfying $f^3 + f = 0$, Quart. J. Math. Oxford, 15(1964) 217-222.

REZIME

KONFORMNO DIFEOMORFNO PRESLIKAVANJE IZMEDJU DVE f–MNOGOSTRUKOSTI

U radu je ispitivano konformno difeomorfno preslikavanje izmedju dve f–-mnogostrukosti i veza izmedju Nijenhuisovih tenzora takvih mnogostrukosti.

Struktura f' na mnogostrukosti $\mathcal{M}'^n$ je integrabilna ako i samo ako je struktura f na $\mathcal{M}^n$ integrabilna, gde su $\mathcal{M}^n$, $\mathcal{M}'^n$ konformno difeomorfne Rimanove f–mnogostrukosti sa tenzorima strukture f i f' respektivno.

Received by the editors June 7, 1990.

[3] Hicks, N.J., Notes on Differential Geometry, D. Van Nostrand Company, INC., Princeton, New Jersey, 1965.

[4] Singh, K.D. and Srivastava, R.C., Almost-Hermitian Manifolds with Torsion, Matematički vesnik, 8(23), 1971, 125-134.

[5] Yano, K., On structure defined by a tensor field f of type $(1,1)$ satisfying $f^3 + f = 0$, Tensor, N.S. 14, 1963, 99-109.

[6] Yano, K., Ishihara S., On integrability conditions of a structure f satisfying $f^3 + f = 0$, Quart. J. Math. Oxford, 15(1964) 217-222.

REZIME

KONFORMNO DIFEOMORFNO PRESLIKAVANJE IZMEĐU DVE f–MNOGOSTRUKOSTI

U radu je ispitivano konformno difeomorfno preslikavanje između dve f–mnogostrukosti i veza između Nijenhuisovih tenzora takvih mnogostrukosti.

Struktura f' na mnogostrukosti M'^n je integrabilna ako i samo ako je struktura f na M^n integrabilna, gde su M^n, M'^n konformno difeomorfne Rimanove f–mnogostrukosti sa tenzorima strukture f i f' respektivno.

Received by the editors June 7, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 89-95

Review of Research
Faculty of Science
Mathematics Series

A COMMON FIXED POINT THEOREM FOR A FAMILY OF MAPPINGS IN CONVEX METRIC SPACES

Olga Hadžić [1]
Institute of Mathematics, University of Novi Sad,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In this paper a generalization of the common fixed point theorem from [2] is proved.

AMS Mathematics Subject Classification (1980): 47H10
Key words and phrases: Common fixed points,convex metric spaces.

1. Introduction

There are many fixed point theorems or common fixed point theorems in convex metric spaces ([1], [2], [3], [4], [7], [8]).

In [2] we proved the following common fixed point theorem.

Theorem 1. *Let (M,d) be a complete, convex metric space, K a nonempty, closed subset of M, $f,S,T : K \to M$ continuous mappings so that $\partial K \subseteq SK \cap TK$, $f(K) \cap K \subseteq SK \cap TK$ and*

$$Tx \in \partial K \Rightarrow f(x) \in K; \ \ Sx \in \partial K \Rightarrow fx \in K.$$

---

[1] This research was supported by Science Fund of Serbia, grant number 0401A, through Matematički institut

If (f,S) *and* (f,T) *are weakly commutative and there exists a nondecreasing function* $q:[0,\infty)\to[0,1)$ *such that*

$$d(fx,fy)\le q(d(Sx,Ty))d(Sx,Ty), \text{for every } x,y\in K$$

then there exists $z\in K$ *so that*

$$z=fz\in\{Tz,Sz\}.$$

If $S,T:M\to M$ *then there exists one and only one* $z\in K$ *such that* $z=fz=Tz=Sz$.

In this paper we shall prove a generalization of Theorem 1 if $S,T:M\to M$.

The notion of a weakly commutative pair of mappings is introduced by Sessa in [6] and the notion of a compatible pair of mappings by Jungck in [5]. There are examples of compatible pairs which are not weakly commutative and weakly commutative pairs which are not commutative.

The next definition is a slight modification of the Jungck definition.

Definition 1. *Let* (M,d) *be a metric space,* $A:\mathcal{D}(A)\to M, S:\mathcal{D}(S)\to M$, $\mathcal{D}(A)\subseteq M,\mathcal{D}(S)\subseteq M$. *The pair* (A,S) *is said to be compatible if for every sequence* $\{x_n\}_{n\in\mathbf{N}}$ *from* $\mathcal{D}(A)\cap\mathcal{D}(S)$ *such that* $\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Sx_n=t\in M$ *and* $Ax_n\in\mathcal{D}(S), Sx_n\in\mathcal{D}(A)$, *for every* $n\in\mathbf{N}$ *the relation*

$$\lim_{n\to\infty}d(ASx_n,SAx_n)=0$$

holds.

2. A common fixed point theorem

The following fixed point theorem is a generalization of Theorem 1 if $S,T:M\to M$.

Theorem 2. *Let* (M,d) *be a complete, convex metric space,* K *a nonempty closed subset of* $M, S,T:M\to M$ *continuous mappings so that* $\partial K\subseteq SK\cap TK$, *for every* $i\in\mathbf{N}$, $A_i:K\to M$ *continuous mappings such that*

$A_iK \cap K \subseteq SK \cap TK$, (A_i, S) and (A_i, T) compatible pairs and there exists a nondecreasing function $q : [0,\infty) \to [0,1)$ such that

$$d(A_ix, A_jy) \le q(d(Sx,Ty))d(Sx,Ty),$$

for every $i \ne j (i,j \in \mathbf{N})$ and every $x, y \in K$.

If for every $i \in \mathbf{N}$ and $x \in K$ the implications

$$Tx \in \partial K \Rightarrow A_ix \in K; \;\; Sx \in \partial K \Rightarrow A_ix \in K$$

hold, then there exists $z \in K$ such that

$$z = Tz = Sz = A_iz, \textit{ for every } \; i \in \mathbf{N}.$$

Proof . Let $x \in \partial K$ and $p_0 \in K$ so that $x = Tp_0$. Then for every $i \in \mathbf{N}$, $A_ip_0 \in K$ and so $A_1p_0 \in SK$ which implies that there exists $p_1 \in K$ such that $Sp_1 = A_1p_0 \in K$. Let $p'_1 = A_1p_0, p'_2 = A_2p_1$. If $p'_2 \in K$ then there exists $p_2 \in K$ so that $Tp_2 = A_2p_1$ and if $p'_2 \notin K$, since M is a convex metric space, there exists $p_2 \in K$ such that $Tp_2 \in \partial K$ and

$$d(Sp_1, Tp_2) + d(Tp_2, A_2p_1) = d(Sp_1, A_2p_1).$$

If we continue this process we obtain sequences $\{p_n\}_{n\in\mathbf{N}}$ and $\{p'_n\}_{n\in\mathbf{N}}$ such that for every $n \in \mathbf{N}$, $p_n \in K$, $p'_{n+1} = A_{n+1}p_n$ and the following implications hold:

(i) $p'_{2n} \in K \Rightarrow p'_{2n} = Tp_{2n}$,
$p'_{2n} \notin K \Rightarrow Tp_{2n} \in \partial K$ and

$$d(Sp_{2n-1}, Tp_{2n}) + d(Tp_{2n}, A_{2n}p_{2n-1}) =$$

$$= d(Sp_{2n-1}, A_{2n}p_{2n-1})$$

(ii) $p'_{2n+1} \in K \Rightarrow p'_{2n+1} = Sp_{2n+1}$,
$p'_{2n+1} \notin K \Rightarrow Sp_{2n+1} \in \partial K$ and

$$d(Tp_{2n}, Sp_{2n+1}) + d(Sp_{2n+1}, A_{2n+1}p_{2n}) = d(Tp_{2n}, A_{2n+1}p_{2n}).$$

Let us prove that there exists $z \in K$ such that $z = \lim_{n\to\infty} Tp_{2n} = \lim_{n\to\infty} Sp_{2n+1}$. The sets P_0, P_1, Q_0, Q_1 will be defined in the following way:

$$P_0 = \{p_{2n}; p'_{2n} = Tp_{2n},\; n \in \mathbf{N}\},$$

$$P_1 = \{p_{2n}; p'_{2n} \neq Tp_{2n},\ n \in \mathbf{N}\},$$

$$Q_0 = \{p_{2n+1}; p'_{2n+1} = Sp_{2n+1},\ n \in \mathbf{N}\},$$

$$Q_1 = \{p_{2n+1}; p'_{2n+1} \neq Sp_{2n+1},\ n \in \mathbf{N}\}.$$

It is easy to see that we have the following possibilities:

$$(p_{2n}, p_{2n+1}) \in P_0 \times Q_0; (p_{2n}, p_{2n+1}) \in P_0 \times Q_1;$$

$$(p_{2n}, p_{2n+1}) \in P_1 \times Q_0.$$

a) $(p_{2n}, p_{2n+1}) \in P_0 \times Q_0$.
Then

$$d(Tp_{2n}, Sp_{2n+1}) = d(A_{2n}p_{2n-1}, A_{2n+1}p_{2n}) \leq$$
$$\leq q[d(Sp_{2n-1}, Tp_{2n})]d(Sp_{2n-1}, Tp_{2n}).$$

b) $(p_{2n}, p_{2n+1}) \in P_0 \times Q_1$.
Then

$$d(Tp_{2n}, Sp_{2n+1}) = d(Tp_{2n}, A_{2n+1}p_{2n}) - d(Sp_{2n+1}, A_{2n+1}p_{2n})$$
$$\leq d(Tp_{2n}, A_{2n+1}p_{2n}) = d(A_{2n}p_{2n-1}, A_{2n+1}p_{2n}) \leq$$
$$\leq q[d(Sp_{2n-1}, Tp_{2n})]d(Sp_{2n-1}, Tp_{2n}).$$

c) $(p_{2n}, p_{2n+1}) \in P_1 \times Q_0$.
Then

$$d(Tp_{2n}, Sp_{2n+1}) \leq d(Tp_{2n}, A_{2n}p_{2n-1}) + d(A_{2n}p_{2n-1}, Sp_{2n+1})$$
$$= d(Tp_{2n}, A_{2n}p_{2n-1}) + d(A_{2n}p_{2n-1}, A_{2n+1}p_{2n}) \leq$$
$$\leq d(Tp_{2n}, A_{2n}p_{2n-1}) + q[d(Sp_{2n-1}, Tp_{2n})]d(Sp_{2n-1}, Tp_{2n})$$
$$\leq d(Sp_{2n-1}, Tp_{2n}) + d(Tp_{2n}, A_{2n}p_{2n-1}) =$$
$$= d(Sp_{2n-1}, A_{2n}p_{2n-1}).$$

Since $p_{2n} \in P_1$ implies that $p_{2n-1} \in Q_0$ we have that $Sp_{2n-1} = A_{2n-1}p_{2n-2}$ and so

$$d(Tp_{2n}, Sp_{2n+1}) \leq d(Sp_{2n-1}, A_{2n}p_{2n-1}) =$$
$$= d(A_{2n-1}p_{2n-2}, A_{2n}p_{2n-1}) \leq q[d(Tp_{2n-2}, Sp_{2n-1})].$$
$$.d(Tp_{2n-2}, Sp_{2n-1}).$$

It can be proved similarly that the following implications hold:

$$(p_{2n-1}, p_{2n}) \in Q_0 \times P_0 \Rightarrow d(Sp_{2n-1}, Tp_{2n}) \leq$$

$$\leq q[d(Tp_{2n-2}, Sp_{2n-1})].d(Tp_{2n-2}, Sp_{2n-1});$$

$$(p_{2n-1}, p_{2n}) \in Q_1 \times P_0 \Rightarrow d(Sp_{2n-1}, Tp_{2n}) \leq$$

$$q[d(Tp_{2n-2}, Sp_{2n-3})]d(Tp_{2n-2}, Sp_{2n-3});$$

$$(p_{2n-1}, p_{2n}) \in Q_0 \times P_1 \Rightarrow d(Sp_{2n-1}, Tp_{2n}) \leq$$

$$\leq q[d(Tp_{2n-2}, Sp_{2n-1})]d(Tp_{2n-2}, Sp_{2n-1}).$$

It is easy to prove that

$$d(Tp_{2n}, Sp_{2n+1}) \leq [q(\delta)]^{n-1}.\delta, \tag{1}$$

$$d(Sp_{2n+1}, Tp_{2n+2}) \leq [q(\delta)]^{n}\delta, \tag{2}$$

where $\delta = \max\{d(Tp_2, Sp_3), d(Tp_2, Sp_1)\}$.

Since $q(\delta) < 1$, (1) and (2) imply that sequences $\{Tp_{2n}\}_{n\in \mathbf{N}}$ and $\{Sp_{2n+1}\}_{n\in \mathbf{N}}$ are Cauchy sequences in K. Since M is complete there exists $z \in K$ so that $z = \lim_{n\to\infty} Tp_{2n} = \lim_{n\to\infty} Sp_{2n+1}$. Further, there exists at least one subsequence $\{Tp_{2n_k}\}_{k\in \mathbf{N}}$ or $\{Sp_{2m_k+1}\}_{k\in \mathbf{N}}$ such that for every $k \in \mathbf{N}$, $p_{2n_k} \in P_0$ or $p_{2m_k+1} \in Q_0$. Hence, let us suppose that $p_{2n_k} \in P_0$, $k \in \mathbf{N}$. From the definition of the set P_0 it follows that $Tp_{2n_k} = A_{2n_k} p_{2n_k-1}$, $k \in \mathbf{N}$.

We shall prove that for every $m \in \mathbf{N}$

$$d(z, A_m z) \leq q(L) d(z, Tz) \tag{3}$$

for a positive number L. Since $\lim_{k\to\infty} d(Sp_{2n_k-1}, Tz) = d(z, Tz)$, there exists $L > 0$ such that

$$d(Sp_{2n_k-1}, Tz) \leq L, \quad k \in \mathbf{N}.$$

Since $Tp_{2n_k} = A_{2n_k} p_{2n_k-1}$, $k \in \mathbf{N}$ we have for $2n_k \neq m$:

$$d(Tp_{2n_k}, A_m z) = d(A_{2n_k} p_{2n_k-1}, A_m z) \leq q[d(Sp_{2n_k-1}, Tz)]\times$$

$$\times d(Sp_{2n_k-1}, Tz) \leq q(L) d(Sp_{2n_k-1}, Tz).$$

When $k \to \infty$ we obtain that (3) holds. We shall prove that $\lim_{k\to\infty} A_m p_{2n_k} = z$. Suppose that $2n_k \neq m$.
Then

$$d(A_m p_{2n_k}, Tp_{2n_k}) = d(A_m p_{2n_k}, A_{2n_k} p_{2n_k-1}) \leq$$
$$\leq q(L')d(Tp_{2n_k}, Sp_{2n_k-1})$$

where $d(Tp_{2n_k}, Sp_{2n_k-1}) \leq L'$, $k \in \mathbf{N}$. Hence $\lim_{k\to\infty} d(A_m p_{2n_k}, Tp_{2n_k}) = 0$ which implies that $\lim_{k\to\infty} A_m p_{2n_k} = z$. Since $Tp_{2n_k} \in K$ and $T : M \to M$ from the relation $\lim_{k\to\infty} d(A_m p_{2n_k}, Tp_{2n_k}) = 0$ it follows, using the compatibility of A_m and T, that

$$\lim_{k\to\infty} d(T(A_m p_{2n_k}), A_m(Tp_{2n_k})) = 0.$$

This implies that $Tz = A_m z$, for every $m \in \mathbf{N}$ since T and A_m are continuous. From (3) we obtain that $d(z, A_m z) \leq q(L)d(z, A_m z)$, for every $m \in \mathbf{N}$ and since $q(L) < 1$ we have that $z = A_m z$. It remains to be proved that $z = Sz$. Suppose that $s \neq m$. Then we have for every $k \in \mathbf{N}$ that

$$d(A_s p_{2n_k-1}, A_m p_{2n_k}) \leq q(L')d(Sp_{2n_k-1}, Tp_{2n_k})$$

and since $\lim_{k\to\infty} d(Sp_{2n_k-1}, Tp_{2n_k}) = 0$ we obtain that $\lim_{k\to\infty} A_s p_{2n_k-1} = z$, for every $s \in \mathbf{N}$. We shall prove that $\lim_{k\to\infty} A_m Sp_{2n_k-1} = Sz$. Since (A_m, S) are compatible and $\lim_{k\to\infty} Sp_{2n_k-1} = \lim_{k\to\infty} A_m p_{2n_k-1} = z$ it follows that

$$\lim_{k\to\infty} d(A_m Sp_{2n_k-1}, SA_m p_{2n_k-1}) = 0.$$

Since S is continuous and $\lim_{k\to\infty} A_m p_{2n_k-1} = z$ we obtain that $\lim_{k\to\infty} A_m Sp_{2n_k-1} = Sz$. Further, we have that for $m \neq s$ and every $k \in \mathbf{N}$

$$d(A_s p_{2n_k}, A_m Sp_{2n_k-1}) \leq q(L")d(Tp_{2n_k}, SSp_{2n_k-1}))$$

which implies, when $k \to \infty$ that $d(z, Sz) \leq q(L")d(z, Sz)$. Here $L"$ is such that $d(Tp_{2n_k}, S(Sp_{2n_k-1})) \leq L"$ for every $k \in \mathbf{N}$. Since $q(L") < 1$ it follows that $z = Sz$ and so

$$z = Tz = Sz = A_m z, \text{ for every } m \in \mathbf{N}.$$

If z is in K then it is enough to suppose that $T, S : K \to M$ since from the relations $\lim_{k\to\infty} A_m p_{2n_k-1} = \lim_{k\to\infty} A_m p_{2n_k} = z$ it follows that there exists $k_0 \in \mathbf{N}$ such that

$$A_m p_{2n_k-1} \in K, \;\; A_m p_{2n_k} \in K, \;\; k \geq k_0$$

and every $m \in \mathbf{N}$.

References

[1] Assad, N.A., Kirk, W.A.: Fixed point theorems for set-valued mappings of contractive type, Pacific J.Math., Vol. 43, No. 3(1972), 553-562.

[2] Hadžić,O.: Some common fixed point theorems in convex metric spaces, Univ. u Novom Sadu, Zb. Rad. Prirod.- Mat.Fak.Ser.Mat., 15, 2(1985), 1-13.

[3] Hadžić,O.: On coincidence points in convex metric spaces, Univ. u Novom Sadu, Zb. Rad. Prirod.- Mat.Fak.Ser.Mat., 19, 2 (1989), 233-240.

[4] Itoh,S.: Multivalued generalized contractions and fixed point theorems, Comm.Math. Univ. Carolinae, 18(2)(1977), 247-258.

[5] Jungck,G.: Compatible mappings and common fixed points, Internat. J.Math. Math.Sci., 9(4)(1986), 771-779.

[6] Sessa,S.: On a weak commutativity condition of mappings in fixed point consideration, Publ.Inst.Math. (Beograd), 32(46)(1982), 147-153.

[7] Takahashi,W.: A convexity in metric space and nonexpansive mappings I, Kodai Math.Sem.Rep., 22(1970), 142-149.

[8] Talman,L.: Fixed points for condensing multifunctions in metric spaces with convex structure, Kodai Math. Sem. Rep., 29(1977), 62-70.

REZIME

TEOREMA O ZAJEDNIČKOJ NEPOKRETNOJ TAČKI ZA FAMILIJU PRESLIKAVANJA U KONVEKSNIM METRIČKIM PROSTORIMA

U ovom radu dokazano je uopštenje teoreme o zajedničkoj nepokretnoj tački iz rada [2].

Received by the editors August 15, 1990.

References

[1] Assad, N.A., Kirk, W.A.: Fixed point theorems for set-valued mappings of contractive type, Pacific J.Math., Vol. 43, No. 3(1972), 553-562.

[2] Hadžić, O.: Some common fixed point theorems in convex metric spaces, Univ. u Novom Sadu, Zb. Rad. Prirod.- Mat.Fak.Ser.Mat. 15, 2(1985), 1-13.

[3] Hadžić, O.: On coincidence points in convex metric spaces, Univ. u Novom Sadu, Zb. Rad. Prirod.- Mat.Fak.Ser.Mat., 19, 2 (1989), 233-240.

[4] Itoh, S.: Multivalued generalized contractions and fixed point theorems, Comm. Math. Univ. Carolinae, 18(2)(1977), 247-258.

[5] Jungck, G.: Compatible mappings and common fixed points, Internat. J.Math. Math. Sci. 9(4)(1986), 771-779.

[6] Sessa, S.: On a weak commutativity condition of mappings in fixed point considerations, Publ.Inst. Math. (Beograd), 32(46)(1982), 149-153.

[7] Takahashi, W.: A convexity in metric space and nonexpansive mappings I. Kodai Math. Sem.Rep., 22(1970), 142-149.

[8] Tallman, L.: Fixed points for condensing multifunctions in metric spaces with convex structure, Kodai Math. Sem. Rep., 29(1977), 62-70.

REZIME

TEOREMA O ZAJEDNIČKOJ NEPOKRETNOJ TAČKI ZA FAMILIJE PRESLIKAVANJA U KONVEKSNIM METRIČKIM PROSTORIMA

U ovom radu dokazano je uopštenje teoreme o zajedničkoj nepokretnoj tački iz rada [3].

Received by the editors August 15, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 97-105

Review of Research
Faculty of Science
Mathematics Series

A FIXED POINT THEOREM FOR NONEXPANSIVE MAPPINGS IN RANDOM NORMED SPACES

Olga Hadžić
Institute of Mathematics, University of Novi Sad,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In this paper a probabilistic generalization of Göhde's result from [2] is proved.

AMS Mathematics Subject Classification (1980): 47H10
Key words and phrases: Fixed points, probabilistic metric spaces.

1. Introduction

The notion of a random normed space was introduced by A.N.Sherstnev [7] and many fixed point theorems in random normed spaces are obtained ([3], [4], [5]). The aim of this paper is to prove a generalization of the well known result of Göhde about the existence of a fixed point of a nonexpansive mapping in Banach spaces which has a compact iterate.

2. Preliminaries

Let Δ denote the set of all distribution functions F such that $F(0) = 0$ (F is a nondecreasing, left continuous mapping from the set of real numbers $\mathbf{R}$ into $[0,1]$ so that $\sup_{x\in\mathbf{R}} F(x) = 1$).

A random normed space $(S, \mathcal{F}, t)$ is an ordered triple where S is a real or complex vector space, t is a T-norm which is stronger then the T-norm

$t_m (t \geq t_m$ and $t_m(a,b) = \max\{a+b-1, 0\})$ and the mapping $F : S \to \Delta$ satisfies the following conditions, where

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

a) $F_p = H \Leftrightarrow p = \theta (\theta$ is the neutral element of $S)$,

b) For every $p \in S$, every $u > 0$ and every $r \in K \backslash \{0\}$ (K is the scalar field): $F_{rp}(u) = F_p(\frac{u}{|r|})$.

c) For every $(p,q) \in S \times S$ and every $(u,v) \in (0,\infty) \times (0,\infty)$

$$F_{p-q}(u+v) \geq t(F_p(u), F_q(v)).$$

If the T-norm t is continuous then S is a topological vector space with the fundamental system of neighbourhoods of zero given by

$$U(\epsilon, \lambda) = \{u; u \in S, F_u(\epsilon) > 1 - \lambda\}.$$

If $(S, \mathcal{F}, t)$ is a random normed space and A a nonempty subset of S the probabilistic diameter of A is the function $D_A(.) : [0,\infty) \to [0,1]$ given by [1]

$$D_A(u) = \sup_{s<u} \inf_{p,q \in A} F_{p,q}(s), \; u \in [0,\infty).$$

A set $A \subseteq S$ is probabilistic bounded if and only if $\sup_{u \in [0,\infty)} D_A(u) = 1$. In [1] and [8] the notions of the Kuratowski and the Hausdorff functions of noncompactness are given.

Definition 1. *Let A be a probabilistic bounded subset of S. The Kuratowski function of noncompactness* $\alpha_A(.) : [0,\infty) \to [0,1]$ *is given by* $\alpha_A(u) = \sup\{s \mid s > 0$, *there is a finite family* $A_j (j \in J)$ *such that* $A = \cup_{j \in J} A_j$ *and* $D_{A_j}(u) \geq s$, *for every* $j \in J\}, u \in [0,\infty)$.

The Kuratowski function has the following properties:

1) $\alpha_A \in \Delta$,

2) $\alpha_A(u) \geq D_A(u)$, for every $u \in [0,\infty)$,

3) $\theta \neq A \subset B \subset S \Rightarrow \alpha_A(u) \geq \alpha_B(u)$, for every $u \in [0,\infty)$,

4) $\alpha_{A\cup B}(u) = \min\{\alpha_A(u), \alpha_B(u)\}$, for every $u \in [0,\infty)$,

5) $\alpha_A(u) = \alpha_{\bar{A}}(u)$, for every $u \in [0,\infty)$, where $\bar{A}$ is the closure of A,

6) $\alpha_A = H \Rightarrow A$ is precompact .

Definition 2. *Let A be a probabilistic bounded subset of S. The Hausdorff function of noncompactness $\beta_A(.) : [0,\infty) \to [0,1]$ is given by*

$\beta_A(u) = \sup\{r \mid r > 0$, *there exists a finite subset A_f of S such that $\tilde{F}_{A,A_f}(u) \geq r\}$ where for every two probabilistic bounded subsets A and B of S*

$$\tilde{F}_{A,B}(u) = \sup_{s<u} \inf_{x\in A} \sup_{y\in B} F_{x,y}(s),\ \ u \in [0,\infty)).$$

The function β has properties 1) - 6) (in which β stands instead of α).

Let $(S,\mathcal{F},t)$ be a random normed space, K a probabilistic bounded subset of S and $T : K \to S$. If $T(K)$ is probabilistic bounded subset of S and for every $B \subset K$ the following implication holds:

$$\gamma_{T(B)}(u) \leq \gamma_B(u),\ \text{for every}\ \ u > 0 \Rightarrow B\ \ \text{is precompact}$$

where $\gamma_A(.) \in \{\alpha_A(.), \beta_A(.)\}$, we say that the mapping T is γ densifying on the set K.

Definition 3. *Let X be a topological space, $F : X \to X$ and M a nonempty subset of X. The set M is said to be an attractor for F if for every $x \in X$*

$$\overline{\cup_{n\in\mathbf{N}} F^n(x)} \cap M \neq \emptyset.$$

3. A fixed point theorem

D.Göhde proved in [2] that every nonexpansive mapping $T : X \to X$, where X is a Banach space, with the property that $\overline{T^{n_0}(X)}$ is compact for some $n_0 \in \mathbf{N}$, has at least one fixed point.

The next theorem is a probabilistic generalization of Göhde's. **Theorem** *Let $(S,\mathcal{F},t)$ be a complete random normed space with a continuous T-norm t, A a nonempty, convex and probabilistic bounded subset of S and $T : A \to A$ so that the following conditions are satisfied:*

(i) For every $x, y \in A$ and every $s \in \mathbf{R}^+ = [0, \infty)$

$$F_{Tx-Ty}(s) \geq F_{x-y}(s).$$

(ii) There exists an attractor $M \subset A$ for the mapping T and the mapping T is densifying on M in respect to γ where $\gamma(.) \in \{\alpha(.), \beta(.)\}$. Then there exists $x \in A$ so that $x = Tx$. Proof. Let $x_0 \in A$ and $k_n \in (0,1)(n \in N)$ so that $\lim_{n\to\infty} k_n = 1$. For every $n \in N$ and every $x \in A$, let $T_n x = k_n Tx + (1-k_n)x_0$. Then $T_n x \in A$ and

$$F_{T_n x - T_n y}(k_n s) = F_{k_n Tx - k_n Ty}(k_n s) =$$

$$= F_{Tx-Ty}(s) \geq F_{x-y}(s)$$

for every $n \in N$, every $x, y \in A$ and every $s \in \mathbf{R}^+$. From [4] it follows that there exists $x_n \in A$ so that $x_n = T_n x_n$, for every $n \in \mathbf{N}$.

Further

$$F_{x_n - Tx_n}(s) = F_{k_n Tx_n + (1-k_n)x_0 - Tx_n}(s) =$$

$$= F_{(k_n-1)(Tx_n - x_0)}(s) = F_{Tx_n - x_0}\left(\frac{s}{1-k_n}\right).$$

Since A is probabilistic bounded it follows that for every $s \in \mathbf{R}^+$

$$\lim_{n\to\infty} F_{x_n - Tx_n}(s) = 1. \tag{1}$$

Since M is an attractor for T, it follows that for every $n \in \mathbf{N}$ there exists $y_n \in M$ so that

$$y_n \in \overline{\cup_{m\in\mathbf{N}} T^m x_n}. \tag{2}$$

Let us prove that $\lim_{n\to\infty} F_{y_n - Ty_n}(s) = 1$, for every $s \in \mathbf{R}^+$. Let $r \in (0,1)$ and $s > 0$. We shall prove that there exists $n(s,r) \in \mathbf{N}$ so that for every $n > n(s,r) : F_{y_n - Ty_n}(s) > 1 - r$. Since the mapping t is continuous there exists $u \in (0,1)$ so that:

$$1 \geq x, y, z > u \Rightarrow t(x, t(y,z)) > 1 - r.$$

Further, from (2) we conclude that for every $n \in \mathbf{N}$ there exists $m_n \in \mathbf{N}$ such that

$$F_{y_n - T^{m_n} x_n}\left(\frac{s}{3}\right) > u.$$

Since for every $(n,k) \in \mathbf{N} \times \mathbf{N}$:

$$F_{T^k x_n - T^{k+1} x_n}(s) \geq F_{x_n - Tx_n}(s)$$

we have that:

$$\begin{aligned} F_{y_n - Ty_n}(s) &\geq t(F_{y_n - T^{m_n} x_n}(\frac{s}{3}), \\ &t(F_{T^{m_n} x_n - T^{m_n+1} x_n}(\frac{s}{3}), F_{T^{m_n+1} x_n - Ty_n}(\frac{s}{3}))) \\ &\geq t(F_{y_n - T^{m_n} x_n}(\frac{s}{3}), t(F_{x_n - Tx_n}(\frac{s}{3}), F_{T^{m_n} x_n - y_n}(\frac{s}{3}))). \end{aligned}$$

Let $n(s,r) \in \mathbf{N}$ be such that for every $n > n(s,r)$

$$F_{x_n - Tx_n}(\frac{s}{3}) > u.$$

The existence of such a number $n(s,r)$ follows from (1) .

Then for every $n > n(s,r), F_{y_n - Ty_n}(s) > 1 - r$ which implies that

$$\lim_{n \to \infty} F_{y_n - Ty_n}(s) = 1.$$

We shall prove that the set $\overline{\{y_n \mid n \in \mathbf{N}\}}$ is compact. Since the mapping T is densifying on M in respect to γ is it enough to prove that for every $u \in \mathbf{R}^+$

(3) $$\gamma_{T[\{y_n \mid n \in \mathbf{N}\}]}(u) = \gamma_{\{y_n \mid n \in \mathbf{N}\}}(u).$$

First, we shall suppose that $\gamma = \beta$. In order to prove (3) we shall prove that

(4) $$\beta_{\{y_n \mid n \in \mathbf{N}\}}(u) \leq \beta_{T[\{y_n \mid n \in \mathbf{N}\}]}(u), \text{ for every } u \in \mathbf{R}^+$$

and

(5) $$\beta_{T[\{y_n \mid n \in \mathbf{N}\}]}(u) \leq \beta_{\{y_n \mid n \in \mathbf{N}\}}(u), \text{ for every } u \in \mathbf{R}^+.$$

Since the function β is left continuous we shall prove that for every $u > 0$ and every $s \in (0,u)$

(6) $$\beta_{\{y_k \mid k \in \mathbf{N}\}}(u - s) \leq \beta_{\{Ty_k \mid k \in \mathbf{N}\}}(u).$$

Then from (6) we obtain (4). If $\beta_{\{y_k \mid k \in \mathbf{N}\}}(u - s) = 0$ then (6) holds and hence we shall suppose that $\beta_{\{y_k \mid k \in \mathbf{N}\}}(u - s) > 0$. In order to prove (6) we shall prove the implication

(7) $$0 < r < \beta_{\{y_k \mid k \in \mathbf{N}\}}(u - s) \Rightarrow r \leq \beta_{\{Ty_k \mid k \in \mathbf{N}\}}(u).$$

From the inequality $r < \beta_{\{y_k|k\in \mathbf{N}\}}(u-s)$ and the definition of the function β it follows that there exists a finite subset A_f of S such that

$$\tilde{F}_{\{y_n|n\in \mathbf{N}\},A_f}(u-s) > r.$$

So, for every $n \in \mathbf{N}$ there exists $z(n) \in A_f$ such that $F_{y_n,z(n)}(u-s) > r$. We shall prove that for every $\delta_1 \in (0,r)$

$$\beta_{\{Ty_k|k\in \mathbf{N}\}}(u) \geq r - \delta_1. \tag{8}$$

Since $t(1,r) = r$ from the continuity of the mapping t it follows that there exists $\delta_2 \in (0,1)$ so that

$$1 \geq h > 1 - \delta_2 \Rightarrow t(h,r) > r - \delta_1.$$

From the relation $\lim_{n\to\infty} F_{y_n - Ty_n}(s) = 1$, for every $s > 0$ it follows that there exists $n(s,\delta_2) \in \mathbf{N}$ such that $F_{y_k - Ty_k}(\frac{s}{2}) > 1 - \delta_2$ for every $k > n(s,\delta_2)$ and so

$$F_{Ty_k - z(k)}(u - \frac{s}{2}) \geq t(F_{Ty_k - y_k}(\frac{s}{2}), F_{y_k - z(k)}(u-s)) \geq$$
$$\geq t(F_{Ty_k - y_k}(\frac{s}{2}), r) > r - \delta_1.$$

Since $\beta_{\{Ty_k|k\in \mathbf{N}\}}(u) = \beta_{\{Ty_k|k>n(s,\delta_2)\}}(u)$ we obtain (8) and so the inequality $r \leq \beta_{\{Ty_k|k\in \mathbf{N}\}}(u)$ holds. Similarly we can prove that

$$\beta_{\{y_k|k\in \mathbf{N}\}}(u) \geq \beta_{\{Ty_k|k\in \mathbf{N}\}}(u)$$

and so

$$\beta_{\{y_k|k\in \mathbf{N}\}}(u) = \beta_{\{Ty_k|k\in \mathbf{N}\}}(u)$$

for every $u \in \mathbf{R}$. Since T is densifyng on M in respect to β we conclude that the set $\overline{\{y_k \mid k \in \mathbf{N}\}}$ is compact. Suppose that $\gamma = \alpha$ and prove that for every $u \in \mathbf{R}$ and $s \in (0,u)$

$$\alpha_{\{y_k|k\in \mathbf{N}\}}(u-s) \leq \alpha_{\{Ty_k|k\in \mathbf{N}\}}(u). \tag{9}$$

Let $0 < r < \alpha_{\{y_k|k\in \mathbf{N}\}}(u-s)$. We shall prove that for every $\delta_1 \in (0,r)$

$$r - \delta_1 \leq \alpha_{\{Ty_k|k\in \mathbf{N}\}}(u) \tag{10}$$

which implies (9) .

Using the definition of the function α we obtain that there exists a finite family $\{Y_1, Y_2, ..., Y_n\}(Y_i \subset S,\ i \in \{1, 2, ..., n\})$ such that $\{y_k \mid k \in \mathbf{N}\} = \cup_{i=1}^{n} Y_i$ where

$$(11) \qquad D_{Y_i}(u - s) > r, \text{ for every } \ i \in \{1, 2, ..., n\},$$

This implies that for every $x, y \in Y_i, F_{x-y}(u - s) > r$. Let $\delta \in (0, r)$. Further, let $\delta_2 \in (0, 1)$ be such a number that the following implication holds:

$$1 \geq v, w > 1 - \delta_2 \Rightarrow t(v, t(r, w)) > r - \delta_1.$$

Since $t(1, t(r, 1)) = r$ such an element δ_2 exists.

Let, for every $j \in \{1, 2, ..., n\}, Z_j$ be a subset of S which is defined by

$$Z_j = \{z \mid z \in S,\ F_{z-y}(\frac{s}{4}) > 1 - \delta_1 \text{ for some } \ y \in Y_j\}.$$

If $n(s, \delta_2) \in \mathbf{N}$ is such that for every $k > n(s, \delta_2)$

$$F_{y_k - Ty_k}(\frac{s}{4}) > 1 - \delta_2$$

then

$$\{Ty_k \mid k > n(s, \delta_2)\} \subset \cup_{j=1}^{n} Z_j.$$

It can be proved that $D_{Z_j}(u) > r - \delta_1$ for every $j \in \{1, 2, ..., n\}$ and so

$$\alpha_{\{Ty_k \mid k > n(s, \delta_2)\}}(u) = \alpha_{\{Ty_k \mid k \in \mathbf{N}\}}(u) \geq r - \delta_1.$$

This proves (9) and so $\alpha_{\{y_k \mid k \in \mathbf{N}\}}(u) \leq \alpha_{\{Ty_k \mid k \in \mathbf{N}\}}(u)$, for every $u \in \mathbf{R}$. Similarly we can prove that

$$\alpha_{\{Ty_k \mid k \in \mathbf{N}\}}(u) \leq \alpha_{\{y_k \mid k \in \mathbf{N}\}}(u)$$

and so $\alpha_{\{y_k \mid k \in \mathbf{N}\}}(u) \leq \alpha_{\{Ty_k \mid k \in \mathbf{N}\}}(u)$, for every $u \in \mathbf{R}$.

Since the mapping T is γ densifying on M it follows that the set $\overline{\{y_n \mid n \in \mathbf{N}\}}$ is compact. If $\lim_{k \to \infty} y_{n_k} = y$ then from $\lim_{n \to \infty} y_n - Ty_n = 0$ it follows that $Ty = y$ since the mapping T is continuous.

Corollary *Let $(S, \mathcal{F}, t)$ be a complete random normed space with a continuous T-norm t, A a nonempty, closed, convex and probabilistic bounded subset of S and T a probabilistic nonexpansive mapping of A into A (i.e.(i) from Theorem holds). If there exists $n_0 \in \mathbf{N}$ such that $\overline{T^{n_0}(A)}$ is compact* *then there exists $x \in A$ such that $x = Tx$.*

Proof . We shall prove that all the conditions of the Theorem are satisfied, where $M = \overline{T^{n_0-1}(A)}$. It is obvious that M is an attractor for T. Suppose that B is an arbitrary subset from the set $\overline{T^{n_0-1}(A)}$ such that

$$\gamma_{T(B)}(u) \leq \gamma_B(u), \text{for every} \quad u > 0.$$

From the continuity of the mapping T it follows that $T(B) \subset \overline{T(T^{n_0-1}(A))}$ and so

$$1 = \gamma_{\overline{T^{n_0}(A)}}(u) \leq \gamma_{T(B)}(u) \leq \gamma_B(u), \text{ for every } u > 0.$$

This means that $\gamma_B = H$ which implies that B is precompact and hence T is $\gamma-$ densifying on M.

References

[1] Bocsan,Gh. and Constantin,Gh.: The Kuratowski function and some applications to the probabilistic metric spaces, Sem. Teor. Funct. si Mat.Apl., Timisoara, R.S.Romania, No. 1, 1973.

[2] Göhde,D.: Zum Princip der kontraktiven Abbildung, Math. Nachr. 30(1965), 251-258.

[3] Hadžić,O.: A fixed point theorem for multivalued mappings in random normed spaces, L'Anal. Numer. Theor. Approx., 81(1978), 49-52.

[4] Hadžić,O.: Some theorems on the fixed points in probabilistic metric and random normed spaces, Boll.Unione Mat. Ital., (6), 1-B (1982), 381-391.

[5] Hadžić,O.: Fixed Point Theory in Topological Vector Spaces, University of Novi Sad, Institute of Mathematics, Novi Sad.

[6] Schweizer,B. and Sklar A. : Statistical metric spaces, North-Holland Series in Probability and Applied Mathematics, 5, 1983.

[7] Sherstnev,A.N.: The notion of random normed spaces, DAN USSR 149 (2)(1963), 280-283.

[8] Tan,D.H.: On probabilistic condensing mappings, Rev. Roum. Math. Pures Appl. 26, 10 (1981), 1305-1317.

REZIME

TEOREMA O NEPOKRETNOJ TAČKI ZA NEEKSPANZIVNA PRESLIKAVANJA U SLUČAJNIM NORMIRANIM PROSTORIMA

U radu je dokazana verovatnosna generalizacija Göhdeovog rezultata iz [2].

Received by the editors August 15, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 107-116

Review of Research
Faculty of Science
Mathematics Series

ASYMPTOTICS OF NONOSCILLATORY SOLUTIONS OF A SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

Henry Howard
Department of Mathematics, University of Kentucky
Lexington, KY 40506, USA

Vojislav Marić and **Zvezdana Radašin**
Faculty of Technical Science, University of Novi Sad,
Trg Dositeja Obradovića 6, 21000 Novi Sad, Yugoslavia

Abstract

Let $f(x)$ be continuous for $x > 0$; we consider the existence and asymptotic behavior for $x \to \infty$ of nonoscillatory solutions of the equation

(0.1) $$y" + f(x)y = 0.$$

In our analysis there are neither hypotheses concerning the sign of the function $f(x)$ nor hypotheses concerning the absolute integrability of $f(x)$ over (a, ∞). First we prove a sufficient condition for the existence of nonoscillatory solutions of (1.1) and then obtain asymptotic formulae for such solutions.

AMS Mathematics Subject Classification (1980): 34C15
Key words and phrases: Asymptotics, nonoscillatory solutions, differential equations.

1. Introduction

Result similar to the ones presented here are given by A. R. Its [1] and by J. Mahony [2]. Both Its and Mahony consider the asymptotic behavior of oscillatory solutions of (0.1) as well as the asymptotic behavior of nonoscillatory solutions. Its assumes

$$f(x) = x^{\beta} P(x^{1+\alpha}) + cx^{-2},$$

where P is a smooth, periodic function of period 1 such that

$$\int_0^1 P(t)dt = 0,$$

with α, β, and c real numbers, and with

$$\beta = \alpha - 1 \text{ and } \int_0^1 x^{-2}\{\int_0^x tP(t)dt\}^2 dx < (c + 1/4)(1+\alpha)^2.$$

Mahony sketches at the end of his paper how his results for the interesting case where $f(x) = \sin(ax)/x$ can be extended to cover the cases in which

$$f(x) = \sin(ax)/x + \sum_2^{\infty} b_r x^{-r}$$

and in which

$$f(x) = p(x)/x,$$

where p is a periodic function of period τ, again of zero mean.

Our results neither contain nor are contained in the result of Its and Mahony. In particular the condition in both Its and Mahony concerning the zero mean of a periodic part of the coefficient function f is not needed in our reasonings.

Regarding the existence of nonoscillatory solutions, we refer to Willett's paper, [3], where sufficient conditions are obtained for this. We are primarily interested in asymptotics and our conditions imply results pertinent to that end. Note also the conditions in our nonoscillation theorem are simple and easy to verify.

We use successive approximations in our proofs. Note we are only concerned with one solution of equation (0.1) and the construction of a second linearly independent solution follows by usual Wronskian techniques.

2. Results

First we prove existence of nonoscillatory solutions for equation (0.1).

Theorem 1. *Let*

$$h(x) = \int_x^\infty f(x)dt \tag{2.1}$$

and let there exist a positive continuous function g, and an $x_0 > 0$, such that $g(x) \to 0$ as $x \to \infty$, while

$$|h(x)| \le g(x), \quad x \ge x_0, \tag{2.2}$$

and

$$\int_x^\infty g^2(t)dt \le cg(x), \quad x \ge x_0, \tag{2.3}$$

with c a constant such that

$$0 < c < 1/4. \tag{2.4}$$

Then the equation (0.1) has nonoscillatory solutions for $x \ge x_0$.

Proof. We prove first that the integral equation

$$u(x) = -\int_x^\infty (u(t) - h(t))^2 dt \tag{2.5}$$

has a solution $z(x)$ such that

$$z(x) = O(g(x)), \quad x \to \infty. \tag{2.6}$$

We shall use this z to construct nonoscillatory solutions of (0.1). Introduce a sequence $\{z_n(x)\}$, defined for $x \ge x_0$, as

$$z_0(x) = 0, \;\; z_n(x) = -\int_x^\infty (h(t) - z_{n-1}(t))^2 dt, \;\; n = 1, 2, \dots . \tag{2.7}$$

First we show that

$$|z_n(x)| < 4cg(x), \quad x \ge x_0. \tag{2.8}$$

Obviously inequality (2.8) is valid for $n = 1$ since, by use of hypotheses (2.2) and (2.3),

$$|z_1(x)| = \int_x^\infty h^2(t)dt \le \int_x^\infty g^2(t)dt \le cg(x), \quad x \ge x_0.$$

If we suppose that (2.8) holds for $n = k$ then, applying (2.2), (2.3) and (2.4), we have

$$|z_{n+1}(x)| \le \int_x^\infty (z_n(t)| + |h(t)|)^2 dt < (4c+1)^2 \int_x^\infty g^2(t)dt < 4cg(x)$$

for $x \geq x_0$. Hence, by induction, (2.8) holds for all integers $n \geq 0$.

Next we prove that $\{z_n(x)\}$ is a uniformly convergent sequence of continuous functions. Continuity follows from the definition of $z_n(x)$ and the continuity of $h(x)$. In order to show the convergence, we prove the inequality

$$(2.9) \qquad |z_{n+1}(x) - z_n(x)| \leq \frac{(4c)^{n+1}}{4} g(x), \;\; n = 0, 1, \dots \; .$$

for $x \geq x_0$. By (2.2), (2.3) and the definition of $z_0(x)$, inequality (2.9) is valid for $n = 0$. Assume, for $x \geq x_0$, that (2.9) holds for $n = k - 1$, so

$$(2.10) \qquad |z_k(x) - z_{k-1}(x)| \leq \frac{(4c)^k}{4} g(x);$$

then one gets from

$$|z_{k+1}(x) - z_k(x)| \leq \int_x^\infty |z_k(t) - z_{k-1}(t)|(|z_k(t)| + |z_{k-1}(t)| + 2|h(t)|)dt,$$

and use of (2.2), (2.3), (2.4), (2.8) and (2.10), that

$$|z_{k+1}(x) - z_k(x)| < \int_x^\infty \frac{(4c)^k}{4}(8c + 2) g^2(t)dt < \frac{(4c)^{k+1}}{4} g(x)$$

for $x \geq x_0$, which proves (2.9) for all integers $n \geq 0$. Since

$$(2.11) \qquad z_{n+1}(x) = \sum_{k=0}^{n} (z_{k+1}(x) - z_k(x)),$$

we conclude from (2.9) that $\{z_n(x)\}$ converges uniformly on $[x_0, \infty)$ and the function

$$(2.12) \qquad z(x) = \lim_{n \to \infty} z_n(x)$$

is continuous. Also, from (2.9) and (2.11) there follows, for $x \geq x_0$,

$$(2.13) \qquad |z_{n+1}(x)| \leq \frac{g(x)}{4} \sum_{k=0}^{n} (4c)^{k+1} < \frac{c}{1 - 4c} g(x).$$

The appraisal of (2.6) now follows at once.

We are left with the proof that $z(x)$ is a solution of (2.5). Since, by (2.2) and (2.8),

$$(z_n(x) - h(x))^2 < (4c + 1)^2 g^2(x)$$

for $x > x_0$, and the integral on the left side of (2.3) is convergent, we get, using the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty}\int_x^\infty (z_n(t)-h(t))^2dt = \int_x^\infty (z(t)-h(t))^2dt,$$

and it follows that $z(x)$ is a solution of equation (2.5).

Finally, to construct a solution of (0.1), we introduce the function $\xi(x)$ by

$$\xi'(x)/\xi(x) = h(x) - z(x), \quad x \geq x_0. \tag{2.14}$$

By a straightforward calculation we show that $\xi(x)$ satisfies the equation

$$\xi"(x) + \{z'(x) - (z(x)-h(x))^2 - h'(x)\}\xi(x) = 0 \quad .$$

for $x \geq x_0$. Noting that $z(x)$ is a solution of equation (2.5) and that, by (2.1),

$$h'(x) = -f(x),$$

we conclude that $\xi(x)$ is a solution of (0.1). If we integrate (2.14) over (a, x), with $a \geq x_0$, we obtain a solution of (0.1) given by

$$\xi(x) = \xi_0 \exp\{\int_a^x (h(t)-z(t))dt\}, \tag{2.15}$$

which is obviously a nonoscillatory one. This proves Theorem 1.

Now we give a theorem concerning the asymptotic behavior of the solution of (0.1) just constructed.

Theorem 2. *Let the functions h and g be as defined as in Theorem 1. If*

$$\int_x^\infty g^2(t)dt \leq c(x)g(x), \quad x \geq x_0, \tag{2.16}$$

where c is a monotonely decreasing functions such that

$$0 < c(x) \leq x < 1/4, \tag{2.17}$$

and if, for some positive integer n,

$$\int_0^\infty c^n(x)g(x)dx < \infty, \tag{2.18}$$

then the solution (2.15) of (0.1) satisfies the asymptotic equality

$$y(x) \sim A\exp(\int_a^x (h(t) - z_{n-1}(t))dt), \; x \to \infty, \tag{2.19}$$

where $A =$ constant and the z_n are defined by (2.7).

Proof. Notice that all hypotheses of Theorem 1 hold. For the sequence $z_n(x)$ put

$$r_n(x) = \sum_{k=n}^{\infty} (z_k(x) - z_{k-1}(x)). \tag{2.20}$$

For the function z defined by (2.12) we have

$$z(x) = z_{n-1}(x) + r_n(x). \tag{2.21}$$

In the same manner as in the proof of inequality (2.9) it can be shown (by use of the fact the c is monotonely decreasing) that

$$|z_k(x) - z_{k-1}(x)| \le \frac{(4c(x))^k}{4} g(x), \quad k = 1, 2, \dots\ , \tag{2.22}$$

for $x \ge x_0$. Now, from (2.17), (2.20) and (2.22), there follows

$$|r_n(x)| \le \frac{g(x)}{4} \sum_{k=n}^{\infty} (4c(x))^k = \frac{4^{n-1} c^n(x)}{1 - 4c(x)} g(x) \le \frac{4^{n-1}}{1 - 4c} c^n(x) g(x). \tag{2.23}$$

But condition (2.18) shows that the integral

$$\int_a^{\infty} r_n(x) dx$$

converges absolutely and, by (2.15) and (2.21), we obtain the asymptotic behavior given by (2.19).

3. Examples

Example 1 Let

$$f(x) = \frac{a + b \sin x}{x^{\alpha} \ln^{\beta} x}, \ \ \alpha \ge 1.$$

We shall treat two cases and obtain asymptotic formulae which hold for $x \to \infty$. For the first case suppose $a \ne 0$ while b is an arbitrary constant. Now, for large x,

$$h(x) = \frac{a}{\alpha - 1} x^{1-\alpha} \ln^{-\beta} x + O(x^{1-\alpha} \ln^{-\beta-1} x),$$

for $\alpha > 1$ and β arbitrary. For any fixed $\epsilon > 0$ we take

$$g(x) = \frac{|a| + \epsilon}{\alpha - 1} x^{1-\alpha} \ln^{-\beta} x.$$

Thus, for large x ,

$$\int_x^\infty g^2(t)dt = \frac{(|a|+\epsilon)^2}{(2\alpha-3)(\alpha-1)^2}x^{3-2\alpha}\ln^{-2\beta}x + O(x^{3-2\alpha}\ln^{-2\beta-1}x).$$

If $\alpha > 2$, assumptions (2.3) and (2.4) hold, for any a, b and β. Moreover, since there exists a constant $M > 0$ such that the function

$$c(x) = Mx^{2-\alpha}\ln^{-\beta}x$$

fulfills the conditions of Theorem 2, (with (2.18) being satisfied for $n = 1$), we conclude

$$y(x) \sim A\exp(\int_a^x h(t)dt).$$

Since

$$\int_a^\infty h(x)dx < \infty,$$

the solution tends to a nonzero constant.
If $\alpha = 2$ and $\beta > 0$ then we take $c(x) = M\ln^{-\beta}x$ and note

$$\int_a^\infty c^n(x)g(x)dx = \int_a^\infty O(x^{-1}\ln^{-(n+1)\beta}x)dx$$

converges for sufficiently large n. If $\beta > 1/2$ we take $n = 1$ in (2.18) and get the following results:

$$\text{if } \beta > 1, \quad y \sim \text{ constant };$$

$$\text{if } \beta = 1, \quad y \sim A(\ln x)^a;$$

$$\text{if } \frac{1}{2} < \beta < 1, \ \ y \sim A\exp(\frac{a}{1-\beta}\ln^{1-\beta}x).$$

If $1/3 < \beta \leq 1/2$ condition (2.18) is fulfilled for $n = 2$ and one gets,

$$\text{if } \beta = 1/2, \quad y \sim A\exp(2a\ln^{1/2}x)(\ln x)^{a^2};$$

$$\text{if } 1/3 < \beta < 1/2, \ y \sim A\exp(\frac{a}{1-\beta}\ln^{1-\beta}x)\exp(\frac{a^2}{1-2\beta}\ln^{1-2\beta}x),$$

and so on; the asymptotic behavior can be obtained for any positive β.

If $\alpha = 2$, and $\beta = 0$ the conditions of Theorem 1 are fulfilled for $|a| < 1/4$ and we have the existence of nonoscillatory solutions of our equations, but we cannot find a function $c(x)$ satisfying condition (2.18) of Theorem 2. Thus in this case we are not able to obtain the asymptotic behavior of solutions. This ends our discussion of the first case.

For the second case suppose $a = 0$. Now we have, for large x

$$h(x) = bx^{-\alpha}(ln^{-\beta}x)\cos x + O(x^{-\alpha-1}\ln^{-\beta}x),$$

$$g(x) = (|b| + \epsilon)x^{-\alpha}\ln^{-\beta}x,$$

$$\int_x^\infty g^2(t)dt = \frac{(|b|+\epsilon)^2}{2\alpha-1}x^{1-2\alpha}\ln^{-2\beta}x + O(x^{1-2\alpha}\ln^{-2\beta-1}x),$$

$$c(x) = Mx^{1-\alpha}\ln^{-\beta}x.$$

If $\alpha > 1$, the conditions of Theorem 2 hold for any β, and if we take $n = 1$ in (2.18) it is easily seen that the solution tends to a nonzero constant as $x \to \infty$. If $\alpha = 1$ and $\beta > 1/2$, the situation is the same. If $\alpha = 1$ and $\beta = 1/2$, take $n = 2$ in (2.18) and get

$$y \sim A(\ln x)^{b^2/2}.$$

If $\alpha = 1$ and $1/3 < \beta < 1/2$ take $n = 2$ in (2.18) and get

$$y \sim A\exp(\frac{b^2}{2(1-2\beta)}\ln^{1-2\beta}x);$$

if $\alpha = 1$ and $1/4 < \beta \leq 1/2$, take $n = 3$ in (2.18) and get

$$y \sim A\exp(b^2\ln^{1/2}x)(\ln x)^{b^4/4}.$$

If $\alpha = 1$ and $1/5 < \beta < 1/4$ take $n = 4$ in (2.18) and get

$$y \sim A\exp(\frac{b^2}{2(1-\beta)}\ln^{1-2\beta}x)\exp(\frac{b^4}{4(1-4\beta)}\ln^{1-4\beta}x).$$

If $\alpha = 1$ and $1/6 < \beta \leq 1/5$, thte behavior is the same as above, although one has to take $n = 5$.

If $\alpha = 1$ and $\beta = 0$ our method gives us only the existence of nonoscillatory solutions (and not the asymptotic behavior) for $|b| < 1/4$. However, it is shown by Its that in this case solutions are nonoscillatory for $|b| < 1/\sqrt{2}$ and the asymptotic behavior is also determined.

Example 2 Let

$$f(x) = ax^{\alpha}\cos(e^{\beta x}), \quad a \neq 0, \beta > 0.$$

Notice that, if $\alpha > 0$,

$$\sup_{x \geq x_0} f(x) = -\inf_{x \geq x_0} f(x) = \infty.$$

By partial integration we get

$$h(x) = -\frac{a}{\beta}x^{\alpha}e^{-\beta x}\sin(e^{\beta x}) - \frac{a}{\beta}\int_x^{\infty}(\alpha t^{\alpha-1} - \beta t^{\alpha})e^{-\beta t}\sin(e^{\beta x})dt;$$

there exists positive constants M_1, M_2 such that

$$g(x) = M_1 x^{\alpha}e^{-\beta x}$$

with

$$\int_x^{\infty} g^2(t)dt < c(x)g(x),$$

where

$$c(x) = M_2 x^{\alpha}e^{-\beta x}.$$

Therefore, Theorem 2 holds (take $n = 1$ in (2.18)) and it follows that (0.1) has a nonoscillatory solution which tends to a nonzero constant for any a, α and $\beta > 0$.

References

[1] Its, A. R., Asymptotic behavior of the solution of the radial Schroedinger equation with oscillating potential for the zero value of the energy (Russian), Scattering theory, Theory of oscillations (Russian), 183-184, Problems in Math. Phys, 9, Leningrad Univ., Leningrad, (1979), pp. 30-41.

[2] Mahony, J. J., Asymptotic results for the solutions of a certain differential equation, J. Australian Math. Soc., (1972), Vol. 13, part 2, pp. 147-159.

[3] Willett, D., On the oscillatory behavior of the solutions of second linear differential equations, Ann. Polon. Math. XXI, (1969), pp. 175-194.

REZIME

ASIMPTOTIKA NEOSCILATORNIH REŠENJA LINEARNE DIFERENCIJALNE JEDNAČINE DRUGOG REDA

Neka je $f(x)$ neprekidna funkcija za $x > 0$. U radu je razmatrana egzistencija i asimptotsko ponašanje za $x \to \infty$ neoscilatornog rešenja jednačine

$$y'' + f(x)y = 0.$$

Pri ovoj analizi ne koriste se pretpostavke kako o znaku funkcije $f(x)$ tako ni o apsolutnoj integrabilnosti funkcije $f(x)$ nad (a, ∞). Prvo je dobijen dovoljan uslov za postojanje neoscilatornih rešenja navedene diferencijalne jednačine, a zatim asimptotska formula za ova rešenja.

Received by the editors November 6, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 117-128

Review of Research
Faculty of Science
Mathematics Series

QUASIASYMPTOTIC EXPANSIONS AND THE STIELTJES TRANSFORMATION

Danica Nikolić - Despotović and **Stevan Pilipović**
Institute of Mathematics, University of Novi Sad,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

Using the notation of the quasiasymptotic expansion of a tempered distribution at ∞ an ordinary asymptotic expansion of the Stieltjes transformation of function from $L^1_{loc}(0,\infty)$ is obtained with the ordinary, asymptotic expansion of the form $\sum_{i=1}^{\infty} \frac{a_i}{t^i}$, $t \to \infty$.

AMS Subject Classification (1980): 44A15, 44F12
Key words and phrases: Stieltjes transformation of distributions, quasiasymptotic expansion of distributions.

1. Notations and known results

Denote by S the space of rapidly decreasing smooth functions defined on the real line $\mathbf{R}$, supplied with the usual topology. Its dual, the space of tempered distributions is S' and S'_+ is its subspace with elements supported by $[0,\infty)$.

Recall [4] a continuous positive function L defined on $(a,\infty), a > 0$, is slowly varying at infinity if for $\lambda > 0$

$$\lim_{t\to\infty} \frac{L(\lambda t)}{L(t)} = 1$$

We denote by $\sum_\infty$ the set of all the slowly varying (in short sv) function at ∞.

The quasiasymptotic behaviour and quasiassymptotic expansion at ∞ of an $f \in S'_+$ were studied by Vladimirov, Drožinov and Zavialov [1]. In [3] Pilipović gave a slight modification of the quasi asymptotic expansion at ∞.

Recall, $f \in S'_+$ has the quasiasymptotic behaviour at ∞ with respect to $k^\nu L(k)$, $\nu \in \mathbf{R}$, $L \in \sum_\infty$, with the limit $g \in S'$ if

$$\lim_{k\to\infty} < \frac{f(kt)}{k^\nu L(k)}, \varphi(t) >=< g(t), \varphi(t) >, \varphi \in S,$$

$< .,. >$ is the dual pairing between S' and S. In this case we write $f \overset{q}{\sim} g$ at ∞ with respect to $k^\nu L(k)$.

Let us recall that the family of homogeneous distributions $f_{\nu+1}$, $\nu \in \mathbf{R}$, is defined by

$$f_{\nu+1}(t) = \begin{cases} \frac{H(t)t^\nu}{\Gamma(\nu+1)}, & \nu > -1, \\ f^{(n)}_{\nu+n+1}(t), & \nu \le -1, \ n+\nu > -1, \ n \in \mathbf{N}, \end{cases}$$

where H is Heaviside's function; Γ is the gamma function.

Let $\nu \in \mathbf{R}$, $A > 0$, and $L \in \sum_\infty$. We put

$$f_{L,A,\nu+1}(t) = \begin{cases} \frac{H(t-A)t^\nu L(t)}{\Gamma(\nu+1)} & \nu > -1, \\ f^{(n)}_{L,A,\nu+n+1}(t), & \nu \le -1, \ n+\nu > -1, \ n \in \mathbf{N}, \end{cases}$$

where n is the smallest integer for which $n + \nu > -1$. This definition includes, for example, distribution $\delta^{(k)}(t-a)$, $a > 0$, $(H(t-a)\ln t)^{(k)}$, $a > 0$, $k \in \mathbf{N}$. For $A = 0$ we use the notation $f_{L,\nu+1}$.

We have

$$f_{L,A,\nu+1}(t) \overset{q}{\sim} f_{\nu+1}(t), t \in \mathbf{R}, \text{ at } \infty \text{ with respect to } k^\nu L(k).$$

Let us denote by Λ the set N or the set of the form $\{1, 2, \ldots, N\}$, $N \in \mathbf{N}$. In the second case we shall also use the symbol Λ_N.

We say [2] that an $f \in S'_+$ has the quasiaysmptotic expansion at ∞ with respect to $\{(k^{\nu_i} L_i),\ i \in \Lambda\}$, $L_i \in \sum_\infty$, if there are complex numbers

$a_i \neq 0$ and $A_i \in \mathbf{R}$, $i \in \Lambda$, such that for any $m \in \Lambda$

$$\frac{f(kt) - \sum_{i=1}^{m} a_i f_{L_i, A_i, \nu_i+1}(kt)}{k^{\nu_m} L_m(k)} \to 0 \text{ in } S'.$$

In that case we write

$$(1) \qquad f \overset{q.e}{\sim} \sum_{i \in \Lambda} a_i f_{L_i, A_i, \nu_i+1} \text{ at } \infty \text{ with respect to } \{(k^{\nu_i} L_i),\ i \in \Lambda\}.$$

The Stieltjes transformation of distribution was considered in [2].

Let $f \in S'$. We say that $f \in \mathcal{J}'(r)$, $r \in \mathbf{R} \backslash (-\mathbf{N})$ if there exits an $m \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and a locally integrable function F such that

$$(2) \qquad \text{a) } f = F^{(m)}.$$

$$\text{b) } \int_{-\infty}^{+\infty} \mid F(t)(t+z)^{-r-m-1} \mid dt < \infty, \text{ for } \ \mathrm{Imz} \neq 0.$$

We also need the definition of the space $I'(r)$.

$I'(r)$ is the space of all $f \in S'$ for wich (a) holds and instead of (b), we suppose that there exist $C = C(F)$ and $\varepsilon = \varepsilon(F)$ such that

$$(3) \qquad |F(t)| \leq C(1+|t|)^{r+m-\varepsilon}, t \in \mathbf{R}.$$

The Stieltjes transformation S_r of index r, $r \in \mathbf{R} \backslash (-\mathbf{N})$ of a distribution $f \in \mathcal{J}'(r)$ with the properties given in (2) is a complex valued function given by

(4)

$$\begin{aligned}(S_r f)(z) &= (r+1)_m \int_{-\infty}^{+\infty} F(t)(t+z)^{-r-m-1} dt \\ &= (r+1)_m < F(t), (t+z)^{-r-m-1} >, \text{ for } Imz \neq 0,\end{aligned}$$

where $(a)_n = a(a+1)\dots(a+n-1)$, $n \in \mathbf{N}$ and $(a)_0 = 1$, $a \in \mathbf{R}$. It is easy to see that $S_r f$ is a holomorphic function of the complex variable z in the domain $\mathbf{C} \backslash (-\infty, +\infty)$. We shall use the following formulas from [3]. If $r > -1$, then

$$(5) \qquad (S_r \frac{H(t-1)}{t})(k) = (r+1) \int_1^{\infty} \frac{\ln t}{(k+t)^{r+2}} dt;$$

$$(6)\qquad (S_r \frac{H(t-1)}{t^s})(k) = \frac{1}{(s-1)!}[(s-2)! - \frac{1}{(1+k)^{r+1}} - \frac{(s-3)!(r+1)}{(1+k)^{r+2}} + \frac{(-1)^{s-2}(r+1)_{s-2}}{(1+k)^{r+s-1}} + \dots + (-1)^{s-1}(r+1)_s \int_1^\infty \frac{\ln\, t}{(t+k)^{r+s+1}}dt],\ k > 1,\ s \ge 2.$$

Let $f \in S'_+$ and

$$f(x) \overset{q.e}{\sim} \sum_{i \in \Lambda} a_i \delta^{(i)}(x).$$

Then, because of $S_r(\delta^{(j)}(t))(x) = -\frac{(r+1)\dots(r+j)}{x^{r+j+1}},\ r > -1, j \in \mathbf{N}_0$ we get

$$(S_r f)(x) \sim \sum_{i \in \Lambda} a_i \frac{(r+1)_i}{x^{r+i+1}},\ x \to \infty.$$

2. Application

Our intension in this paper is to give the application for the classical Stieltjes transformation. From the assertions which are to follow, one can easily derive the coresponding assertions for the distributional Stieltjes transformation. But we shall emphasize the fact that for the classical Stieltjes transformation we obtain new "classical" results by using the abstract theory of the preceding section.

We assume that $r \in \mathbf{N}_0$. First we shall prove two lemmas which will be used in the proof of the main Proposition 4.

Lemma 1. *Let* $s \in \mathbf{N}, k > 1$. *Then*

$$(7)\qquad \int_1^\infty \frac{\ln t}{t(k+t)^{r+s+1}}dt = \frac{1}{k^{r+s}}\,(-\frac{1}{r+s}\ln k + \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{jk^j}\,\frac{\Gamma(r+s+j)}{\Gamma(j+1)\Gamma(r+s+1)} + \frac{1}{r+s}B_s)$$

where

$$B_s = \frac{(-1)^{r+s-1}}{\Gamma(r+s)}(\frac{1}{z}\ln\frac{z}{1+z})^{(r+s-1)}|_{z=1} +$$

$$+\int_1^\infty \frac{\ln u}{(1+u)^{r+s+1}}du.$$

Proof. We have

$$\int_1^\infty \frac{\ln t dt}{(t+k)^{r+s+1}} = \frac{1}{k^{r+s}}(\int_{1/k}^\infty \frac{\ln u}{(1+u)^{r+s+1}}du + \ln k \int_1^\infty \frac{du}{(1+u)^{r+s+1}} =$$

$$= \frac{1}{k^{r+s}}(\int_{1/k}^1 \frac{\ln u}{(1+u)^{r+s+1}}du + \int_1^\infty \frac{\ln u}{(1+u)^{r+s+1}}du + \frac{\ln k}{(r+s)}\ \frac{1}{(1+(1/k))^{r+s}})$$

and by the partial integration in the integral from $\frac{1}{k}$ to 1 we get

$$\int_1^\infty \frac{\ln t}{(t+k)^{r+s+1}}dt = \frac{1}{k^{r+s}}\ (\frac{1}{r+s}\int_{1/k}^1 \frac{du}{u(1+u)^{r+s}} + \int_1^\infty \frac{\ln u}{(1+u)^{r+s+1}}du).$$

Consider the function $F(z) = \int_{1/k}^1 \frac{du}{u(z+u)}$, $z > 0$. We have

$$F(z) = \frac{1}{z}(\ln u - \ln(z+u))\ |_{1/k}^1 = \frac{1}{z}(\ln(1+kz) - \ln(1+z)) =$$

$$= \frac{1}{z}(\ln k + \ln\frac{z}{1+z} - \ln(1+\frac{1}{kz})).$$

Take k such that $kz > 1$. Thus we get

$$F(z) = \frac{1}{z}\ln k + \frac{1}{z}\ln\frac{z}{1+z} + \sum_{j=1}^\infty \frac{(-1)^{j-1}}{jk^j z^{j+1}}.$$

After differentiation $(r+s-1)$ times we get

$$(F(z))^{(r+s-1)} = (-1)^{r+s-1}(r+s-1)!\int_{1/k}^1 \frac{du}{u(z+u)^{r+s}} =$$

$$= (-1)^{r+s}\frac{(r+s-1)!\ln k}{z^{r+s}} + (\frac{1}{z}\ln\frac{z}{1+z})^{(r+s-1)} +$$

$$+\sum_{j=1}^\infty \frac{(-1)^{j-1}}{jk^j}\ \frac{\Gamma(j+r+s)(-1)^{r+s-1}}{\Gamma(j+1)z^{j+r+s}}$$

and thus by putting $z = 1$ we get

$$\int_{1/k}^1 \frac{du}{u(1+u)^{r+s}} = \ln k + \frac{(-1)^{r+s-1}}{\Gamma(r+s)}\ (\frac{1}{z}\ln\frac{z}{1+z})^{(r+s-1)}|_{z=1} +$$

$$+\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{jk^j}\frac{\Gamma(j+r+s)}{\Gamma(j+1)\Gamma(r+s)}$$

this implies (7).

We shall also the following formula

$$\frac{1}{(1+k)^s}=\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(s+j)}{\Gamma(s)\Gamma(j+1)}\,\frac{1}{k^{s+j}},\ k>1,\ s\in\mathbf{N}. \tag{8}$$

Let

$$T_1=\frac{a_2}{1}+\frac{a_3}{2}+\ldots+\frac{a_{n-1}}{n-2},\ T_2=\frac{a_3}{2!}+\frac{a_4}{3\cdot 2}+\ldots+\frac{a_{n-2}}{(n-2)(n-3)},$$

$$T_3=\frac{a_4}{3!}+\frac{a_5}{4\cdot 3\cdot 2}+\ldots+\frac{a_{n-1}(n-5)!}{(n-2)!},\ldots,T_{n-2}=\frac{a_{n-1}}{(n-2)!}.$$

Lemma 2. *Let* $a_i\neq 0, i\in\Lambda_{n-1}$

$$\sum_{i=1}^{n-1}(S_r\frac{H(t-1)}{t^i})(k)=\frac{1}{k^{r+1}}(a_1B_1+T_1)+\sum_{m=1}^{n-1}(-1)^{m-1}\frac{a_m(r+1)_m\ln k}{(m-1)!k^{r+m}(r+m)}+$$

$$+\sum_{\ell=2}^{n-2}(-1)^{\ell}\frac{\Gamma(r+\ell)}{k^{r+\ell}}(\sum_{i=1}^{\ell-1}\frac{a_i(r+1)_i}{(\ell-i)\Gamma(\ell+1-i)\Gamma(r+i)(i-1)!}-$$

$$-\sum_{j=0}^{\ell-1}\frac{(r+1)_jT_{j+1}}{\Gamma(\ell-j)\Gamma(r+1+j)}+a_{\ell}\frac{(r+1)_{\ell}}{(\ell-1)!}B^*_{\ell})+$$

$$+\frac{(-1)^{n-1}\Gamma(r+n-1)}{k^{r+n-1}}(\sum_{i=1}^{n-2}\frac{a_i(r+1)_i}{(n-1-i)\Gamma(n-i)\Gamma(r+i)(i-1)!}-$$

$$-\sum_{j=0}^{n-3}\frac{(r+1)_jT_{j+1}}{\Gamma(n-1-j)\Gamma(r+j)}-\frac{a_{n-1}(r+1)_{n-1}B_{n-1}}{(n-2)!})+$$

$$+\frac{(-1)^{n}\Gamma(r+n)}{k^{r+n}}(\sum_{i=1}^{n-1}\frac{a_i(r+1)_i}{(n-i)\Gamma(n+1-i)\Gamma(r+i)(i-1)!}-$$

$$-\sum_{j=0}^{n-3}\frac{(r+1)_jT_{j+1}}{\Gamma(n-j)\Gamma(r+1+j)})+0(\frac{1}{k^{r+n+1}}).$$

Proof. From formulas (5), (6), (7) and (8) we get $(k > 1)$

$$S_r \frac{H(t-1)}{t}(k) = \frac{1}{k^{r+1}}(\ln k + B_1 + \sum_{j=1}^{\infty} \frac{(-1)^j}{jk^j}\,\frac{\Gamma(r+j+1)}{\Gamma(j+1)\Gamma(r+1)},$$

$$(S_r \frac{H(t-1)}{t^s})(k) = \frac{1}{(s-1)}\,\{(s-2)!\,\sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(r+j+1)}{\Gamma(r+1)\Gamma(j+1)}\,\frac{1}{k^{r+1+j}} -$$

$$-\,(s-3)!(r+1)\sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(r+j+2)}{\Gamma(r+2)\Gamma(j+1)}\frac{1}{k^{r+2+j}} + \ldots$$

$$+(-1)^{s-2}(r+1)_{s-2}\sum_{j=2}^{\infty}\frac{(-1)^j\Gamma(r+s-1+j)}{\Gamma(j+1)\Gamma(r+s-1)}\frac{1}{k^{r+s-1+j}} +$$

$$+(-1)^{s-1}(r+1)_s \frac{1}{k^{r+s}}[\frac{1}{r+s}\ln k +$$

$$+\frac{1}{r+s}B_s + \sum_{j=1}^{\infty}\frac{(-1)^j\Gamma(r+s+j)}{jk^j\Gamma(j+1)\Gamma(r+s+1)}\,]\}$$

where $r = 2, 3, \ldots, n-1$. We shall denote by $B_s^* = \frac{1}{r+s}B_s$.

This implies

$$H(k) = \sum_{i=1}^{n-1} a_i (S_r \frac{H(t-1)}{t^i})(k) =$$

$$= a_1[\frac{\ln k}{k^{r+1}} + \frac{B_1}{k^{r+1}} + \sum_{j=1}^{\infty}\frac{(-1)^j\Gamma(r+j+1)}{\Gamma(j+1)\Gamma(r+1)jk^{r+j+1}}] +$$

$$+\,a_2[\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(r+j+1)}{\Gamma(r+1)\Gamma(j+1)}\,\frac{1}{k^{r+1+j}} - \frac{(r+1)_2}{k^{r+2}}\,\frac{\ln k}{r+2} - \frac{(r+1)_2}{k^{r+2}}B_2^* -$$

$$-(r+1)_2\sum_{j=1}^{\infty}\frac{(-1)^{j-1}\Gamma(r+j+2)}{jk^{j+r+2}\Gamma(j+1)\Gamma(r+3)}] +$$

……………………

$$+\frac{a_s}{(s-1)!}[\sum_{i=0}^{s-2}(-1)^i(s-2-i)!(r+1)_i\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(r+1+j+i)}{\Gamma(j+1)\Gamma(r+1+i)}\,\frac{1}{k^{r+1+i+j}} +$$

$$+\,\frac{(-1)^{s-1}(r+1)_s\ln k}{k^{r+s}(r+s)} + (-1)^{s-1}(r+1)_s\frac{B_s^*}{k^{r+s}} +$$

$$+ (-1)^{s-1}(r+1)_s \sum_{j=1}^{\infty} \frac{(-1)^{j-1}\Gamma(r+s+j)}{jk^{r+s+j}\Gamma(j+1)\Gamma(r+s+1)} +$$

$$+ \frac{a_{n-1}}{(n-2)!} [\sum_{i=0}^{n-3} (-1)^i (n-3-i)!(r+1)_i \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(r+1+j+i)}{\Gamma(j+1)\Gamma(r+1+i)} \frac{1}{k^{r+1+i+j}} +$$

$$+ (-1)^{n-2}(r+1)_{n-1} \frac{\ln k}{k^{r+n-1}(r+n-1)} + (-1)^{n-1} \frac{(r+1)_{n-1}}{k^{r+n-1}} B^*_{n-1} +$$

$$+ (-1)^{n-2}(r+1)_{n-1} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}\Gamma(r+n+1+j)}{jk^{r+n-1+j}\Gamma(j+1)\Gamma(r+n)} =$$

$$= \frac{1}{k^{r+1}} [a_1 B_1 + (a_2 + \frac{a_3}{2} + \ldots + \frac{a_{n-1}}{n-2})] + a_1 \frac{\ln k}{k^{r+1}} + \frac{1}{k^{r+2}} (-a_1 \frac{\Gamma(r+2)}{\Gamma(2)\Gamma(r+1)} -$$

$$- a_2(r+1)_2 B^*_2 - a_2 \frac{\Gamma(r+2)}{\Gamma(r+1)\Gamma(2)} - a_3 \frac{\Gamma(r+2)}{\Gamma(r+1)\Gamma(2)} - \frac{a_3(r+1)}{2} \frac{\Gamma(r+2)}{\Gamma(1)\Gamma(r+2)} -$$

$$- \frac{a_4}{3} \frac{\Gamma(r+2)}{\Gamma(r+1)\Gamma(2)} - \frac{a_4}{3!}(r+1) \frac{\Gamma(r+2)}{\Gamma(1)\Gamma(r+2)} - \frac{a_{n-1}}{(n-2)} \frac{\Gamma(r+1)}{\Gamma(1)\Gamma(r+1)} -$$

$$- \frac{a_{n-1}}{(n-2)!}(n-4)! \frac{(r+1)\Gamma(r+2)}{\Gamma(r+2)\Gamma(1)}) - a_2(r+1)_2 \frac{\ln k}{k^{r+2}} - (r+1)a_2 \frac{B_2}{k^{r+2}} + \ldots +$$

$$+ \frac{(-1)^{s-2}}{k^{r+s}} [\frac{(r+1)\Gamma(r+s)}{\Gamma(s)\Gamma(r+1)(s-1)} a_1 + \frac{(r+1)_2 \Gamma(r+s) a_2}{\Gamma(s-1)\Gamma(r+2)(s-2)} + \ldots +$$

$$+ \frac{(r+1)_{s-1} a_{s-1}}{(s-2)!} \frac{\Gamma(r+s)}{\Gamma(2)\Gamma(r+s-1)}) + \frac{\Gamma(r+s)}{\Gamma(s)\Gamma(r+1)} T_1 +$$

$$+ \frac{(r+1)\Gamma(r+s)}{\Gamma(s-1)\Gamma(r+2)} T_2 + \ldots + \frac{\Gamma(r+s)(r+1)_{\ell-1}}{\Gamma(1)\Gamma(r+s)} T_s + \frac{a_s(r+1)_s}{(s-1)!} B^*_s] + \ldots +$$

$$+ \frac{(-1)^{n-2}(r+1)_{n-1} a_{n-1}}{(n-2)!(r+n-1)k^{r+n-1}} \ln k + \ldots$$

After some calculation we get

$$H(k) = \frac{1}{k^{r+1}}(a_1 B_1 + T_1) + a_1 \frac{\ln k}{k^{r+1}} - \frac{1}{k^{r+2}} [\frac{\Gamma(r+2)}{\Gamma(2)\Gamma(r+1)} a_1 + \frac{\Gamma(r+2)}{\Gamma(2)\Gamma(r+1)} T_1 +$$

$$+ \frac{\Gamma(r+2)(r+1)T_2}{\Gamma(1)\Gamma(r+2)} - a_2(r+1)_2 B^*_2] - a_2(r+1)_2 \frac{\ln k}{k^{r+2}} +$$

$$+\frac{\Gamma(r+3)}{k^{r+3}}\,[-\sum_{i=1}^{2}\frac{a_i(r+1)_i}{(3-i)\Gamma(4-i)\Gamma(r+i)(i-1)!}+\sum_{i=0}^{2}\frac{(r+1)_jT_{j+1}}{\Gamma(3-j)\Gamma(r+1+j)}+$$

$$+\frac{a_3(r+1)_3}{2}B_3^*\,]+\ldots$$

$$\ldots+\frac{(-1)^{n-2}\Gamma(r+n-2)}{k^{r+n-2}}\,[\sum_{i=1}^{n-3}\frac{a_i(r+1)_i}{(n-2-i)\Gamma(n-1-i)\Gamma(r+i)(i-1)!}-$$

$$-\sum_{i=0}^{n-3}\frac{(r+1)_jT_{j+1}}{\Gamma(n-2-j)\Gamma(r+1+j)}-\frac{a_{n-2}(r+1)_{n-2}B_{n-2}^*}{(n-3)!}\,]+$$

$$+(-1)^{n-2}\frac{(r+1)_{n-1}\ln k\,a_{n-1}}{(n-2)!(r+n-1)k^{r+n-1}}+$$

$$+(-1)^{n-1}\frac{\Gamma(r+n-1)}{k^{r+n-1}}\,(\sum_{i=0}^{n-2}\frac{a_i(r+1)_i}{(n-1-i)\Gamma(n-i)\Gamma(r+i)(i-1)!}-$$

$$-\sum_{j=0}^{n-3}\frac{(r+1)_jT_{j+1}}{\Gamma(n-1-j)\Gamma(r+j)}-\frac{a_{n-1}(r+1)_{n-1}}{(n-2)!}B_{n-1}^*\,)+\ldots$$

This proves the assertion.

For the next proposition we also need the following assertion from [3].

Lemma 3. *If* $h\in L^1_{loc}\,(0,\infty)$ *and* $h(t)\sim\frac{a_n}{t^n}$, $t\to\infty$. *Then*

$$(S_rh)(k)=a_n\,(\,\frac{m_0}{k^{r+1}}-\frac{(r+1)m_1}{k^{r+2}}+\ldots+(-1)^{n-2}\frac{(r+1)_{n-2}m_{n-2}}{k^{r+n-1}}+$$

$$+\frac{(-1)^{n-2}}{(n-1)!}\,\frac{(r+1)_n}{r+n}\,\frac{\ln k}{k^{r+n}}\,)\,+0\,(\,\frac{\ln k}{k^{r+n}}\,),\ k\to\infty,$$

where $m_i=\int_0^\infty t^i\frac{h(t)}{i!}dt,\ i=0,\ldots,n-2$.
Now, from the previous two Lemmas we obtain

Proposition 1. *Let* $f\in L^1_{loc}\,(0,\infty)$ *and let* $f(t)\sim\sum_{i=1}^{\infty}\frac{a_i}{t^i}$, $t\to\infty$. *Then for* $n\geq 2$ *we have*

$$(S_rf(t))(k)\sim\frac{1}{k^{r+1}}(a_1B_1+T_1+a_nm_0)+\sum_{m=1}^{n-1}\frac{(-1)^m a_m(r+1)_{m-1}\ln k}{(m-1)!k^{r+m}}+$$

$$+ \sum_{\ell=2}^{n-2} \frac{(-1)^{\ell}}{k^{r+\ell}} \, (\Gamma(r+\ell) \sum_{i=1}^{\ell-1} \frac{a_i(r+1)_i}{(\ell-1)\Gamma(\ell+1-i)\Gamma(r+i)(i-1)!} -$$

$$- \sum_{j=0}^{\ell-1} \frac{(r+1)_j T_{j+1}}{\Gamma(\ell-j)\Gamma(r+1+j)} + \frac{a_\ell (r+1)_\ell B^*_\ell}{(1-1)!} - m_{\ell-1}(r+1)_{\ell-1}) +$$

$$+ \frac{(-1)^{n-1}\Gamma(r+n-1)}{k^{r+n-1}} \, (\sum_{i=1}^{n-2} \frac{a_i(r+1)_i}{(n-1-i)\Gamma(n-i)\Gamma(r+i)(i-1)!} -$$

$$- \sum_{j=0}^{n-3} \frac{(r+1)_j T_{j+1}}{\Gamma(n-1-j)\Gamma(r+j)} - \frac{a_{n-1}(r+1)_{n-1} B^*_{n-1}}{(n-1)!} - m_{n-1}(r+1)_{n-2}) +$$

$$+ \frac{(-1)^{n-1}}{(n-1)!}(r+1)_{n-1} \frac{\ln k}{k^{r+n}} + o\,(\, \frac{\ln k}{k^{r+n}} \,), \, as \; k \to \infty$$

where

$$m_i = \int_0^\infty t^i \, \frac{f(t) - \sum_{j=1}^{n-1} \frac{a_j H(t-1)}{t^j}}{i!} dt, \quad i = 1, \ldots, n-2.$$

Proof. From the assumptions we have

$$f(t) - \sum_{i=1}^{n-1} a_i \frac{H(t-1)}{t^i} \sim \frac{a_n}{t^n}, \; t \to \infty.$$

By Lemma 3 we get

$$(S_r f(t))(k) - \sum_{i=1}^{n-1} a_i (S_r \frac{H(t-1)}{t^i})(k) \; =$$

$$= \; a_n \sum_{\ell=0}^{n-2} (-1)^{\ell} \frac{m_\ell (r+1)_\ell}{k^{r+\ell+1}} + \frac{(-1)^{n-1}}{(n-1)!} \, \frac{(r+1)_{n-1}}{k^{r+n}} \ln k + o\,(\frac{\ln k}{k^{n+r}})$$

Now, by Lemma 2, we have

$$(S_r f(t))(k) = \frac{1}{k^{r+1}}(a_1 B_1 + T_1) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} a_m (r+1)_m \ln k}{(m-1)! k^{r+m}} +$$

$$+ \sum_{\ell=2}^{n-2} \frac{(-1)^{\ell}\Gamma(r+\ell)}{k^{r+\ell}} \, (\sum_{i=1}^{\ell} \frac{a_i (r+1)_i}{((\ell-i)\Gamma(\ell+1-i)\Gamma(r+i)(i-1)!} -$$

$$-\sum_{j=0}^{\ell-1}\frac{(r+1)_i T_{j+1}}{\Gamma(\ell-j)\Gamma(r+j)}+\frac{a_\ell(r+1)_\ell B_\ell^*}{(n-2)!})+$$

$$\frac{(-1)^{n-1}\Gamma(r+n-1)}{k^{r+n-1}}\,(\sum_{i=1}^{n-1}\frac{a_i(r+1)_i}{(n-1-i)\Gamma(n-i)\Gamma(r+i)(i-1)!}-$$

$$-\sum_{j=0}^{n-3}\frac{(r+1)_j T_{j+1}}{\Gamma(n-1-j)\Gamma(r+j)}-\frac{a_{n-1}(r+1)_{n-1}B_{n-1}^*}{(n-2)!})+o\,(\frac{\ln k}{k^{r+n}})+$$

$$+\,a_n\sum_{\ell=0}^{n-2}(-1)^\ell\frac{m_\ell(r+1)_\ell}{k^{r+\ell+1}}+\frac{(-1)^{n-1}}{(n-1)!}\,\frac{(r+1)_{n-1}}{k^{r+n}}\ln k.$$

This implies the assertion.

Clearly the cases when r is not form $\mathbf{N}_0$ and when instead of $\sum_{i=0}^{\infty}\frac{a_i}{t^i}$ we have $\sum_{i=1}^{\infty}\frac{a_iL_i(t)}{t^i}$ in the assumptions of Proposition 4, then the corresponding assertion can be derived in a similar but technically more complicated way.

References

[1] V.S. Vladimirov, Ju.N. Drožinov, B.I. Zavialov: Multidimensional Tauberian Theorems for Generalized Functions, *Nauka, Moscow,* (1986) (In Russian).

[2] J. Lavoine, D.P. Misra: Abelian theorem for the distributional Stiletjes transformation, *Proc.Camb.Phil.Soc.* 86 (1979), 287-293.

[3] S. Pilipović: Quasiasymptotic expansion and the Laplace transformation, (to appear in Appl. Analysis.)

[4] E.Seneta: Regularly varying functions (Lecture Notes Math. 508). *Berlin-Heidelberg-New York, Springer-Verlag* 1976.

REZIME

KVAZIASIMPTOTSKI RAZVOJ I STIELTJESOVA TRANSFORMACIJA

Koristeći pojam kvaziasimptotskog razvoja temperirane distribucije u ∞ dobijamo, Tvrdjenje 4, (običan) asimptotski razvoj Stieltjesove transformacije funkcije $f \in L^1_{loc}\ (0,\infty)$ koja ima asimptotski razvoj oblika $\sum_{i=1}^{\infty} \frac{a_i}{t^i}$, $t \to \infty$.

Recived by the editors August 16, 1989.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 129-133

Review of Research
Faculty of Science
Mathematics Series

REPRESENTATION FOR A CLASS OF NON – LINEAR FUNCTIONALS OF GAUSSIAN MARTINGALES

Zoran A. Ivković
Faculty of Mathematics, University of Belgrade,
Studentski trg 16, 11000 Beograd, Yugoslavia

Abstract

Let $\{X(t), t > 0\}$ be a continuous Gaussian martingale and let $\mathcal{H}^*$ be the mean-square linear closure of all the one-dimensional polynomials $\{P_n(X(t)),\ n = \overline{1,\infty}, t > 0\}$. For $Y \in \mathcal{H}^*$, there is the representation $Y = \int_0^\infty \Phi(t, X(t))\, dX(t)$, $\|\Phi(t, X(t))\| \in \mathcal{L}_2(\|dX(t)\|^2)$.

AMS Mathematics Subject Classification (1980): 60G12, 60G15.
Key words and phrases: Stochastic integral, Hermite polynomial.

1. Introduction

Let $\{X(t), t > 0\}, X(0) = 0$, be a real mean-square continuous martingale and let the Hilbert space $\mathcal{H}_1$ be the mean-square linear closure of $\{X(t), t > 0\}$. It is a well-known fact (see, for instance, [2], Ch. IX) that any $Y, Y \in \mathcal{H}_1$ has the representation

$$Y = \int_0^\infty \varphi(t) dX(t), \tag{1}$$

where the non-random function $\varphi(u), u > 0$, belongs to $\mathcal{L}_2(dF), F(t) = EX^2(t) = \|X(t)\|^2$. In this paper we shall consider the continuous Gaussian martingale $\{X(t), t > 0\}$ and the Hilbert space $\mathcal{H}^*$ -the mean-square linear closure of all the one-dimensional polynomials $\{P_n(X(t)), n = \overline{1,\infty}, t > 0\}$. (We assume that all the random variables are centered at the expectations). Remark that $\mathcal{H}^*$ is the subspace of $\mathcal{H}$ -the linear closure of all the polynomials $\{P_n(X(t_1)\ldots, X(t_n)), n = \overline{1,\infty}, t_1, \ldots, t_n > 0\}$. For example, the element $X(t)X(s) - F(min\{s,t\}), s \neq t$, does not belong to $\mathcal{H}^*$. We shall show that $Y \in \mathcal{H}^*$ has the representation

$$Y = \int_0^\infty \Phi(t, X(t))dX(t), \tag{2}$$

where $\|\Phi(t, X(t))\|$ belongs to $\mathcal{L}_2(dF)$. The integral of form (2) is considered in the sense of [2], Ch.IX.

The proof proceeds according to the technique of Hermite polynomials $H_p(X_1, \ldots, X_p)$ of Gaussian variables $X_1, \ldots, X_p$ (see, for instance, [3]). We write $H_p(X) = H_p\underbrace{(X_1, \ldots, X)}_{k}$

Lemma 1. *For* $0 < t < t_1, \Delta X = X(t_1) - X(t), p \geq 2$

$$[H_p(X(t) + \Delta X) - H_p(X(t)] - pH_{p-1}(X(t))\Delta X = \tag{3}$$

$$= A_2(\Delta X)^2 + \ldots + A_p(\Delta X))^p,$$

where A_k *are independent of* ΔX.

Proof. Recall the relation $H_p(X+Y) = \sum_{k=0}^{p} \left(\begin{array}{c} p \\ k \end{array}\right) H_p\underbrace{(X, \ldots, X}_{k}\underbrace{Y, \ldots,}_{p-k}$ [4], and the factorization property of Hermite polynomials: if X and Y are independent, then $H_p\underbrace{(X, \ldots, X}_{k}\underbrace{Y, \ldots, Y)}_{p-k} = H_k(X)H_{p-k}(Y)$. Since $X(t)$ and ΔX are independent, we have

$$H_p(X(t) + \Delta X) = H_p(X(t)) + pH_{p-1}(X(t))H_1(\Delta X) +$$

$$+ \left(\begin{array}{c} p \\ p-2 \end{array}\right) H_{p-2}(X(t))H_2(\Delta X) + \ldots$$

$$\ldots + pH_1(X(t))H_{p-1}(\Delta X) + H_p(\Delta X). \qquad \square$$

2. Representation

Proposition. 1. *The space $\mathcal{H}^*$ coincides with the set of integrals $\int_0^\infty \Phi(t, X(t))$ $dX(t)$, where the random function $\Phi(t, X(t))$ satisfies $\|\Phi(t, X(t))\| \in \mathcal{L}_2(dF)$ (It is not necessarily that $E\Phi = 0$).*

Proof. Let $\mathcal{H}_p^*, p = \overline{1, \infty}$, be the mean-square linear closure of $\{H_p(X(t)), t > 0\}(\mathcal{H}_1^* = \mathcal{H}_1)$. Then, by the orthogonality of Hermite polynomials of different degrees, it holds that

$$\mathcal{H}^* = \sum_{p=1}^{\infty} \bigotimes \mathcal{H}_p^* \tag{4}$$

Let E_t be the conditional expectation with respect to δ-field generated by $\{X(u), u \leq t\}$. From

$$E_s H_p(X(t)) = H_p(E_s X(t)) = H_p(X(s)), s < t, \text{ [3]} \tag{5}$$

it follows that $H_p(X(t)), t > 0\}$ is a martingale. In this way $\mathcal{H}_p^*$ coincides with the set of integrals

$$\int_0^\infty \varphi_p(t) dH_p(X(t)), \varphi_p(t) \in \mathcal{L}_2(\|dH_p(X(t))\|^2). \tag{6}$$

By Lemma $dH_p(X(t)) = pH_{p-1}(X(t))dX(t), H_0(\cdot) = 1.$ Then (6) becomes

$$\int_0^\infty \varphi_p(t) pH_{p-1}(X(t))dX(t). \tag{7}$$

Remark that the measure $\|dH_p(X(t))\|^2 = \|pH_{p-1}(X(t))\|^2 dF(t)$ is equivalent by absolute continuity to the measure $dF(t)$. Let $Y \in \mathcal{H}^*$. By (4) and (6)

$$Y = \sum_{p=1}^{\infty} \int_0^\infty \varphi_p(t) pH_{p-1}(X(t))dX(t) = \int_0^\infty \{\sum_{p=1}^{\infty} \varphi_p(t) pH_{p-1}(X(t))\} dX(t).$$

Let us denote $\Phi(t, X(t)) = \sum_{p=1}^{\infty} \varphi_p(t) pH_{p-1}(X(t))$. The random function $\Phi(t, X(t)), t > 0$, is such that $\|\Phi(t, X(t)\| \in \mathcal{L}_2(dF)$.
Conversely, consider the random function $\Phi(t, X(t)), t > 0$, such that $\int_0^\infty$ $\|\Phi(t, X\ (t))\|^2 dF(t) < \infty$.
For almost every t (with respect to dF), $\|\Phi(t, X(t))\|^2$ is finite. The random variable $\Phi(t, X(t)), t \in \sup dF$ is decomposable by the complete

orthogonal system $\{H_p(X(t)), p = \overline{0,\infty}\}$ in the space of all the random variables of the finite variances and measurable to $X(t)$. Let $\Phi(t, X(t)) = \sum_{p=0}^{\infty} \varphi_p(t) H_p(X(t))$ be this decomposition.
The integral

$$Y = \int_0^\infty \Phi(t, X(t)) dX(t) = \int_0^\infty \{\sum_{p=0}^{\infty} \varphi_p(t) H_p(X(t))\} dX(t) =$$

$$= \sum_{p=0}^{\infty} \int_0^\infty \varphi_p(t) \frac{1}{p+1} dH_{p+1}(X(t))$$

belonges to $\mathcal{H}^*$. □

3. Processes as curves in $\mathcal{H}^*$. Linear completeness.

Let us denote $\mathcal{H}_t^* = E_t \mathcal{H}^*$. By (5) the subspace $\mathcal{H}_t^*$ is the linear closure of $\{P_n(X(u)), n = \overline{1,\infty}, u \leq t\}$ and $\mathcal{H}_p^*$ reduces the family of projections $\{E_t, t > 0\}$. It follows immediately that the spectral type of $\{E_t, t > 0\}$ in $\mathcal{H}^*$ is dF and the multiplicity of dF is ∞. (For these notions see, for instance, the classical paper [1]). Consider the process $\{Y(t), t > 0\}$in æ* defined by

$$Y(t) = \int_0^t \Phi(t, u, X(u)) dX(u), \|\Phi(t, \cdot, X(\cdot)\| \in \mathcal{L}_2(dF).$$

Let us benote by $\mathcal{H}_1(Y; t)$ then mean-square linear closure of $\{Y(u), u \leq t\}$. We say that the process $\{Y(t)\}$ is linearly complete in $\mathcal{H}^*$ if $\mathcal{H}_1(Y; t) = \mathcal{H}_t^*$ for each $t > 0$. The characterization of the completenes of $\{Y(t)\}$ in the terms of $\Phi(t, u, X(u))$ seems to be of some interest. We say that the family of random functions $\{\Phi(t, u, X(u)), t > 0\}$ is complete in $\mathcal{H}^*$ if from $\int_0^t < \Phi(t, u, X(u), \Psi(u, X(u)) > dF(u) = 0$, for each $t > 0$, follows that $\in t_0^\infty \|\Psi(u, X(u))\|^2 dF(u) = 0$. ($< \cdot, \cdot >$ denotes the inner product in $\mathcal{H}^*$.

Proposition. 2. *The relation $\mathcal{H}_1(Y; t) = \mathcal{H}_t^*$ for each $t > 0$ holds if and only if the family $\{\Phi(t, u, X(u)), t > 0\}$ is complete in $\mathcal{H}^*$.*

Proof. Let $Z = \int_0^\infty \Psi(u, X(u)) dX(u) \in \mathcal{H}^*$. Then $< Y(t), Z >= \int_0^t < \Phi(t, u, X(u), \Psi(u, X(u)) > dF(u)$. If $< Y(t), Z >= 0$ for each $t > 0$, then Z

is orthogonal to $\mathcal{H}_1(Y)$. So, if $Z \in \mathcal{H}_1(Y;t)$ it follows that $\|Z\|^2 = \int_0^\infty \|\Psi(u, X(u))\|^2 dF(u) = 0$ or the family $\{\Phi(t; u, X(u)), t > 0\}$ is complete in $\mathcal{H}^*$. Conversely, if $\{\Phi(t, u, X(u)), t > 0\}$ is complete, it follows that necessarily $Z = 0$ or $\mathcal{H}_1(Y;t) = \mathcal{H}_t^*$. Remark that the completeness of $\{\Phi(t, u, X(u)), t > 0\}$ in $\mathcal{H}^*$ is equivalent to the completeness of the system of non-random functions $\{\varphi_p(t, n), t > 0, p = \overline{1, \infty}\}in\mathcal{H}^*$, because $\Phi(t, u, X(u)) = \sum_{p=1}^\infty \varphi_p(t, u) p H_{p-1}(X\ (u))$.

References

[1] H. Cramér: Stochastic processes as curves in Hilbert Sspaces, Probability Theory and its Applications, Vol. 9 (1964), 195-204.

[2] J.L.Doob: Stochastic Processes, John Wiley and Sons, 1953.

[3] Z. Ivković and Z. Lozanov: Hermite Polynomials of Gaussian Processe, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 12(1982), 105-115.

[4] Z. Ivković: On Hermite polynomials of gaussion Markov processes and fields, to apear in Bull. Acad. Sci. Arts No. 17 (Belgrade).

REZIME

REPREZENTACIJA JEDNE KLASE NELINEARNIH FUNKCIONALA GAUSOVSKIH MARTINGALA

Neka je $\{X(t), t > 0\}$ neprekidni Gausovski martingal i neka je $\mathcal{H}^*$ srednje kvadratna linearna zatvorenost svih jedno-dimenzionalnih polinoma $\{P_n(X(t)), n =$ $\overline{1, \infty}, t > 0\}$. Za $Y \in \mathcal{H}^*$ važi reprezentacija $Y = \int_0^\infty \Phi(t, X(t)) dX(t), \|\Phi(t, X(t))\|$ $\mathcal{L}_2(\|dX(t)\|^2)$.

Recived by Editors July 15, 1989.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 135-144

Review of Research
Faculty of Science
Mathematics Series

AN INTEGRAL GENERATED BY A DECOMPOSABLE MEASURE

Endre Pap
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

An integral using the pseudo-addition $\oplus$, pseudo-multiplication $\otimes$ and the $\bigoplus$ –decomposable measure is introduced. The method used is similar to the procedure of the construction of Lebesgue integral.

AMS Mathematics Subject Classification (1980): 28A25
Key words and phrases: Pseudo-addition, pseudo-multiplication, decomposable measure.

1. Introduction

There are many different ways for defining integrals with respect to non-additive set functions (some of them in [1], [2], [3], [4], [7], [8], [9]). We continue our investigations of $\bigoplus$ –decomposable measures initiated in paper [5] based on the pseudo-addition $\oplus$ and pseudo-multiplication $\otimes$ defined on $[a,b] \subset [-\infty,+\infty]$. In this paper we shall examine the corresponding integral using a construction similar to that of the Lebesgue integral.

2. Decomposable measures

Let $[a,b]$ be a closed (in some cases semiclosed) subinterval of $[-\infty,+\infty]$. We shall consider a partial order $\leq$ on $[a,b]$, which can be the usual order of the real line, but it can also be another order. All future considerations will be with respect to the order $\leq$.

The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a,b]\times[a,b] \to [a,b]$ which is commutative, nondecreasing (with respect to $\leq$) associative and either a or b is a zero element, denoted by 0, i.e. for each $x \in [a,b]$

$$0 \oplus x = x \quad \text{holds.}$$

Let $[a,b]_+ = \{x : x \in [a,b], \ \ x \geq 0\}$.

The operation $\otimes$ (pseudo-multiplication) is a function $\otimes : [a,b]\times[a,b] \to [a,b]$ which is commutative, positively nondecreasing, i.e. $x \leq y$ implies $x \otimes z \leq y \otimes z, \ z \in [a,b]_+$, associative and for which there exist a unit element $\mathbf{1} \in [a,b]$, i.e. for each $x \in [a,b]$

$$\mathbf{1} \otimes x = x.$$

We suppose, further, $0 \otimes x = 0$ and that $\otimes$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e.

$$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z).$$

Examples can be found in paper [5]. Some of them are:

$$x \oplus y = (x^p + y^p)^{\frac{1}{p}},\ p > 0 \quad \text{and} \quad x \otimes y = x \cdot y \ \text{on } [0,+\infty]$$

$$\text{or} \quad x \oplus y = \max\{x,y\} \quad \text{and} \quad x \otimes y = x + y \ \text{on } [-\infty,+\infty).$$

Pseudo-addition $\oplus$ is idempotent if for any $x \in [a,b]$

$$x \oplus x = x \quad \text{holds.}$$

Let X be a non-empty set. Let $\sum$ be a $\sigma-$algebra of subsets of X.

A set function $m : \sum \to [a,b]_+$ (or semiclosed interval) is a $\bigoplus-$decomposable measure if there hold

$$m(\emptyset) = 0 \quad \text{(if } \oplus \text{ is not idempotent)}$$

$$m(A \cup B) = m(A) \oplus m(B)$$

for $A, B \in \sum$ such that $A \cap B = \emptyset$.
In the case when $\oplus$ is idempotent, it is possible that m is not defined on the empty set.

A $\bigoplus$ −decomposable measure m is σ - $\bigoplus$ −decomposable if

$$m(\bigcup_{i=1}^{\infty} A_i) = \bigoplus_{i=1}^{\infty} m(A_i)$$

hold for any sequence (A_i) of pairwise disjoint sets from $\sum$.

Proposition 1. *If $m : \sum \to [a,b]_+$ is a $\bigoplus$ −decomposable measure with respect to the idempotent pseudo-addition $\oplus$, then we have*

$$m(A \cup B) = m(A) \oplus m(B)$$

for any $A, B \in \sum$.

3. Integral

Let m be a σ - $\bigoplus$ −decomposable measure. A function $f : X \to [a,b]$ is measurable from below if for any $c \in [a,b]$ the sets $\{x : f(x) \leq c\}$ and $\{x : f(x) < c\}$ belong to $\sum$. f is measurable, if it is measurable from below and the sets $\{x : f(x) \geq c\}$ and $\{x : f(x) > c\}$ belong to $\sum$.

Let f and g be two functions defined on X and with values in $[a,b]$. Then, we define for any $x \in X$

$$(f \oplus g)(x) = f(x) \oplus g(x) \ ,$$

$$(f \otimes g)(x) = f(x) \otimes g(x)$$

and for any $c \in [a,b]$

$$(c \otimes f)(x) = c \otimes f(x).$$

We suppose further that $([a,b],\oplus)$ and $([a,b],\otimes)$ are complete lattice ordered semigroups. A complete lattice means that for each set $A \subset [a,b]$ bounded from above (below) there exists $\sup A$ ($\inf A$). Further, we suppose that $[a,b]$ is endowed with a metric d compatible with sup and inf and which satisfies at laest one of the following conditions:

(a) $d(x \oplus y,\ x' \oplus y') \le d(x,x') + d(y,y')$

(b) $d(x \oplus y,\ x' \oplus y') \le \max\{d(x,x'),\ d(y,y')\}$.

Both conditions (a) and (b) imply that :

$$d(x_n, y_n) \to 0 \quad \text{implies} \quad d(x_n \oplus z,\ y_n \oplus z) \to 0.$$

Condition (b) implies

$$d(\bigoplus_{i=1}^{n} x_i,\ \bigoplus_{j=1}^{n} y_j) \le \min_j \max_i d(x_i, y_j).$$

We suppose further the monotonicity of the metric d, i.e.

$$x \le z \le y \quad \text{implies} \quad d(x,y) \ge \max\{d(y,z), d(x,z)\}.$$

Let ε be a positive real number, and $B \subset [a,b]$. A subset $\{l_i^\varepsilon\}$ is a $\varepsilon-$net if for each $x \in B$ there exists l_i^ε such that $d(l_i^\varepsilon, x) \le \varepsilon$. If we have $l_i^\varepsilon \le x$, then we shall call $\{l_i^\varepsilon\}$ a lower $\varepsilon-$net. If $l_i^\varepsilon \le l_{i+1}^\varepsilon$ holds, then $\{l_i^\varepsilon\}$ is monotone.

We define the characteristic function

$$\chi_A(x) = \begin{cases} 0, & x \notin A \\ 1, & x \in A\ . \end{cases}$$

A mapping $e : X \to [a,b]$ is an elementary (measurable) function if it has the following representation

$$e = \bigoplus_{i=1}^{\infty} a_i \otimes \chi_{A_i} \quad \text{for} \quad a_i \in [a,b]$$

and $A_i \in \sum$ disjoint if $\oplus$ is not idempotent.

Theorem 1. *Let $f : X \to [a,b]$ be a measurable from below function if the pseudo-addition is idempotent, or f is measurable and for each positive real number ε there exists a monotone $\varepsilon-$net in $f(X)$. Then, there exist a sequence (φ_n) of elementary functions such that, for each $x \in X$,*

$$d(\varphi_n(x), f(x)) \to 0 \quad \textit{uniformly as} \quad n \to \infty.$$

Proof. Suppose first that $\oplus$ is not idempotent.We take a lower monotone $\varepsilon-$net $\{f_i^\varepsilon\}$ on $f(X)$.

Let

$$e^\varepsilon = \bigoplus_{i=1}^{\infty} f_i^\varepsilon \otimes \chi_{X_i^\varepsilon},$$

where $X_i^\varepsilon = \{x : f_{i+1}^\varepsilon > f(x) \geq f_i^\varepsilon\}$.
For each point x from X there exists $f_i^\varepsilon(x)$ such that

$$d(f_i^\varepsilon(x), f(x)) \leq \varepsilon,$$

where $f_i^\varepsilon(x) = f_i^\varepsilon \otimes \chi_{X_i^\varepsilon}(x)$. Hence, by the monotonicity of d and $e^\varepsilon(x) \geq f_i^\varepsilon(x)$, we obtain

$$d(e^\varepsilon(x), f(x)) < \varepsilon, \quad x \in X.$$

Taking $\varepsilon = \frac{1}{n}$ we define the desired sequence (φ_n) as $\varphi_n = e^{\frac{1}{n}}$.

If the pseudo-addition $\oplus$ is idempotent, then we take in the preceding procedure a lower $\varepsilon-$net $\{f_i^\varepsilon\}$ and

$$X_i^\varepsilon = \{x : f(x) \geq f_i^\varepsilon\}.$$

We have used that $e^\varepsilon(x) \leq f(x)$ holds, which follows by

$$e^\varepsilon(x) \leq e^\varepsilon(x) \oplus f(x) \leq \bigoplus_{i=1}^{\infty}((f_i^\varepsilon \otimes \chi_{X_i^\varepsilon}(x)) \oplus f(x)) \leq f(x).$$

The integral of a simple function $s = \bigoplus_{i=1}^{n} a_i \otimes \chi_{A_i}$ for $a_i \in [a,b]$ with disjoint $A_1, A_2, ...A_n$, if $\oplus$ is not idempotent, is defined by

$$\int_X^{\oplus} s \otimes dm := \bigoplus_{i=1}^{n} a_i \otimes m(A_i).$$

The integral of an elementary function

$$e = \bigoplus_{i=1}^{\infty} a_i \otimes \chi_{A_i} \quad \text{for} \quad a_i \in [a,b] \ (i \in N) \text{ with } (A_i)$$

disjoint if $\oplus$ is not idempotent, is defined by

$$\int_X^{\oplus} e \otimes dm := \bigoplus_{i=1}^{\infty} a_i \otimes m(A_i). \tag{1}$$

The integral of a bounded measurable (from below for $\oplus$ idempotent) function $f : X \to [a,b]$, for which, if $\oplus$ is not idempotent for each $\varepsilon > 0$, there exists a monotone ε-net in $f(X)$, is defined by

$$\int_X^{\oplus} f \otimes dm := \lim_{n\to\infty} \int_X^{\oplus} \varphi_n(x) \otimes dm, \tag{2}$$

where (φ_n) is the sequence of elementary functions constructed in Theorem 1.

Theorem 2.. *The integral defined in (2) is independent of the choice of sequence* (φ_n).

Theorem 3. *Let* $\oplus$ *and* $\otimes$ *be continuous and* $\oplus$ *infinitely commutative and associative. Then the integrals defined by (1) and (2) have the following properties:*

(i) $\int_X^{\oplus}(f \oplus g) \otimes dm = \int_X^{\oplus} f \otimes dm \oplus \int_X^{\oplus} g \otimes dm$,

(ii) $\int_X^{\oplus}(c \otimes f) \otimes dm = c \otimes \int_X^{\oplus} f \otimes dm$

for any $c \in [a,b]$.

Proof. (*i*) Let f and g be elementary functions, i.e.

$$f = \bigoplus_{i=1}^{\infty}(a_i \otimes \chi_{A_i}), \quad g = \bigoplus_{i=1}^{\infty} b_i \otimes \chi_{B_i},$$

where (A_i) and (B_i) are (if $\oplus$ is not idempotent, disjoint) partitions of X in $\sum$. Hence, $f \oplus g$ is also an elementary function and

$$f \oplus g = \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} (a_i \oplus b_j) \otimes \chi_{A_i \cap B_j}.$$

By (1) we obtain

$$\int_X^{\oplus} (f \oplus g) \otimes dm = \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} (a_i \oplus b_j) \otimes m(A_i \cap B_j) =$$

$$= \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} (a_i \otimes m(A_i \cap B_j)) \oplus \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} b_j \otimes m(A_i \cap B_j) =$$

$$= \bigoplus_{i=1}^{\infty} (a_i \otimes m(A_i \cap \cup_{j=1}^{\infty} B_j)) \oplus \bigoplus_{j=1}^{\infty} (b_j \otimes m(\cup_{i=1}^{\infty} A_i \cap B_j) =$$

$$= \bigoplus_{i=1}^{\infty} (a_i \otimes m(A_i)) \oplus \bigoplus_{j=1}^{\infty} (b_j \otimes m(B_j)) =$$

$$= \int_X^{\oplus} f \otimes dm \oplus \int_X^{\oplus} g \otimes dm$$

Now, let f and g be measurable (from below if $\oplus$ is idempotent). Let (φ_n) and (ψ_n) be the corresponding sequences from Theorem 1 to f and g, respectively. The integral

$$\int_X^{\oplus} (f \oplus g) \otimes dm$$

exists, since it can be defined by the sequence $(\varphi_n(x) \oplus \psi_n(x))$

$$\int_X^{\oplus} (f \oplus g) \otimes dm = \lim_{n \to \infty} \int_X^{\oplus} (\varphi_n(x) \oplus \psi_n(x)) \otimes dm$$

and $(\varphi_n(x))$ and $(\psi_n(x))$ satisfy, for any x,

$$d(\varphi_n(x), f(x)) \to 0 \quad \text{and} \quad d(\psi_n(x), g(x)) \to 0.$$

Hence, since d satisfies (a) or (b)

$$d(\varphi_n(x) \oplus \psi_n(x),\ f(x) \oplus g(x)) \to 0.$$

Now, we have

$$\int_X^{\oplus}(f\oplus g)\otimes dm=\lim_{n\to\infty}\int_X^{\oplus}(\varphi_n\oplus\psi_n)\otimes dm=$$

$$=\lim_{n\to\infty}(\int_X^{\oplus}\varphi_n\otimes dm\oplus\int_X^{\oplus}\psi_n\otimes dm)=$$

$$=\lim_{n\to\infty}\int_X^{\oplus}\varphi_n\otimes dm\oplus\lim_{n\to\infty}\int_X^{\oplus}\psi_n\otimes dm=$$

$$=\int_X^{\oplus}f\otimes dm\oplus\int_X^{\oplus}g\otimes dm.$$

Property (ii) easily follows by the continuity of $\otimes$.

Example 1. *For any function g bounded above we can define*

$$m(A)=\sup_{x\in A}g(x)\qquad A\in\mathcal{B},$$

where $\mathcal{B}$ is the Borel $\sigma-$algebra on $[-\infty,\infty)$.
Taking $\oplus=\max=\sup$, $\otimes=+$, we obtain

$$\int_{\mathbf{R}}^{\oplus}f\otimes dm=\sup_{x\in\mathbf{R}}(f(x)+g(x)),$$

f bounded above.

If $\oplus$ is a strict pseudo-addition with a monotone generator g, $g\circ m:\sum\to[0,g(c)]$ and $c\in[a,b]$ is an additive measure then we have (see [3], [5]*) for the simple function*

$$\int_X^{\oplus}s\otimes dm=g^{-1}(\sum_{i=1}^{n}g(a_i)\cdot(g\circ m)(A_i))$$

and for the measurable function f

$$\int_X^{\oplus}f\otimes dm=g^{-1}(\int_X(g\circ f)\cdot dx),$$

where $dx=d(g\circ m)$ is the Lebesgue measure and $u\otimes v=g^{-1}(g(u)\cdot g(v))$.

Example 2. ([5]) *If $c>0$, then we define*

$$u \oplus v = -c\ln(e^{-\frac{u}{c}} + e^{-\frac{v}{c}}) \quad \textit{and}$$

$$u \otimes v = u + v.$$

The corresponding integral is

$$\int_R^{\oplus} f \otimes dm = -c\ln \int_{\mathbf{R}} e^{(-\frac{f}{c})} dx.$$

References

[1] D'Apuzzo, L., Squillante, M., Ventre, A.G.S., A survey on decomposable measures, Control and Cybernetics (to appear).

[2] Choquet, G., Theory of capacities, Ann. Inst. Fourier (Grenoble), 5 (1953-1954), 131-292.

[3] Ichihashi, H., Tanaka, M., Asai,K., Fuzzy Integrals Based on Pseudo-additions and Multiplications, J. Math. Anal. Appl. 130 (1988), 354 -364.

[4] Maslov, V.P., Asymptotical methods for solving pseudo-differential equations, (in Russian), Moscow, 1987.

[5] Pap, E., Decomposable measures and applications on nonlinear partial differential equations, Rend. del Circolo Mat. di Palermo (to appear).

[6] Schweizer, B., Sklar, A., Associative functions and abstract semigroups, Publ. Math. Debrecen 10 (1963), 69-81.

[7] Sugeno, M., Theory of fuzzy integrals and its applications, Ph.D. Thesis, Tokyo, Institute of Technology, 1974.

[8] Weber, S., $\perp$−decomposable measures and integrals for Archimedean t−conorm, J. Math. Anal.Appl. 101 (1984), 114-138.

[9] Weber, S., Measures of fuzzy sets and measures of fuzzines, Fuzzy Sets and Systems 13 (1984), 247-271.

[10] Weber, S., Two integrals and some modified version - critical remarks, Fuzzy Sets and Systems 20 (1986), 97-105.

REZIME

INTEGRAL GENERISAN DEKOMPOZABILNOM MEROM

Uveden je integral pomoću pseudo-sabiranja $\oplus$, pseudo-množenja $\otimes$ i $\bigoplus$–-dekompozabilne mere. Korišćena je metoda bliska konstrukciji Lebesgueovog integrala.

Received by the editors November 15, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 145-153

Review of Research
Faculty of Science
Mathematics Series

CONTINUITIES OF MULTI-PRESEMINORMS

Árpád Száz
Institute of Mathematics,
Lajos Kossuth University,
H-4010 Debrecen, Hungary

Abstract

By using the results of [8], it is shown that a multi-preseminorm on a product preseminormed space is boundedly uniformly continuous if and only if it is continuous at the origin.

A particular case of this assertion allows the easy proof of some essential improvements of the fundamental theorems of [1, p. 72] and [7] on continuities of multilinear maps.

AMS Mathematical Subject Classification (1980): 46Axx
Key words and phrases: Preseminormed spaces, bounded nets, bounded uniform continuity, multi-preseminorms, multilinear maps.

1. Prerequisites

Instead of topological vector spaces it is ofter more convenient to use preseminormed spaces [4].

A preseminormed space over $K = R$ or C is a pair $X(\mathcal{P}) = (X, \mathcal{P})$ consisting of a vector space X over K and a nonvoid family $\mathcal{P}$ of preseminorms on X. A preseminorm on X is a subadditive real-valued function p on X such that $p(\lambda x) \leq p(x)$ for all $|\lambda| \leq 1$ and $x \in X$, and $\lim_{\lambda \to 0} p(\lambda x) = 0$ for all $x \in X$. Note that these latter properties imply, in particular, that $p(x) \geq p(0) = 0$

for all $x \in X$, and $p(\lambda x) = p(x)$ for all $| \lambda | = 1$ and $x \in X$.
Among the various basic tools definiable in $X(\mathcal{P})$, we shall actually need only null nets and bounded nets [8].
A net (x_α) in $X(\mathcal{P})$ is a null net if $\lim_\alpha p(x_\alpha) = 0$ for all $P \in \mathcal{P}$. Moreover, a net (x_α) in $X(\mathcal{P})$ converges to a point x in $X(\mathcal{P})$ if $(x_\alpha - x)$ is a null net. And two nets (x_α) and (y_α) in $X(\mathcal{P})$ are coherent [6], if $(x_\alpha - y_\alpha$ is a null net. Note that several useful properties of null nets in $X(\mathcal{P})$ can be easily derived from the usual properties of null nets in R by using the above properties of preseminorms.
A net (x_α) in $X(\mathcal{P})$ is bounded if

$$\lim_{\lambda \to 0} \overline{\lim}_\alpha p(\lambda x_\alpha) = 0$$

for all $p \in \mathcal{P}$. Moreover, a nonvoid subset A of $X(\mathcal{P})$ is bounded if the identity function of A is bounded as a net whenever A is considered to be directed, such that $x \leq y$ for all $x, y \in A$. Note that A is therefore bounded, if and only if

$$\lim_{\lambda \to 0} \sup_{x \in A} p(\lambda x) = 0$$

for all $p \in \mathcal{P}$. And thus nets contained in bounded subsets of $X(\mathcal{P})$ are bounded.
Under this new definition all Cauchy nets in $X(\mathcal{P})$ become bounded. And null nets and bounded nets in $X(\mathcal{P})$ become equivalent tools. Since a net (x_α) in $X(\mathcal{P})$ is bounded if and only if $(\lambda_\beta y_\beta)$ is a null net in $X(\mathcal{P})$, whener (λ_β) is a null net in K and (y_β) is a subnet of (x_α). And a net (x_α) in $X(\mathcal{P})$ is a null net if and only of each subnet (y_β) of (x_α) has a subnet (z_γ) for which there exists an unbounded net (λ_γ) in K such that $(\lambda_\gamma z_\gamma)$ is still a bounded net in $X(\mathcal{P})$. Having the above definition, it is also possible to introduce an important new continuity property of functions.
A function f from a subset D of $X(\mathcal{P})$ into another preseminormed space $Y(\mathcal{Q})$ is boundedly uniformly continuous, if $(f(x_\alpha))$ and $(f(y_\alpha))$ are coherent nets in $Y(\mathcal{Q})$, whenever (x_α) and (y_α) are bounded coherent nets in D. Recall that f may be called uniformly continuous if it takes coherent nets into coherent nets. Thus, if f uniformly continuous, then f is also boundedly uniformly continuous. Moreover, note that if f is boundedly uniformly continuous, then f is necessarily continuous and the restrictions of f to bounded subsets of D are uniformly continuous.
In the sequel, we shall also need a straightforward notion of a product preseminormed space from [5].

If $X_i(\mathcal{P}_i)$ is a preseminormed space for each i in a nonvoid set I, and moreover

$$X = X_{i\in I} X_i \text{ and } \mathcal{P} = \bigcup_{i\in I} \mathcal{P}_i o \pi_i,$$

where π_i is the projection of X onto X_i and $\mathcal{P}_i o\pi_i = \{po\pi_i : p \in \mathcal{P}_i\}$, then the preseminormed space $X(\mathcal{P})$ is the Cartesian product of the spaces $X_i(\mathcal{P}_i)$ and the notation

$$X(\mathcal{P}) = X_{i\in I} X_i(\mathcal{P}_i)$$

is used. An important consequence of this definition is that a net (x_α) in $X(\mathcal{P})$ is a null net (bounded net), if and only if each of its coordinate nets $(x_{\alpha i})$ is a null net (bounded net).

2. Multi-preseminorms

To estabilish easily the continuity properties of multilinear maps, it seems convenient to introduce an appropriate notion of a multi-preseminorm.

Definition 2.1. *Let X_i be a vector space over K for all $i = 1, 2, ..., n$ and*

$$X = X_{i=1}^n X_i.$$

For each $x = (x_i) \in X$ and $i = 1, 2, ..., n$, denote by ϕ_{xi} the function defined on X_i by

$$\phi_{xi}(t) = (x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_n).$$

A real-valued function p on X will be called a multi- preseminorm on X if
(i) $p_{xi} = po\phi_{xi}$ is a preseminorm on X_i for all $x \in X$ and $i = 1, 2, ..., n$.
(ii) $p_{xi}(\lambda x_i) = p_{xk}(\lambda x_k)$ for all $\lambda \in K, x \in X$ and $i, k = 1, 2, ..., n$.

This rather restrictive definition of a multi-preseminorm is mainly motivated by the next simple example which can easily be extended to multilinear relations.

Example 2.2. If f is a multilinear map of $X = X_{i=1}^{n} X_i$ into a vector space Y and q is a preseminorm on Y, then $p = qof$ is a multi-preseminorm on X.

Another ample reason for having Definition 2.1. is that it allows us to naturally extend several useful properties of preseminorms to multi-preseminorms. First, we shall prove a rather complicated, but important inequality for multi-preseminorms whose one-dimensional particular case is, however, quite trivial.

Lemma 2.3. *If p is a multi-preseminorm on $X = X_{i=1}^{n} X_i$, and moreover $x, y \in X$ and $I = \{1, 2, ..., n\}$, then*

$$p(x) \leq \sum_{A \subset I} p(\chi_A(x-y) + (\chi_I - \chi_A)y),$$

where χ_A is the characteristic function of A on I and the multiplication is taken in the usual pointwise sense.

Proof. By Definition 2.1., it is clear that

$$p(x) \leq p(x_1, ..., x_{n-1}, x_n - y_n) + p(x_1, ..., x_{n-1}, y_n),$$

and the mapping

$$z \epsilon X_{i=1}^{n-1} X_i \to p(z_1, ..., z_{n-1}, s)$$

is a multi-preseminorm on $X_{i=1}^{n-1} X_i$ for all $s \in X_n$.
Thus, if the corresponding inequality holds for $n-1$, then

$$\begin{aligned} p(x) \leq & \sum_{A \subset I \setminus \{n\}} p(\chi_A(x-y) + (\chi_{I \setminus \{n\}} - \chi_A)y + \chi_{\{n\}}(x-y)) + \\ & + \sum_{A \subset I \setminus \{n\}} p(\chi_A(x-y) + (\chi_{I \setminus \{n\}} - \chi_A)y + \chi_{\{n\}}y) = \\ & = \sum_{A \subset I \setminus \{n\}} p(\chi_{A \cup \{n\}}(x-y) + (\chi_I \setminus \chi_{A \cup \{n\}})y) + \\ & \sum_{A \subset I \setminus \{n\}} p(\chi_A(x-y) + (\chi_I - \chi_A)y) \end{aligned}$$

whence the stated inequality follows immediately.

Remark 2.4. Note that this lemma depends only on the multi-subadditivity of p.

Next, we shall prove a very unusual statement about continuities of multi-preseminorms which depends only on the easy consequence of Definition 2.1. that $p(\lambda x) = p(x)$ whenever $\lambda \epsilon K^n$, such that $\prod_{i=1}^n \mid \lambda_i \mid = 1$.

Lemma 2.5. *If p is a multi-preseminorm on the product preseminormed space*

$$X(\mathcal{P}) = X_{i=1}^n X_i(\mathcal{P}_i),$$

such that p is continuous at the origin of $X(\mathcal{P})$ and (x_α) is a bounded net in $X(\mathcal{P})$, such that $(x_{\alpha i})$ is a null net in $X_i(\mathcal{P}_i)$ for some $i = 1, 2, ..., n$, then $(p(x_\alpha))$ is a null net in R.

Proof. To prove this, we need only show that each subnet $(p(y_\beta))$ of $(p(x_\alpha))$ has a subnet $(p(z_\gamma))$ such that $(p(z_\gamma))$ is a null net in R. If (y_β) is a subnet of (x_α), then, because of the assumption that $(x_{\alpha i})$ is a null net in $X_i(\mathcal{P}_i)$, it is clear that $(y_{\beta i})$ is also a null net in $X_i(\mathcal{P}_i)$. Thus, by [8, Lemma 3.3], there exist a subnet (z_γ) of (y_β) and a net (λ_γ) of positive numbers with $\lim_\gamma \lambda_\gamma = \infty$ such that $(\lambda_\gamma z_{\gamma i})$ is still a null net in $X_i(\mathcal{P}_i)$. Define $\omega_\gamma \epsilon R^n$ for all γ such that

$$\omega_{\gamma i} = \lambda_\gamma \quad \text{and}$$

$$\omega_{\gamma k} ={}^{n-1}\sqrt{\lambda_\gamma^{-1}} \quad \text{if } k \in \{1, 2, ..., n\} \setminus \{i\}.$$

Then, because of $\prod_{k=1}^n \omega_{\gamma k} = 1$, we clearly have

$$p(z_\gamma) = p(\omega_\gamma z_\gamma)$$

for all γ. Moreover, because of the assumed boundedness of (x_α) and some basic properties of bounded nets listed in Section 1, it is clear that $(\omega_\gamma z_\gamma)$ is a null net in $X(\mathcal{P})$. Thus, because of the assumed continuity of p at o and the above equality, $(p(z_\gamma))$ is a null net in R, and the proof is complete.

Remark 2.6. For an easy illustration of this lemma, one can at once consider the bipreseminorm p defined on R^2 by $p(x) = \mid x_1 \mid\mid x_2 \mid$.

Combining Lemma 2.3. and Lemma 2.5., we can now easily prove the following useful criterion for the bounded uniform continuity of multi-preseminorms.

Theorem 2.7. *If p is a multi-preseminorm on the product preseminormed space*

$$X(\mathcal{P}) = X_{i=1}^n X_i(\mathcal{P}_i),$$

then the following assertions are equivalent:
(i) p is boundedly uniformly continuous;
(ii) p is continuous at the origin of $X(\mathcal{P})$.

Proof. Since the implication $(i) \Rightarrow (ii)$ is apparent, we need only show that (ii) also implies (i). For this, assume that (ii) holds and (x_α) and (y_α) are bounded coherent nets in $X(\mathcal{P})$. Then, by Lemma 2.3., we clearly have

$$| p(x_\alpha) - p(y_\alpha) | \leq \sum_{\emptyset \neq A \subset I} (p(\chi_A(x_\alpha - y_\alpha) +$$

$$+ (\chi_I - \chi_A) y_\alpha) + p(\chi_A(y_\alpha - x_\alpha) + (\chi_I - \chi_A) x_\alpha))$$

for all α. Hence, by the assumption and Lemma 2.5., it is clear that

$$\lim_\alpha | p(x_\alpha) - p(y_\alpha) | = 0.$$

That is, $(p(x_\alpha))$ and $(p(y_\alpha))$ are coherent nets in R, and thus (i) also holds.

Remark. To see that this theorem includes Lemma 2.5., note that if (x_α) is as in Lemma 2.5., then by defining $y_\alpha = (\chi_I - \chi_{\{i\}}) x_\alpha$ for all α, we get a bounded net (y_α) in $X(\mathcal{P})$ such that (x_α) and (y_α) are coherent nets in $X(\mathcal{P})$ and $p(y_\alpha) = 0$ for all α.

3. An application

By using a convenient reformulation of [1, (18.3) Lemma] and a particular case of Lemma 2.5., we can now easily prove as essential improvement of [1, (18.2) Theorem].

Theorem 3.1. *If f is a multilinear map from a product preseminormed space*

$$X(\mathcal{P}) = X_{i=1}^n X_i(\mathcal{P}_i)$$

into an arbitrary preseminormed space $Y(\mathcal{Q})$, *then the following assertions are equivalent:*
(i) f is boundedly uniformly continuous;
(ii) f is continuous at the origin of $X(\mathcal{P})$;
(iii) $q o f$ *is continuous at the origin of* $X(\mathcal{P})$ *for all* $q \in \mathcal{Q}$.

Proof. It is clear that (i) implies (ii), and (ii) implies (iii).
To prove that (iii) also implies (i), assume that (iii) is true, and let (x_α) and (y_α) be bounded coherent nets in $X(\mathcal{P})$, and moreover $q \in \mathcal{Q}$ and $p = qop$. Then, similarly as in Lemma 2.3., we have

$$f(x_\alpha) - f(y_\alpha) = \sum_{\emptyset \neq A \subset I} f(\chi_A(x_\alpha - y_\alpha) + (\chi_I - \chi_A)y_\alpha),$$

and hence

$$q(f(x_\alpha) - f(y_\alpha)) \leq \sum_{\emptyset \neq A \subset I} p((\chi_A(x_\alpha - y_\alpha) + (\chi_I - \chi_A)y_\alpha)$$

for all α. On the other hand, by Lemma 2.5., it is clear that

$$\lim_\alpha p(\chi_A(x_\alpha - y_\alpha) + (\chi_I - \chi_A)y_\alpha) = 0,$$

whenever $\emptyset \neq A \subset I$. Consequently, we have

$$\lim_\alpha q(f(x_\alpha) - f(y_\alpha)) = 0.$$

And this proves (i).

Remark 3.2. By [8, Theorems 2.2, 2.11 and 2.13] it is clear that a boundedly uniformly continuous map also preserves Cauchy nets.
Thus, the above theorem also improves the important particular case of a recent theorem of Jürg Rätz [3, Theorem 11] when the valuated field is K. For an easy illustration of this theorem, we can at once state

Theorem 3.3. *If $X(\mathcal{P})$ is a preseminormed space over K, then the scalar multiplication defined on $K \times X$ by*

$$(\lambda, x) \to \lambda x$$

is a boundedly uniformly continuous map of $K(||) \times X(\mathcal{P})$ into $X(\mathcal{P})$ which is uniformly continuous if and only if $\mathcal{P} = \{0\}$.

Proof. If $((\lambda_\alpha, x_\alpha))$ is a null net in $K(||) \times X(\mathcal{P})$ and $p \in \mathcal{P}$, then because of the inequality

$$p(\lambda_\alpha x_\alpha) \leq ([| \lambda_\alpha |] + 1)p(x_\alpha)$$

where [ ] denotes the integer part, it is clear that $(\lambda_\alpha x_\alpha)$ is a null net in $X(\mathcal{P})$. Thus, by Theorem 3.1, the scalar multiplication in $X(\mathcal{P})$ is always boundedly uniformly continuous.

On the other hand if $\mathcal{P} \neq \{0\}$, then there exist $p \in \mathcal{P}$ and $x \in X$ such that $p(x) \neq 0$. Thus, by defining

$$\lambda_n = n + n^{-1},\ \mu_n = n \text{ and } x_n = y_n = nx$$

for all positive integers n, we can get coherent sequences $((\lambda_n, x_n))$ and $((\mu_n, y_n))$ in $K(||) \times X(\mathcal{P})$ such that $(\lambda_n x_n)$ and $(\mu_n y_n)$ are not coherent sequences in $X(\mathcal{P})$. Thus, in this case, the scalar multiplication in $X(\mathcal{P})$ is not even sequentially uniformly continuous.

Finally, we note that if $\mathcal{P} = \{0\}$, then any two nets in $X(\mathcal{P})$ are coherent, and thus the uniform continuity of the scalar multiplication in $X(\mathcal{P})$ holds in a trivial manner.

Remark 3.4. Note that the addition in $X(\mathcal{P})$ and the members of $\mathcal{P}$ are always uniformly continuous.

References

[1] S.K.Berberian, Lecture in Functional Analysis and Operator Theory, Springer-Verlag, Berlin 1974.

[2] J.L.Kelley, General Topology, Van Nostrand Reinhold Company, New York, 1955.

[3] J.Rätz, On continuous multilinear mappings, C.R.Math.Rep. Acad.Sci. Canada 8(1986), 219-224.

[4] Á.Száz, Preseminormed spaces, Publ.Math. Debrecen 30(1983), 217-224.

[5] Á.Száz, Projective generation of preseminormed spaces by linear relations, Studia Sci. Math. Hungar., 23(1988), 297-313.

[6] Á.Száz, Coherences instead of convergences, Proceedings of the Conference on Convergence and Generalized Functions, Katowice 1983, Institute of Mathematics, Polish Academy of Sciences, Warsaw 1984, pp. 141-148.

[7] Á.Száz, Nets of multilinear maps, Abstracts and Program of the Conference on Generalized Functions, Convergence Structures and their Applications, Dubrovnik 1987, Institute of Mathematics, Novi Sad 1987, p. 55.

[8] Á.Száz, Bounded nets in preseminormed spaces, Period. Math. Hungar., to appear.

REZIME

NEPREKIDNOST MULTI – PRESEMINORMI

Korišćenjem rezultata iz [8], pokazano je da multipreseminorma nad proizvod preseminormiranom prostoru je ograničeno uniformno neprekidna ako i samo ako je neprekidna u nuli.

Specijalan slučaj ovog tvrdjenja omogućava jednostavan dokaz nekih bitnih poboljšanja fundamentalnih teorema iz [1,p.72] i [7] o neprekidnosti multilinearnih preslikavanja.

Received by the editors December 20, 1989.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 155-166

Review of Research
Faculty of Science
Mathematics Series

ON SEBASTIAO E SILVA'S ORDER OF GROWTH OF DISTRIBUTIONS

Arpad Takači
Institute of Mathematics, University of Novi Sad,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

Sebastiao e Silva introduced in [7] the "order of growth" of distributions, which fitted very well into his axiomatic approach to the distribution theory. This notion enabled him to define a limit of distributions (both at finite and infinite points), the Landau "oh" symbols for them, and most important, the definite integral which led naturally to the convolution and Fourier transformation. In this paper the "equivalence at infinity" (analysed in [10]) is compared with the "order of growth" of distributions, and using both notions an asymptotic expansion of distributions is applied to the distributional Stieltjes transformation in the sense of [3].

AMS Mathematics Subject Classification (1980): 46F12
Key words and phrases: Order of growth of distributions, asymptotic expansion, Stieltjes transformatrion.

1. Order of growth and equivalence at infinity of distributions

The letters m and n will always denote natural, while α and $\rho > -1$ will denote real numbers. The set of negative integers will be denoted by Z_-. The distributional detivation operator in x will be denoted by D.

We rewrite Sebastiao e Silva's extension of the Landau symbols first:

Definition 1.1. *([7], 8.3) Let I be an unbounded interval to the right and let $r \in C^{\infty}(I)$. We write $T = O(r)$ (respectively $T = o(r)$) as $x \to +\infty$ iff there exist an $a \in \mathbf{R}$, a continuous function F and an $n \in \mathbf{N}_0$ such that*

$$T = rD^nF \text{ on } (a,\infty),$$

and

$$(1) \qquad \frac{F(x)}{x^n} \text{ is bounded on } (a,\infty), \text{ (respectively } \lim_{x\to+\infty} \frac{F(x)}{x^n} = 0)$$

If $r = 1$ on I we get the definition of " $\lim\limits_{x \to +\infty} T = 0$", or, more generally, if the limit in (1) is equal to $\frac{c}{n!}$, then we write $\lim\limits_{x \to \infty} T \overset{D'}{=} c$. As can be expected, this notion generalizes the usual Landau symbols and limits for continuous functions, while the opposite is not true. For instance, $\sin x = o(1)$ (i.e. $\lim\limits_{x \to \infty} \sin x \overset{D'}{=} 0$) in the sense of Definition 1. In the same manner one can define the Landau symbols and the limits at $-\infty$ or at finite points. The latter was defined by Lojasiewicz in 1957 (see [4]).

The case $r(x) = x^{\alpha}, x > 0$ for $\alpha > -1$ is the archetype of the functions r. For such an r we have

Lemma 1.1. *([7], 8.4) If T is $O(x^{\alpha})$ (resp. $o(x^{\alpha})$) as $x \to \infty$, then DT is $O(x^{\alpha-1})$) (resp. $o(x^{\alpha-1})$) as $x \to \infty$ for any $\alpha \in \mathbf{R}$.*

A more precise notion was used in [2], [3] and [10], which is an asymptotic behaviour of distributions:

Definition 1.2. *A distribution T is* **equivalent at infinity** *with a regularly varying function $r(x) = x^{\alpha}L(x)$ if*

i. for $\alpha \notin Z_-$ there exist $a \in \mathbf{R}, n \in \mathbf{N}_0, n + \alpha > 0$ and a continuous function F on $\mathbf{R}$ such that $T = D^nF$ on (a,∞) and

$$\lim_{x\to\infty} \frac{F(x)}{x^{n+\alpha}L(x)} = \frac{1}{(n+1)\dots(n+\alpha)} =: C_{\alpha,n}$$

ii. for $\alpha \in Z_-$ there exist $a \in \mathbf{R}$ and a continuous function F on $\mathbf{R}$ such that $T = F$ on (a,∞) and..

$$\lim_{x\to+\infty} \frac{F(x)}{x^nL(x)} = 1.$$

We then write $T \overset{E}{\sim} r(x)$ as $x \to \infty$.

We recall that a function $r(x) = x^\alpha L(x)$ is termed regularly varying at infinity if the function $L : (a, \infty) \to (0, \infty)$ is measurable and satisfies the condition

$$lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1. \text{ for each } \lambda > 0.$$

(L is then called slowly varying at infinity.)

We shall now prove a few "expected" properties of these notions.

Theorem 1.1. *If a distribution T is equivalent at infinity with $r(x) = x^\alpha$, then $T = O(x^\alpha)$ and $T \neq o(x^\alpha)$ as $x \to \infty$.*

Proof. The statement is obvious for $\alpha \in Z_-$. So, let $T \overset{E}{\sim} x^\alpha$ as $x \to \infty, \alpha \notin Z_-$ and let a, n and F be as in Definition 1.2. This means that on (a, ∞)

$$T = D^n(C_{\alpha,n} x^{\alpha+n}(1 + \omega(x))),$$

where ω is a continuous function such that $\omega(x) \to 0$ as $x \to \infty$.

From the Leibniz formula it follows that

$$\begin{aligned} T &= D^{n-1}[x^{\alpha+n-1}(C_{\alpha,n-1}(1 + \omega(x)) + C_{\alpha,n} D(x \cdot \omega(x)) - \omega(x))] = \\ &= D^{n-1}(x^{\alpha+n-1} D(\omega_1(x))) \text{ on } (a, \infty), \end{aligned}$$

where ω_1 is again a continuous function such that $|\omega_1(x)| \leq C_1 x$ for x sufficiently large. Continuiting in this manner, we get

$$T = x^\alpha D^n(\omega_n(x))$$

where the continuous function ω_n satisfies

$$|\omega_n(x)| \leq C_n x^n, \text{ for } x \text{ sufficiently large.} \tag{2}$$

So, $T = O(x^n)$.

If $T = o(x^\alpha)$, then $T = x^\alpha D^m G(x)$, where G is a continuous function such that $G(x)/x^m \to 0$ as $x \to \infty$. So, we obtain

$$T = \sum_{k=0}^{m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix} \alpha(\alpha - 1)\ldots(\alpha - k + 1) D^{m-k}(x^{\alpha-k} G(x))$$

on (a,∞) and $x^{\alpha-k}G(x) = o(x^{\alpha+m-k})$ as $x \to \infty$ in the ordinary sense. But, then, it is impossible to find an n and a continuous function F such that

$$F(x) \sim Cx^{\alpha+n} \text{ as } x \to \infty.$$

From the proof of Theorem 1.1. we get

Theorem 1.2. *If $T \overset{E}{\sim} x^{\alpha}$ as $x \to \infty$, then on some interval (a,∞) we have*

$$T = x_{+}^{\alpha} + R,$$

where $R \in D'$ satisfies $R = o(x^{\alpha})$ as $x \to \infty$.

As usual, x_{+}^{α} is the tempered distribution which regularizes the function $H(x)\cdot x^{\alpha}$, H is the Heaviside function.

If x^{α} is replaced with a regularly varying function, then we can only prove

Theorem 1.3. *If $T \overset{E}{\sim} x^{\alpha}L(x)$ as $x \to \infty$, then $T = o(x^{\alpha+\varepsilon})$ and $T = o(x^{\alpha-\varepsilon})$ for any $\varepsilon > 0$, such that $\alpha+\varepsilon$ (resp. $\alpha - \varepsilon$) is not a negative integer if $\alpha \notin Z_{-}$.*

Proof. It is well known ([6]) that for a given $\varepsilon > 0$ a slowly varying function L at infinity satisfies the inequalities

$$(3) \qquad C_1 x^{-\varepsilon} \le L(x) \le C_2 x^{\varepsilon} \text{ for } x \ge a_1$$

for some positive constants a_1, C_1 and C_2 which depend on ε.

We shall give only the proof for $\alpha \notin Z_{-}$ of the statement "$T = O(x^{\alpha+\varepsilon})$", the others are similar. We can write on (a,∞)

$$T = D^{n}(x^{\alpha+n+\varepsilon'}(C_{\alpha,n}\frac{L(x)}{x^{\varepsilon'}} + \frac{L(x)}{x^{\varepsilon'}}\omega(x)))$$

for $a \ge a_1$ and $n \ge -\alpha$ as in Definition 1.2. and $\omega(x) \to 0$ as $x \to \infty$ (we choose $0 < \varepsilon' < \min(1,\varepsilon)$ such that $\alpha + n + \varepsilon' \notin \mathbf{N}$). In the same way as in the proof of Theoren 1.1, using (3), we get

$$T = x^{\alpha+\varepsilon'}D^{n}(\omega_n(x))$$

where $\omega_n(x)$ is a continuous function satisfying (2).

A regularly varying function does not need to be continuous. However, by [6], p.17, for a given slowly varying function L there exists another, infinitely differentiable slowly varying function L_1 such that $L_1(x) \sim L(x)$ as $x \to \infty$ and $L_1(n) = L(n)$ for all sufficiently large integers n. If L_1 has this property, then we have

Theorem 1.4. *If $r_1(x) = x^\alpha L_1(x)$ is an infinitely differentiable regularly varying function at infinity, and $T \overset{E}{\sim} r_1$ as $x \to \infty$, then $T = O(r_1)$ as $x \to \infty$.*

Proof. As in the proof of Theorem 1.1, we can write $T = D^n F$ on (a, ∞) for some $a, n > -\alpha$ and some continuous function F such that $F(x) = C_{\alpha,n} x^{\alpha+n} L_1(x)(1 + \omega(x))$, where $\omega(x) \to 0$ as $x \to \infty$. So we obtain (compare to the proof of Theorem 1.1)

$$T = D^n F = x^\alpha L_1(x) D^n \omega_n(x)$$

where ω_n is a continuous function such that $|\omega_n(x)| \le C_n x^\alpha$ for x sufficiently large; we used the fact that $\lim_{x\to\infty} \frac{xL_1'(x)}{L_1(x)} = 0$ (see [6], pp.6-7).

2. Asymptotic expansion of distributions at infinity

Let a distribution T and an increasing sequence of real numbers $(n_k)_{k=0}^\infty$ be given. We say that T has the **asymptotic expansion** $\sum_{k=0}^\infty A_k^{-n_k}$ at ∞ related to $(x^{-n_k})_{k=0}^\infty$, if

i. either $T \overset{E}{\sim} A_0 x^{-n_0}$ as $x \to \infty$, or $T = o(x^{-n_0})$ (by definition we put $A_0 = 0$);

ii. if the conplex numbers $A_0, \dots, A_m (m \in \mathbf{N}_0)$ are already chosen, then either

$$T_m := T - \sum_{k=0}^m A_k x^{-n_k} \overset{E}{\sim} A_{m+1} x^{-n_{m+1}} \tag{1}$$

as $x \to \infty$ or

$T = o(x^{-n_{m+1}})$; we put then $A_{m+1} = 0$.

Then we write

$$T \overset{AE}{\sim} \sum_{k=0}^{\infty} A_k x^{-n_k} \text{ as } x \to \infty. \tag{2}$$

It is clear that the asymptotic expansion of a distribution T is unique, provided it exists; however two different distributions can have the same expansion. In view of Definition 1.2 it is clear that if

$$T \overset{AE}{\sim} \sum_{m=0}^{\infty} A_m x^{-n_m} \text{ as } x \to \infty,$$

and at least one of the numbers n_m is a natural number with $A_m \neq 0$, then T_m is a locally integrable function on some interval (a, ∞) and the asymptotic expansion of T_m can be taken in the usual sense.

Similarly as the equivalence at infinity, this asymptotic expansion is a local property of a distribution which generalizes the classical asymptotic expansion and preserves some "expected" properties of an asymptotic expansion. For instance, the distribution T_m from (2) is $o(x^{-n_m})$ as $x \to \infty$. Furthermore, the asymptotic expansion can be added or multiplied with nonzero constants. As in the classical case, it is possible to "integrate" the asymptotic expansion (i.e. to convolve it with the Heaviside function), provided that $T \in D'_+$, which means that the support of T is in $[0, \infty]$. However, in general, one can not differentiate it.

On using Theorem 1.2, we obtain

Theorem 2.1. *If $T \in D'_+$ satisffies $T \overset{AE}{\sim} \sum_{k=0}^{\infty} A_k x^{-n_k}$ as $x \to \infty$ where $n_k \notin Z_-$ for each $k = 0, 1, \ldots$, and $n_0 < n_1 < \ldots$, then on some interval (a_m, ∞) $(0 \leq a_0 \leq a_1 \leq \ldots)$*

$$T = \sum_{k=0}^{\infty} A_k x_+^{-n_k} + R_m(x), \text{ where } R_m(x) = o(x^{-n_m}) \text{ as } x \to \infty. \tag{3}$$

From Theorem 1.2 and Theorem 3 form [10] we get an important property of the asymptotic expansion:

Theorem 2.2. *If T satisties the conditions of Theorem 2.1., then $R_m = o(x^{-n_m})$ satisfies*

$$\lim_{\lambda \to \infty} \langle R_m(x), \frac{\varphi(x/\lambda)}{\lambda^{n_m+1}} \rangle = 0 \tag{4}$$

for each $\varphi \in D$ with a support in (a_m, ∞).

A special case of the asymptotic expansion gives

Theorem 2.3. *A distribution $T \in E'$ has the asymptotic expansion at infinity related to any sequence* $(x^{-n_k})_{k=0}^{\infty}, n_0 < n_1 < \ldots,$ *just zero.*

This statement remains true if T is rapidly decreasing at infinity, i.e. for each $\alpha < 0$ there exist continuous functions F_l and natural numbers $n_l, l = 1, \ldots, m$ such that

$$T = \sum_{l=1}^{m} D^{n_l} F_l \text{ on some interval } (a, \infty)$$

and

$$F_l = o(x^{\alpha}) \text{ for } l = 1, \ldots, m.$$

3. Integrals of distributions

The limit of distributions from Section 1 allows a definition of their integral.

Definition 3.1. *Let $[a,b] \subset \mathbf{R}$ be interval and T a distribution. The integral of T over $[a,b]$ is defined by*

$$S - \int_a^b T(x)dx := T^{-1}(b+) - T^{-1}(a-),$$

where T^{-1} is a primitive of T (i.e. $D(T^{-1}) = T$)), provided that the limits

$$T^{-1}(b+) = \lim_{x \to b+0} T^{-1}(x), \ \lim T^{-1}(a-) = \lim_{x \to a-0} T^{-1}(x).$$

exist in the distributional sense.

(The letter "S" stands for Sebastiao E Silva, since he gave this definition in [7]).
Especially the integral over $[a, \infty]$ is defined as

(1)
$$S - \int_a^{\infty} T(x)dx := T^{-1}(+\infty) - T^{-1}(a-),$$

where

$$T^{-1}(+\infty) := \lim_{x\to\infty} T^{-1}(x).$$

It was shown in [7] that this integral generalizes properly the usual definite one. We need the following two statements.

Lemma 3.1. *([7], Section 9) The integral (1) exists if* $T = O((x-a)^{\beta})$ *as* $x \to a+0$ *for some* $\beta > -1$ *and* $T = O(x^{\alpha})$ *as* $x \to \infty$ *for some* $\alpha < -1$.

Lemma 3.2. *(Compare to [8], Section 2.) Let* $T \in D'_{+}$ *and* $\varphi \in C^{\infty}(\mathbf{R})$. *If* $T = O(x^{\alpha})$ *and* $\varphi^{(k)} = O(x^{\beta-k})$ *as* $x \to \infty$, *for* $k = 0, 1, \ldots$, *and some* $\alpha, \beta \in \mathbf{R}$, *then* $T \cdot \varphi = O(x^{\alpha+\beta})$ *and for each* $n \leq 0$ *with the property* $\alpha + \beta < n - 1$ *we have*

$$(2) \qquad S - \int_{-0}^{\infty} \varphi D^n T = (-1)^n \Big(S - \int_{-0}^{\infty} \varphi^{(n)} T\Big)$$

Observe that $\psi(x) = \frac{1}{(x+s)^{\rho+1}}$ satisfies the conditions of this lemma.

4. Distributional Stieltjes transformation

The classical Stieltjes transformsof a function f is defined by the integral

$$\sigma_0(f)(s) = \int_0^{\infty} \frac{f(x)}{x+s} dx, \; s \in \Delta := C/(-\infty, 0],$$

or more generaly

$$(1) \qquad \sigma_{\rho}(f)(s) = \int_0^{\infty} \frac{f(x)}{(x+s)^{\rho+1}} dx, \; s \in \Delta,$$

where $\rho > -1$ is called the order of the transform. A sufficient condition for the convergence of (1) is that f is locally integrable on $[0, \infty)$ and satisfies the condition

$$(2) \qquad f = O(x^{\rho-\xi}) \text{ as } x \to \infty \text{ for some } \xi > 0.$$

There are several definitions of the Stieltjes transform of a distribution T from D'_{+}. In [2] the following one was given:

Definition 4.1. *If $T \in D'_+$ can be written as $T = D^k F$ for some $k \in \mathbf{N}$, continuous function F on $\mathbf{R}$ which is zero on $(-\infty, 0)$ and satisfies the condition.*

$$\sup\{|F(x)x^{-k+\alpha}|, x \geq b\} \leq C < \infty, \tag{3}$$

then the Stieltjes transforms of T is given by

$$\sigma_\rho(T)(s) = (\rho+1)_k \cdot \int_0^\infty \frac{F(x)dx}{(x+s)^{\rho+k+1}}, \; s \in \Delta. \tag{4}$$

Here $(\rho+1)_0 := 1, (\rho+1)_k = (\rho+1)\ldots(\rho+k)$ and $\rho > \alpha$. As usual, we use the same letter in (1) and (4) for the classical and the distributional ("generalized") transform σ_ρ; this makes sense, since if T is defined by a locally integrable function f on $[0, \infty)$ satisfying condition (2), then the two notions coincide.

(In a later paper ([3]), a slightly more general condition on F was given, namely that the integral $\int_b^\infty \frac{|F(x)|dx}{x^{\rho+k+1}}$ converges. However, in what follows condition (3) is sufficient.)

If $F = O(x^{+\rho+k-\xi})$ as $x \to \infty$ for some ξ with the property $\rho + 1 > \xi > 0$, then $T = D^k F = O(x^{\rho-\xi})$. Since, $\lim_{x \to 0} D^l F = 0$, $\lim_{x \to +\infty} \frac{F^{(k-l)}(x)}{x^{\rho+l+1}} = 0$ for $l = 0, 1, \ldots, k$, we have by partial integration (see Lemma 3.2)

$$\begin{aligned} S - \int_{0-}^\infty \frac{T(x)}{(x+s)^{\rho+1}}dx &= S - \int_{0-}^\infty \frac{D^k F(x)}{(x+s)^{\rho+1}}dx = \\ &= S - \int_{0-}^\infty \frac{F(x)dx}{(x+s)^{\rho+k+1}} = \int_0^\infty \frac{F(x)dx}{(x+s)^{\rho+1}} = \sigma_\rho(T)(s). \end{aligned}$$

This suggests defining the Stieltjes transform of $T \in D'_+$ as

$$\sigma_\rho(T)(s) := S - \int_{0-}^\infty \frac{T(x)}{(x+s)^{\rho+1}}dx, \; s \in \Delta. \tag{5}$$

Formula (5) can be used for a greater set of distributions than (4); namely $T = O(x^{\rho-\varepsilon})$ as $x \to \infty$ is a sufficient, but not a necessary condition for the existence of (5). We remark that if $T = B \in E'_+$(distributions with a compact support in $[0, \infty)$), then

$$\sigma_\rho(B)(s) = S - \int_{0-}^\infty \frac{B(x)dx}{(x+s)^{\rho+1}} = \langle B(x), \frac{\eta(x)}{(x+s)^{\rho+1}} \rangle,$$

where $\eta \in C^\infty(\mathbf{R})$ has the properties $\eta(x) = 0$ for $x < -2\varepsilon$ and $\eta(x) = 1$ for $x > -\varepsilon, \varepsilon > 0$. Furthermore if $T = x_+^\alpha$, then (see [2])

$$\sigma_\rho(x_+^\alpha(s)) = B(\rho - \alpha, \alpha + 1)s^{\alpha-\rho}, \rho > \alpha, \tag{6}$$

and for $\alpha \in Z_-, s \in \Delta$. As usual Γ and B are the gamma and beta functions, more precisely their analytic continations. From now on we suppose that s is real and positive.

The following theorem was proved in [5] (compare with (6)):

Theorem 4.1. *If $T \in D'_+$ is equivalent at infinity with the regularly varying function $r(x) = x^\alpha L(x)$ as $x \to \infty$, then*

$$\sigma_\rho(T)(s) \sim B(\rho - \alpha, \rho + 1) \cdot s^{\alpha-\rho} L(s) \text{ as } s \to \infty,$$

provided that $-1 < \alpha < \rho$.

Remark 4.1. The condition $\alpha > -1$ is essential; if $\alpha \leq -1$, then the "quasiasymptotic behaviour" from [1] of distributions is more appropriate for this analysis than equivalence at infinity. Especially if $T = B \in E'_+$, then using the structural theorem from [1] one can show that it has a quasiasymptotic behaviour of some order $-m_0 (m_0 \in \mathbf{N})$ and

$$\sigma_\rho(B)(s) \sim C \cdot s^{-(\rho+m_0)} \text{ as } s \to \infty \tag{7}$$

(see the Abelian theorem from [9]).

We can now prove

Theorem 4.2. *If $-1 < \alpha < \rho$ and $T \in D'_+$ satisfies $T = o(x^\alpha)$, then $\sigma_\rho(T) = o_\rho(s^{\alpha-\rho})$ as $s \to \infty$.*

Proof. From $T = o(x^\alpha)$ as $x \to \infty$ it follows $T = x^\alpha D^n F(x)$ on some interval (a, ∞), for some continuous function F on $\mathbf{R}$ and some $n \in \mathbf{N}$ such that $\lim_{x\to\infty} \frac{F(x)}{x^n} = 0$. We put $G(x) = F(x)$ for $x > a, G(x) = 0$ for $x \leq a$ and $T_1 = T - x^\alpha D^n G(x)$. The support of T_1 is a compact subset of $[0, a]$, so bu Lemma 3.1 from [3] follows

$$|\sigma_\rho(T_1)(s)| \leq C \cdot s^{-(\rho+1)} \text{ as } s \to \infty \text{ for some } C > 0 \tag{8}$$

(see also (7)). From (3.2) it follows that

$$S-\int_{a-}^{\infty}\frac{x^{\alpha}D^{n}G(x)}{(x+s)^{\rho+1}}dx=$$

$$=S-\int_{a-}^{\infty}\frac{C_0x^{\alpha}+C_1\cdot s\cdot x^{\alpha-1}+\cdots+C_ns^nx^{\alpha-n}}{(x+s)^{\rho+n+1}}G(x)dx.$$

for some constants $C_i, i=0,1,\ldots,n$. The last S-integral is also an "ordinary" integral, hence, from Theorem 4.1. it follows that

$$|\sigma_\rho(x^{\alpha}D^{n}G(x))(s)|\leq C\cdot s^{\rho+1}, \tag{9}$$

s sufficiently large, since $G(x)=x^{n}\omega(x)$ and $\omega(x)\to 0$. Hence (7) and (8) imply

$$\lim_{s\to\infty}s^{\rho-\alpha}\sigma_\rho(T)(s)=0,\text{ or }\sigma_\rho(T)(s)=o(s^{\alpha-\rho})$$

as $s\to\infty$.

In a similar way one can prove

Theorem 4.3. *If $-1<\alpha<\rho$ and $T\in D'_{+}$ satisfies $T=O(x^{\alpha})$, then $\sigma_\rho(T)=(s^{\alpha-\rho})$ as $s\to\infty, s\in\mathbf{R}$.*

A consequence of these statements is

Theorem 4.4. *Let $T\overset{AE}{\sim}\sum_{k=0}^{\infty}A_kx^{-n_k}$ as $x\to\infty$ (in the sense of Definition 2.1) for $-\rho<n_0<n_1<\ldots<1$. Then*

$$\sigma_\rho(T)(s)\sim\sum_{k=0}^{\infty}A_ks^{-n_k-\rho} \tag{10}$$

as $s\to\infty$ in the ordinary sense.

References

[1] Drožžinov, Ju. N. and Zav'jalov. B.I., The quasiasymptotic behaviour of generalized functions and Tauberian theorems in a complex domain (in Russsian), Mat. Sb. 102 (1977), 372-390.

[2] Lavoine, J. et Misra, O.P.: Théoremes abéliens pour la transformation de Stieltjes des distributions, C.R. Acad. Sci. Paris, 279 (1974), Ser. A, 99-102.

[3] Lavoine, J. et Misra, O.P.: Abelian theorems for the distributional Stieltjes transformation, Math. Proc. Camb. Phil. Soc. 86 (1979), 287-293.

[4] Lojasiewicz, L.: Sur la valeur et la limit d'une distribution dans un point, Studia mathematica, 16, 1 (1957), 1-36.

[5] Marić, V., Skendžić, M. and Takači, A.: On Stieltjes transform of distributions behaving as regularly varying functions, Acta Sci. Math. 50 (1986), 405-410.

[6] Seneta, E.: Regularly varying functions, Springer Verlag, Berlin, Heidelberg, New York, Lecture Notes in Math. 508 (1976).

[7] Sebastiao e Silva, J.: Integrals and orders of growth of distributions, Cours international sur la théorie des distributions, Lisbonne, Centro de Cálculo Cientifico (1964).

[8] Sebastiao e Silva, J.: Les séries de multipoles des physiciens et la théorie des ultradistributions, Math. Ann. 174 (1967), 109-142.

[9] Takači, A., A note on the distributional Stieltjes transformation, Math. Proc. Camb. Phil. Soc., 94 (1983), 523-527.

[10] Takači A., On the equivalence of distributions at infinity, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., 15,1(1985), 175-187.

REZIME

O SEBASTIAO E SILVA "REDU RASTA" DISTRIBUCIJA

U ovom radu se uporedjuje "red rasta" koji je uveo Sebastiao E Silva sa "ekvivalencijom u beskonačnosti" za distribucije. Pomoću ova dva pojma se uvodi asimptotski razvoj distribucija, koji se primenjuje na uopštenu Stieltjesovu transformaciju.

Received by the editors January 13, 1988.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 167-183

Review of Research
Faculty of Science
Mathematics Series

ON S - QUASI - CONTINUOUS MULTIVALUED MAPS

Janina Ewert and **Tadeusz Lipski**
Department of Mathematics, Pedagogical University
Arciszewskiego 22, 76-200 Stupsk, Poland

Abstract

Among various generalizations of continuity we can see two types; in one of them it is assumed that the inverse images of open sets belong to some class larger than the topology, this way leads - in particular - to quasi - continuous maps [11, 17]. In the second one the inverse images of sets from some subfamily of the given topology are considered. Then, as an example there are s-continuous maps [13, 14, 15,]. Joinging both the above methods the class of s-quasi-continuous maps is introduced.

AMS Mathematics Subject Classification (1980): 54C60, 54C08
Key words and phrases: Multivalued map, quasi-continuity, locally connected space

1. Basic definitions and properties

For a subset A of a topological space X the symbols $Cl(A)$ and $Int(A)$ are used to denote the closure and the interior of A.
A set $A \subset X$ is said to be:
- semi-open if $A \subset Cl\ Int(A)$, [17],
- semi-closed if $X \setminus A$ is semi-open [2, 3].
The sets $Scl(A) = \bigcap\{B : A \subset B, \text{ B is semi-closed}\}$

and $Sint(A) = \bigcup\{B : B \subset A, \text{ B is semi-open}\}$ are called the semi-closure and the semi-interior of A respectively [2, 3].

Now, let F be a multivalued map defined on a topological space X and having values in the non-empty subsets of a topological space Y ; for the simplicity we will write $F : X \to Y$. For any set $V \subset Y$ we denote $F^+(V) = \{x \in X : F(x) \subset V\}$ and $F^-(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$ [1].

A multivalued map $F : X \to Y$ is called upper (lower) s - quasi - continuous at a point $x_0 \in X$, if for each open set $V \subset Y$ having the connected complement and $F(x_0) \subset V$ (resp. $F(x_0) \cap V \neq \emptyset$), there exists a semi- open set $U \subset X$ such that $x_0 \in U \subset F^+(V)$ ($x_0 \in U \subset F^-(V)$). A map F is called upper (lower) s - quasi - continuous if it has this property

at each point.

If, in the above definitions, we omit the condition that $Y \setminus V$ is connected, then we shall obtain the definitions of the upper and the lower quasi - continuity respectively [19, 20].

Any function $f : X \to Y$ can be considered as the multivalued map F given by $F(x) = \{f(x)\}$. In this case we have s - quasi - continuous (quasi - continuous [11, 17]) maps.

The standard proofs of the following two theorems we shall omit.

1.1 Theorem.

A multivalued map $F : X \to Y$ is upper (lower) s - quasi - continuous at $x_0 \in X$ if and only if for each open set $V \subset Y$ with the connected complement and $F(x_0) \subset V$ (resp. $F(x_0) \cap V \neq \emptyset$), and for each neighbourhood U of x_0 there exists an open non-empty set $U_1 \subset U$ such that $U_1 \subset F^+(V)$ (resp. $U_1 \subset F^-(V)$).

1.2 Theorem.

For a multivalued map $F : X \to Y$ the following statements are equivalent:

(a) F is upper (lower) s-quasi - continuous;

(b) for every open set $V \subset Y$ having the connected complement the set

$F^+(V)$ (resp. $F^-(V)$) is semi - open in X;
(c) for every connected closed set $M \subset Y$ the set $F^-(M)$ (resp. $F^+(M)$) is semi - closed;
(d) for every set $B \subset Y$ having the connected closure, the following inclusion holds: $Scl(F^-(B)) \subset F^-(Cl(B))$ (resp. $Scl(F^+(B)) \subset F^+(Cl(B))$);
(e) for every set $A \subset Y$ such that $Y \setminus Int(A)$ is connected, $F^+(Int(A)) \subset Sint(F^+(A))$ (resp. $F^-(Int(A)) \subset Sint(F^-(A))$).

Now, let (Y,T) be a topological space and let ς denote the collection of all the open sets whose complements are connected. Let T^* denote the topology on Y generated by taking ς as a subbase. Obviously, $T^* \subset T$.
For any map with values in Y, it holds that:
quasi - continuity $\rightarrow$ T^*-quasi - continuity $\rightarrow$ s - quasi - continuity
and none of the above implications can be reversed neither for multivalued maps nor for functions.

1.3 Examples.

(a) Let R be the set of real numbers with the natural topology and $x_0 \in R$ an established point. The multivalued map $F : R \to R$ given by

$$F(x) = \begin{cases} [1,2] & \text{for } x < x_0, \\ [3,4] & \text{for } x = x_0, \\ [4,5] & \text{for } x > x_0, \end{cases}$$

is upper and lower s - quasi - continuous but it is neither upper nor lower T^* - quasi - continuous at x_0.
(b) Let X be the set of real numbers with the natural topology and (Y,T) the Sorgenfrey line [4, p. 39]. Then T^* is the co-finite topology on Y [22, Prop. 1]. Let a, b, x_0 be established numbers with $x_0 < a < b$. We define the multivalued map $F : X \to Y$ assuming

$$F(x) = \begin{cases} [a,b] & \text{for } x = x_0, \\ \{x\} & \text{for } x \neq x_0. \end{cases}$$

The map F is upper and lower T^* - quasi - continuous but it is not upper nor lower quasi - continuous.

2. Points of continuity

For a multivalued map F, the symbols $C^+(F), C^-(F)$ are used to denote the sets of all the points at which F is upper or lower semi - continuous respectively.

A multivalued map $F : X \to Y$ is called upper (lower) almost continuous at a point $x_0 \in X$ if for each open set $V \subset Y$ with $F(x_0) \subset V$ (resp. $F(x_0) \cap V \neq \emptyset$) it holds that $x_0 \in IntCl(F^+(V))$, (resp. $x_0 \in IntCl(F^-(V))$), [21]. For single valued maps each of these definitions imply almost continuity in the sense of Husain [9].

2.1 Theorem.

Let X be a topological space, Y a locally connected regular one and let $F : X \to Y$ be a multivalued map.

(a) If F is lower s - quasi - continuous, $F(x)$ is a connected compact for $x \in X$ and F is upper almost continuous at x_0, then F is upper semicontinuous at x_0.

(b) If F is upper s - quasi - continuous, lower almost continuous at x_0, then F is lower semicontinuous at x_0.

Proof. Assume that F is not upper semicontinuous at x_0. Then, there exists an open set $V \subset Y$, $F(x_0) \subset V$ such that each neighbourhood U' of x_0 contains a point x' for which $F(x') \not\subset V$. Because Y is regular locally connected and $F(x_0)$ is a connected compact set, we can choose an open connected set V_1 satisfying

$$F(x_0) \subset V_1 \subset Cl(V_1) \subset V.$$

The set $U = IntCl(F^+(V_1))$ is a neighbourhood of x_0, so it contains a point x_1 such that $F(x_1) \cap (Y \setminus Cl(V_1)) \neq \emptyset$.
Using the lower s - quasi - continuity of F at x_1, we choose an open nonempty set $U_1 \subset U$ such that $F(x) \cap (Y \setminus Cl(V_1)) \neq \emptyset$ for $x \in U_1$. But $U_1 \cap F^+(V_1) \neq \emptyset$, thus for some $x_2 \in U_1$ we have $F(x_2) \subset V_1$, which is impossible. Hence, the proof of (a) is completed.
The proof of (b) is analogus.

2.2 Lemma.

Let X be a topological space, Y a second countable one, and let $F : X \to Y$ be a multivalued map. Then:
(a) The set of all the points at wich F is not lower almost continuous is of the first category [12].
(b) If $F(x)$ is compact for each $x \in X$, then the set of all the points at wich F is not upper almost continuous is of the first category [5, Th. 2.1.]

Combining this lemma with Theorem 2.1 we immediately obtain

2.3 Theorem.

Let X be a topological space, Y a locally connected second countable regular one and let $F : X \to Y$ be a multivalued map. Then:
(a) If F is lower s - quasi - continuous and $F(x)$ is connected compact for each $x \in X$, then $X \backslash C^+(F)$ is of the first category.
(b) If F is upper s -quasi - continuous, then $X \backslash C^-(F)$ is of the first category.

2.4 Corollary.

Let X be a topological space and Y a locally connected regular one. A function $f : X \to Y$ is continuous if and only if it is s -quasi - continuous and almost continuous.

3. Functions of two variables

Let X, Y, Z be topological spaces and $F : X \times Y \to Z$ a multivalued map. For any $x \in X$, $y \in Y$ by F_x, F^y we denote the multivalued maps $F_x : Y \to Z$ and $F^y : X \to Z$ defined by $F_x(y) = F(x, y) = F^y(x)$.

3.1 Theorem.

Let X be a Baire space, Y a locally second countable and let Z be a locally connected regular one. If $F : X \times Y \to Z$ is a multivalued map with compact values such that F_x, F^y are upper s - quasi - continuous for each

$x \in X, y \in Y$ and all the F^y are lower quasi-continuous, then F is upper s - quasi - continuous.

Proof. We apply the method analogous to that in papers [18, 19]. Assume that F is not upper s - quasi - continuous at (x_0, y_0). Then, there exists an open set W containing $F(x_0, y_0)$ and having the connected complement, and there exists a neighbourhood $U \times V$ of (x_0, y_0) such that every non-empty open set $U' \times V' \subset U \times V$ has a point (x, y) for which it holds $F(x', y') \cap (Z \backslash W) \neq \emptyset$. Without loss of generality we can assume that V has a countable base $\{V_n : n \geq 1\}$. The set $M = Z \backslash W$ is closed, $F(x_0, y_0)$ compact, so we can choose disjoint open sets $W', W'' \subset Z$ such that $F(x_0, y_0) \subset W'$ and $M \subset W''$. Since Z is a locally connected space and M a connected subset, there exists an open connected set $W_1 \subset Z$ such that $M \subset W_1 \subset W''$. Hence, $W' \cap Cl(W_1) = \emptyset$ which implies

$$F(x_0, y_0) \subset Z \backslash Cl(W_1) \subset Cl(Z \backslash Cl(W_1)) \subset W.$$

Let $W_0 = Z \backslash Cl(W_1)$; then W_0 is an open set with the connected complement. The map F^{y_0} is upper s - quasi - continuous at x_0, therefore $U_1 = U \cap Int(F^{y_0})^+(W_0)$ is an open non-empty set. Let us put $A_n = \{x \in U_1 : V_n \subset (F_x)^+(W_0)\}$. For $x \in U_1$ we have $F_x(y_0) = F(x, y_0) \subset W_0$. Since F_x is upper s - quasi -continuous, we obtain $Int(F_x)^+(W_0) \cap V \neq \emptyset$, so $V_n \subset (F_x)^+(W_0)$ for some $n \geq 1$. Thus $x \in A_n$ and consequently $U_1 = \bigcup_{n=1}^{\infty} A_n$.

Let $n \geq 1$ be established and let U' be a non-empty open subset of U. Then, for some point $(x_1, y_1) \in U' \times V_n$, it holds that $F(x_1, y_1) \cap (Z \backslash Cl(W_0)) \neq \emptyset$. The lower quasi - continuity of F^{y_1} at x_1 implies the existence of a non - empty open set $U'' \subset U'$ such that

$$F(x, y_1) \cap (Z \backslash Cl(W_0)) \neq \emptyset \text{ for } x \in U''.$$

Hence, $y_1 \notin (F_x)^+(W_0)$ for $x \in U''$. But $y_1 \in V_n$, so $V_n \not\subset (F_x)^+(W_0)$ for $x \in U''$. This means $U'' \cap A_n = \emptyset$ and in consequence U_1 is of the first category. This contradiction finishes the proof.

3.2 Theorem.

Let X be a Baire space, Y a locally second countable and let Z be a locally connected regular one. If $F : X \times Y \to Z$ is a multivalued map such that F_x, F^y are lower s - quasi - continuous for each $x \in X, y \in Y$ and all F^y are upper quasi - continuous, then F is lower s - quasi - continuous.

Proof. Assume that F is not lower s - quasi - continuous at a point (x_0, y_0). Then, there exists an open set $W \subset Z$ with $F(x_0, y_0) \cap W \neq \emptyset$ and $Z \backslash W$ connected, and there exists a neighbourhood $U \times V$ of (x_0, y_0) such that each non-empty open set $U' \times V' \subset U \times V$ contains a point (x', y') for wich $F(x', y') \subset Z \backslash W$. We can assume that V has a countable base $\{V_n : n \geq 1\}$. Let $z_0 \in F(x_0, y_0) \cap W$. Since $M = Z \backslash M$ is a closed connected set, there exists an open connected set W_1 such that $M \subset W_1$ and $z_0 \in Z \backslash Cl(W_1) \subset Cl(Z \backslash CL(W_1)) \subset W$. Denoting $W_0 = Z \backslash Cl(W_1)$, we have an open set with the connected complement and $z_0 \in W_0$. Since F^{y_0} is lower s - quasi - continuous at x_0, the set $U_1 = U \cap Int(F^{y_0})^-(W_0)$ is non-empty. Let $A_n = \{x \in U_1 : V_n \subset (F_x)^-(W_0)\}$. Using the lower s - quasi - continuity of maps F_x at y_0, we can show the equality $U_1 = \bigcup_{n=1}^{\infty} A_n$. For established $n \geq 1$ and an open non-empty set $U' \subset U_1$, there exists a point $(x_1, y_1) \in U' \times V_n$ such that $F(x_1, y_1) \subset Z \backslash Cl(W_0)$. Since F^{y_1} is upper quasi - continuous at x_1, there exists an open non-empty set $U'' \subset U'$ such that

$$F(x, y_1) \subset Z \backslash Cl(W_0) \text{ for } x \in U''.$$

Thus $y_1 \notin (F_x)^-(W_0)$ for $x \in U''$, and $y_1 \in V_n$, so $V_n \not\subset (F_x)^-(W_0)$ for $x \in U''$. This implies $U'' \cap A_n = \emptyset$. Hence, A_n are nowhere dense sets and U_1 is of the first category.

This is the contradiction finishing the proof.

Let us observe that in the above theorems assumptions on the spaces X, Y, Z are not necessary.

3.3 Example.

Let $Z = [0, \infty)$, $M = \{\frac{1}{n} : n \geq 1\}$ and

$$U_n(x) = \begin{cases} Z \cap (x - \frac{1}{n}, x + \frac{1}{n}), & \text{if } x \neq 0, \\ (0, \frac{1}{n}) \backslash M, & \text{if } x = 0. \end{cases}$$

Then $\{U_n(x) : n \geq 1,\ x \in Z\}$ is a neighbourhood system for some topology T on Z. For each number $a > 0$, the topology T restricted to the subspace $(0, a]$ or $[a, \infty)$ coincides with the natural topology, so

$(0.a]$ and $[a,\infty)$ are connected sets in (Z,T). Moreover, 0 belongs to the T-closure of $(0,a]$, therefore $[0,a]$ is connected. Now, let X be the set of real numbers and $\mathcal{N}$ the σ-ideal of sets of the Lebesgue measure zero. Let us consider the topology $\tau = \{(a,\infty) : a \in X\} \cup \{\emptyset, X\}$; then $\tau_{\mathcal{N}} = \{W \setminus H : W \in \tau, H \in \mathcal{N}\}$ is a topology on X [8].

Assume that $g : (X,\tau) \to (Z,T)$ is an s -quasi - continuous function such that $g(x_1) \leq g(x_2)$ for some $x_1, x_2 \in X, x_1 \neq x_2$. Let us take numbers a_1, a_2 such that $g(x_1) < a_1 < a_2 < g(x_2)$. The sets $[0,a_1)$ and (a_2,∞) are open and have the connected complement in (Z,T) so $Int g^{-1}([0,a_1)) \neq \emptyset \neq Int g^{-1}((a_2,\infty))$. From the definition of τ, for some $b \in X$ we have $(b,\infty) \subset g^{-1}([0,a_1)) \cap g^{-1}((a_2,\infty))$, which is impossible. Hence, only constant functions $g : (X,\tau) \to (Z,T)$ are s - quasi - continuous. It is easy to see that the same holds for functions $g : (X,\tau_{\mathcal{N}}) \to (Z,T)$.

Now, let $f : (X \times X, \tau \times \tau_{\mathcal{N}}) \to (Z,T)$ be a function such that all the f_x are s - quasi - continuous and f^y are quasi - continuous. Then, f_x, f^y are constant functions, so f is too. Thus f is s - quasi - continuous. But (X,T) is not a Baire space, $(X,\tau_{\mathcal{N}})$ is not locally second countable and (Z,T) is not regular [4, p. 61] nor locally connected.

4. Limits of sequences of S - quasi - continuous maps

The pointwise convergence does not preserve the s - quasi - continuity even for real functions.

4.1 Example.

In the space R of real numbers with the natural topology we shall consider functions $f_n, f : R \to R$ given by

$$f_n(x) = \begin{cases} -\frac{1}{n} & \text{for } x \in (-\infty,-1) \\ x^{2n+1} & \text{for } x \in [-1,1] \\ \frac{1}{n} & \text{for } x \in (1,\infty) \end{cases}$$

$$f(x) = \begin{cases} -1 & \text{for } x = -1 \\ 1 & \text{for } x = 1 \\ 0 & \text{for } x \in R\backslash\{-1,1\}. \end{cases}$$

Then f_n are s - quasi - continuous functions, $f_n \to f$ and f is not s - quasi - continuous at points -1 and 1.

The other situation is in the case of transfinite sequences. To begin with, we shall give some symbols and notions.
Let Ω be the first uncountable ordinal number.

A transfinite sequence $\{a_\xi : \xi < \Omega\}$ of elements of a topological space Y is said to be convergent to $a \in Y$ if for each neighbourhood U of a there exists $\xi < \Omega$ such that $a_\xi \in U$ for each ξ, $\xi_0 \leq \xi < \Omega$, [16]. In the sequel we shall use the following

4.2 Lemma. [16]

Let Y be a first countable T_1-space. If a transfinite sequence $\{a_\xi : \xi < \Omega\}$ converges to a $\in Y$, then, there exists $\xi_0 < \Omega$ such that $a_\xi = a$ for each ξ, $\xi_0 \leq \xi < \Omega$.

A transfinite sequence $\{f_\xi : \xi < \Omega\}$ of maps of X into a topological space Y is called convergent to a map $f : X \to Y$ if for each $x \in X$ the sequence $\{f_\xi(x) : \xi < \Omega\}$ converegs to $f(x)$; then we write $f = \lim_{\xi<\Omega} f_\xi$.

Now, for a transfinite sequence $\{F_\xi : \xi < \Omega\}$ of multivalued maps the symbol $F = \lim_{\xi<\Omega} F_\xi$ means the convergence of all the sequences $\{F_\xi(x) : \xi < \Omega\}$ for $x \in X$, in the space $S(Y)$ of all the non-empty subsets with the Vietoris topology.

4.3 Theorem.

Let X be a locally separable first countable space and Y a topological one such that the space $C(Y)$ of all the non-empty closed subsets of Y is a first countable T_1-space. Assume that $F, F_\xi : X \to Y$, $\xi < \Omega$, are multivalued maps with closed values and $F = \lim_{\xi<\Omega} F_\xi$. If F_ξ are lower (upper) s - quasi - continuous, then F is lower (upper) s - quasi - continuous.

Proof. Suppose that F is not lower (upper) s - quasi - continuous at a point $x_0 \in X$. Then, there exists an open set $V \subset Y$ with $Y \setminus V$ connected and $F(x_0) \cap V \neq \emptyset$ (resp. $F(x_0) \subset V$), and there exists a neighbourhood U of x_0 such that each open non-empty set $U' \subset U$ contains a point x' for which $F(x') \subset Y \setminus V$ (resp. $F(x') \cap (Y \setminus V) \neq \emptyset$). Without loss of generality, let U be separable. Let $\{z_n : n \geq 1\}$ be a dense subset of U and let $\{W_{nj} : j \geq 1\}$ be an open base at z_n, $W_{nj} \subset U$ for $n, j \geq 1$. So, for each $n, j \geq 1$ we can choose a point $x_{nj} \in W_{nj}$ such that

$$(1) \qquad F(x_{nj}) \subset Y \setminus V, \text{ (resp. } F(x_{nj}) \cap (Y \setminus V) \neq \emptyset).$$

According to Lemma 4.2 there exist numbers $\xi_0, \xi_{nj} < \Omega$, $n, j \geq 1$, such that $F_\xi(x_0) = F(x_0)$ for $\xi, \xi_0 \leq \xi < \Omega$,

$$F_\xi(x_{nj}) = F(x_{nj}) \text{ for } n, j \geq 1, \ \xi_{nj} \leq \xi < \Omega.$$

Thus, we can choose $\alpha < \Omega$ such that $\xi_0 < \alpha, \xi_{nj} < \alpha$ for $n, j \geq 1$. Then, we have $F_\alpha(x_0) = F(x_0)$ and $F_\alpha(x_{nj}) = F(x_{nj})$ for $n, j \geq 1$. Since F_α is lower (upper) s - quasi - continuous at x_0, for some W_{nj} it holds

$$F_\alpha(x) \cap V \neq \emptyset \text{ (resp. } F_\alpha(x) \subset V) \text{ for } x \in W_{nj},$$

which is contradictory to (1). Thus the proof is completed.

Now, we are going to consider the uniform limits of s - quasi - continuous maps. For a uniform space $(Y, **)$ by T_{**} is denoted the topology induced by $**$.

For any point $y \in Y$, a set $A \subset Y$ and $W \in **$ we denote by $W[y] = \{x \in Y : (y, x) \in W\}$ and $W[A] = \bigcup\{W[y] : y \in A\}$.

An open cover $\mathcal{A}$ of Y is said to have the Lebesgue property [10] if there exists $W \in **$ such that $\{W[x] : x \in Y\}$ is a refinement.

4.4 Lemma. [10]

Each binary open cover has the Lebesgue property if and only if for any disjoint closed sets $A, B \subset Y$ there exists $W \in **$ such that $W[A] \cap W[B] = \emptyset$.

We shall use the following property

(*) for each connected closed set $M \subset Y$ and each open set U with $M \subset U$, there exists $W \in **$ such that $W[M] \subset U$.

It follows from Lemma 4.4 that if each binary open cover has the Lebesgue property, then (*) is satisfied. Then converse does not hold.

On pages 10-12 instead of the symbol ** to give the script or German U.

4.5 Example.

In the space R with the natural uniformity, the condition (*) is satisfied. But the open cover consisting of the sets

$$U = R\backslash\{1,2,\ldots\},\ V = \bigcup_{n=1}^{\infty}(n - \frac{1}{n}, n + \frac{1}{n})$$

has not the Lebesque property.

Let $F, F_n, n \geq 1$, be multivalued maps defined on X with values in a uniform space $(Y, **)$. The sequence $\{F_n : n \geq 1\}$ is called uniformly convergent to F, if for each $W \in **$ there is n_0 such that $F_n(x) \subset W[F(x)]$ and $F(x) \subset W[F_n(x)]$ for each $n \geq n_0,\ x \in X$.

4.6 Theorem.

Let X be a topological space, $(Y, **)$ a uniform one with the property (*) and such that (Y, T_{**}) is locally connected. Let $F, F_n : X \to Y, n \geq 1$, be multivalued maps and the sequence $\{F_n : n \geq 1\}$ uniformly converge to F. Then,

(a) If F_n are lower s - quasi - continuous, then F is lower s - quasi - continuous.

(b) If F has the closed values and F_n are upper s - quasi - continuous, then F is upper s - quasi - continuous.

Proof. (a) Let $x_0 \in X, U$ be a neighbourhood of x_0 and let V be an open set with $Y \backslash V$ connected and $F(x_0) \cap V \neq \emptyset$. Let us take $y_0 \in F(x_0) \cap V$, then $y_0 \notin M = Y \backslash V$. Using the regularity of Y and the property (*), we can choose $W \in **$, $W = W^{-1}$ such that $W[y_0] \cap W[M] = \emptyset$. For each $y \in M$ there exists a connected open set V_y satisfying $y \in V_y \subset W[y]$. Because M is connected, the set $V_0 = \bigcup\{V_y : y \in M\}$ is connected and $M \subset V_0 \subset W[M]$. Thus, we have $Cl(V_0) \cap W[y_0] = \emptyset$. Moreover, according to (*) there exists $W_1 \in **, W_1^{-1} = W_1 \subset W$ for which $W_1[M] \subset V_0$. Let n be an established number such that $F_n(x) \subset W_1[F(x)]$ and $F(x) \subset W_1[F_n(x)]$ for each $x \in X$.

Since $y_0 \in F(x_0) \subset W_1[F_n(x_0)]$, for some $z \in F_n(x_0)$ we have $(y_0, z) \in W_1$. From this it follows that $z \notin Cl(V_0)$ and, thus, $F_n(x_0) \cap (Y \backslash Cl(V_0)) \neq \emptyset$. The map F_n is lower s - quasi - continuous at x_0, so there exists an open non-empty set $U_1 \subset U$ such that $F_n(x) \cap (Y \backslash Cl(V_0)) \neq \emptyset$ for $x \in U_1$. Let us take $x \in U_1$ and $y \in F_n(x) \cap (Y \backslash Cl(V_0))$. The condition $y \notin Cl(V_0)$ implies $y \notin W_1[M]$, and $W_1[y] \cap M \neq \emptyset$. From $y \in F_n(x) \subset W_1[F(x)$

] it follows that $(y, z) \in W_1$ for some $z \in F(x)$. Thus $z \notin M$ and consequently, $F(x) \cap (Y \backslash M) \neq \emptyset$. Hence, we have shown $F(x) \cap V \neq \emptyset$ for $x \in U_1$, i.e. F is lower s - quasi - continuous at x_0.

(b) For any $x_0 \in X$ let U be a neighbourhood of this point and $V \subset Y$ an open set with $Y \backslash V$ connected and $F(x_0) \subset V$. The set $M = Y \backslash V$ is connected closed, so applying (*) we can take $W \in **$ such that $W = W^{-1}$ and $W^2[M] \subset Y \backslash F(x_0)$. Hence, we have $W[M] \cap W[F(x_0)] = \emptyset$. Similarly as in the proof of the point (a), we can choose an open connected set V_0 satisfying $M \subset V_0 \subset W[M]$. Thus, $Cl(V_0) \cap W[F(x_0)] = \emptyset$. Using once more the assumption (*), there exists $W_1 \in **, W_1 = W_1^{-1}$ for which $W_1[M] \subset V_0$. Let $n \geq 1$ be an established number such that $F_n(x) \subset W_1[F(x)]$ and $F(x) \subset W_1[F_n(x)]$ for each $x \in X$. Then, we have $F_n(x_0) \subset W_1[F(x_0)] \subset Y \backslash Cl(V_0)$. Since F_n is upper s - quasi - continuous, there exists an open non - empty set $U_1 \subset U$ such that $F_n(x) \subset Y \backslash Cl(V_0)$ for $x \in U_1$. Hence, $F_n(x) \cap W_1[M] = \emptyset$ for $x \in U_1$ and consequently $W_1[F_n(x) \cap M = \emptyset$ for $x \in U_1$. Combining this fact and $F(x) \subset W_1[F_n(x)]$, we obtain $F(x) \subset V$ for $x \in U_1$, which finishes the proof.

5. Remarks on S - quasi - continuous real functions

In this section R denotes the set of real numbers with the natural topology.

A real function f defined on a topological space X is said to be upper (lower) quasi-continuous at a point $x_0 \in X$ if for each neighbourhood U of x_0 and each $\varepsilon > 0$ there exists an open non-empty set $U_1 \subset U$ such that

$$f(x) < f(x_0) + \varepsilon \text{ (resp. } f(x_0) - \varepsilon < f(x)) \text{ for } x \in U_1.$$

A function f is called upper (lower) quasi - continuous if it has this property at each point [6].

The simple examples show that a function which is upper and lower quasi - continuous simultaneously need not be quasi-continuous. But we have

5.1 Remark.

A function $f : X \to R$ is upper and lower quasi-continuous if and only if it is s - quasi - continuous.

We have the following

5.2 Theorem. [6, Th. 6.1 and 6.2]

Let X be a Baire space, Y a second countable one and let $f : X \times Y \to R$ be any function. If f_x, f^y are lower (upper) quasi-continuous for $x \in X, y \in Y$ and f^y are upper (lower) quasi - continuous, then f is lower (upper) quasi - continuous.

Looking carefully at the proofs of these theorems it is easy to see that they are true for a locally second countable space Y. Thus, from the above we obtain immediately.

5.3 Theorem.

Let X be a Baire space, Y a locally second countable one and let $f : X \times Y \to R$ be a function such that f_x, f^y are s - quasi - contionuous for each $x \in X, y \in Y$. Then f is s - quasi - continuous.

Furthermore, we have

5.4 Theorem. [7, Th. 5]

If a function $f : X \to R$ is the pointwise limit of a sequence $\{f_n : n \geq 1\}$ of upper (lower) quasi - continuous functions $f_n : X \to R$, then the set of points at which f is lower (upper) discontinuous is of the first category.

From this it follows simply that:

5.5 Theorem.

If a function $f : X \to R$ is the pointwise limit of a sequence $\{f_n : n \geq 1\}$ of s - quasi - continuous functions $f_n : X \to R$, then the set of points at which f is discontinuous is of the first category.

6. Problems

It is easy to see that for real functions Theorem 6.3 is a stronger result than that which follows from Theorem 3.1 or 3.2. This observation leads to the following

6.1 Problem

Is Theorem 3.1 (resp. 3.2) true if the assumption of lower (upper) quasi - continuity of all the F^y is replaced by lower (upper) s - quasi - continuity?

Let $F_n, F : X \to Y$ be multivalued maps, X, Y topological spaces and let $S(Y)$ be the space of all the non-empty subsets of Y, with the Vietoris topology. We write $F = \lim_{n\to\infty} F_n$, if for each $x \in X$ the sequence $\{F_n(x) : n \geq 1\}$ converges to $F(x)$ in $S(Y)$. We do not know what the limits of upper (lower) s - quasi - continuous maps are. But looking at Theorem 5.5 we obtain

6.2 Problem

Let $F_n, F : X \to Y$ be multivalued maps and $F = \lim_{n\to\infty} F_n$. If F_n are upper (lower) s - quasi - continuous, then, - perhaps under additional assumptions -is the set of points at which F is not lower (upper) semicontinuous of the first category?

References

[1] C. Berge, Espaces topologiques. Fonctions multivoques. Dunod Paris 1966.

[2] S. G. Crossley, S. K. Hildebrand, Semi - closure. Texas J. Sci. 22 (1971), 99-112.

[3] S. G. Crossley, S. K. Hildebrand, Semi - closed sets and semi - continuity in topological spaces. Texas J. Sci. 22 (1971), 123 - 126.

[4] R. Engelking, General topology. Warszawa 1977.

[5] J. Ewert, On almost continuity, Rev. Roumaine Pures Appl. 35 (1990), 33-40

[6] J. Ewert, T. Lipski, Lower and upper quasi - continuous functions. Demonstratio Math. 16 (1983), 85-93.

[7] J. Ewert, M. Przemski, Cliquish, lower and upper quasi - continuous functions. Slupskie Prace Mat. Przyr. 4 (1983), 3 - 12.

[8] H. Hashimoto, On the *-topology and its application. Fund. Math. 91 (1976), 5 - 10

[9] T. Husain, Almost continuous mappings. Prace Matemat. 10 (1966), 1-7.

[10] K. Iseki, On the property of Lebesgue in uniform spaces. IV. Proc. Japan. Acad. 31 (1955), 524 - 525.

[11] S. Kempisty, Sur les fonctions quasicontinues. Fund. Math. 19 (1932), 184 - 197.

[12] P.S. Kenderov, Multivalued mappings and their properties similar to continuity (in Russian). Usp. Mat. Nauk 35 (1980), 194 - 196.

[13] L. K. Kohli, A class of mappings containing all continuous and all semiconnected mappings. Proc. Amer. Math. Soc. 72 (1978). 175 - 181.

[14] L. K. Kohli, S - continuous functions and certain weak forms of regularity and complete regularity. Math. Nachr. 97 (1980). 189 - 196.

[15] L. K. Kohli, S - continuous mappings, certain weak forms of normality and strongly semilocally connected spaces. Math. Nachr. 99 (1980), 69 - 76.

[16] P. Kostyrko, On convergence of transfinite sequences. Mat. Časopis 21 (1971), 233 - 239.

[17] N. Levine, Semi-open sets and semi-comtinuity in topological spaces. Amer. Math. Monthly 70 (1963), 36-41.

[18] N. F. G. Martin, Quasi - continuous functions on product spaces. Duke Math. J. 28 (1961), 39 -44.

[19] T. Neubrunn, On quasicontinuity of multifunctions. Math. Slovaca 32 (1982), 147 - 154.

[20] V. Popa, Asupra unor proprietăti ale multifunctiilor cvasicontinue si aproape continue. St. Cerc. Mat. 30 (1978), 441-446.

[21] V. Popa, H - almost continuous multifunctions. St. Cerc. Mat. 32 (1980), 103 - 109.

[22] I. L. Reilly, M. K. Vamanamurthy, On the topology of semilocal connectedness. Math. Nachr. 129 (1986), 109 - 113.

REZIME

O S – KVAZI – NEPREKIDNIM VIŠEZNAČNIM PRESLIKAVANJIMA

Medju raznim uopštenjima neprekidnosti posmatraju se dva tipa i kod jednog od njih se pretpostavlja da inverzna slika otvorenih skupova pripada jednoj klasi širоj nego topologija – ovaj put vodi – specijalno – kvazi – neprekidnim preslikavanjima [11,17]. Kod drugog inverzne slike podskupova neke podfamilije date topologije su razmatrane. Tada, kao primer postoje s – preslikavanja [13,14,15]. Spajanjem obe ove metode klasa s – kvazi – preslikavanja je uredjena.

Received by the editors November 18, 1988.

Univ. u Novom Sadu
Zb. Rad. Prirod.-Mat. Fak.
Ser. Mat. 20, 1 (1990), 185-193

Review of Research
Faculty of Science
Mathematics Series

WEAK CONTINUITY AND ALMOST CONTINUITY FOR MULTIFUNCTIONS

Valeriu Popa
Department of Mathematics, Higher Education Institute
55000 Bacău, Romania

Abstract

The purpose of the note is to investigate some properties of weakly continuous multifunctions and almost continuous multifunctions and to obatin new characterizations of almost continuous multifunctions using the notion of a semi-open set generalizing the results from [7] - [9].

AMS Mathematics Subjects Classification (1980): 54C60
Key words and phrases: Multifunctions, weakly continuous multifunctions, almost continuous multifunctions.

Weak continuous multifunctions are defined by us in [11] as a generalization of univocal weak continuous applications defined by Levine in [6]. Smithson also defines weak continuous multifunctions in [22]. The properties and some applications of weak continuous multifunctions are given in [1],[2],[11],[12],[14],[16] and [18].

Almost continuous multifunctions are defined by us in [13] as a generalization of univocal almost continuous applications defined by Singal in [20]. The properties of almost continuous multifunctions are studied in [4],[15],[16] and [17].

Let X and Y be two topological spaces. For a multifunction $F : X \longrightarrow Y$, we shall denote by $F^+(G)$ and $F^-(G)$ the upper and lower inverse of the set $G \subset Y$ and thus

$$F^+(G) = \{x \in X : F(x) \subset G.\}; F^-(g) = \{x \in X : F(x) \cap G \neq 0\}.$$

As the paracompact sets can be defined in two way [3], we shall define the paracompact sets in the following way:

Definition 1. *The set M in the topological space X is called strictly paracompact if every covering of M by open sets from X can be refined by a covering by open sets from X, locally finite in X [3].*

It is known from [3] that the notion of the strictly paracompact set and that of the paracompact set defined with the help of the relative topology do not coincide.

Definition 2. *Let X and Y be two topological spaces.*

(a) The multifunction $F : X \longrightarrow Y$ is upper weakly continuous (u.w.c.) respectively, upper almost continuous (u.a.c.) in the point $x_0 \in X$ if for every open set $G \subset Y$ with $F(x_0) \subset G$, there exists an open set $V \subset X$ containing x_0, se that $F(V) \subset ClG$, respectively, $F(V) \subset IntClV$ ($Cl\ G$ stands for the closure of G).

(b) The multifunction $F : X \longrightarrow Y$ is lower weakly continuous (l.w.c.) respectively, lower almost continuous (l.a.c.) in the point $x_0 \in X$ if for every open set $G \subset Y$ with $F(x_0) \cap G \neq 0$, there exists an open set $V \subset X$ conataining x_0, so that $F(x) \cap G \neq 0$, $\forall x \in V$, respectively, $F(x) \cap IntClG \neq 0, \forall x \in V$.

(c) The multifunction $F : X \longrightarrow Y$ is weakly continuous (w.c.), respectively, almost continuous (a.c.), in the point $x_0 \in X$ if F is upper and lower weakly continuous, respectively, upper and lower almost continuous, in the point $x_0 \in X$.

(d) The multifunctional $F : X \longrightarrow Y$ is weakly continuous (u.w.c.,l.w.c.), respectively, almost continuous (u.a.c.,l.a.c.) if it has this property in any point $x \in X$ [11],[13],[22].

The following implications hold

1. u.s.c. $\Rightarrow$ u.a.c. $\Rightarrow$ u.w.c.

2. l.s.c. $\Rightarrow$ u.a.c. $\Rightarrow$ l.w.c. .

If the range Y is a regular space and $F(x), \forall x \in X$ is a strictly paracompact set then

(1') u.s.c. $\Leftrightarrow$ u.a.c. $\Leftrightarrow$ u.w.c.

If the range Y is a regular space, then

(2') l.s.c. $\Leftrightarrow$ l.a.c. $\Leftrightarrow$ l.w.c. [11], [13], [14], [22]

Definition 3. *A topological space X is almost regular if for any point $x \in X$ and each regular closed set $A \subset X$ with $x\overline{\in}A$ there exist disjoint open sets U and V such that $x \in U$ and $A \subset V$ [21].*

Every regular space is almost regular, but the converse is not always true [21].

Definition 4. *A topological is said to be almost compact if every open cover admits a finite subfamily, the closures of whose members cover the space [19].*

Theorem 1. *Let X and Y be two topological spaces. If $F : X \longrightarrow Y$ is a surjective multifunction with an X compact space and Y T_4 space, so that:*

1) F is punctually compact.
2) F is u.w.c.,

then F is u.a.c.

Proof. Let $V = \{V_i : i \in I\}$ be any open cover of $Y.F(x)$ being compact $\forall x \in X$, there exists a finite subfamily $V' = \{V_{i_k} : k = 1, ..., n\}$ of V such that $F(x) \subset \bigcup_{k=1}^{n} V_{i_k}$. Let $V_x = \bigcup_{k=1}^{n} V_{i_k}$, then $F(x) \subset V_x$. The family $\{V_x : x \in X\}$ is then an open cover of $Y.F$ being v.w.c.. By Theorem 6, implication $(1) \Rightarrow (2)$ from [11] it follows that $F^+(V_x)c$ Int $F^+(ClV_x)$. F being a surjectiove multifunction the family $\{F^+(V_x)\}$ is a covering of Y and the family $\{Int\ F^+(ClV_x) : x \in X\}$ is an open cover of Y. Since X is compact it has a finite family $\{IntF^+(ClV_{x_j}) : j = 1, 2, .., m\}$ such that $X = \bigcup_{j=1}^{m} IntF^+(ClV_{x_j})$. F being a surjection, then

$$Y = F(X) = F(\bigcup_{j=1}^{m} IntF^+(ClV_{x_j}) \subset \bigcup_{j=1}^{m} F(F^+(ClV_{x_j}) \subset \bigcup_{j=1}^{m} ClV_{x_j}.$$

Hence we have $Y = \bigcup_{j=1}^{m} ClV_{x_j}$. This implices that Y is almost compact. Every almost compact and T_4-space is almost regular [10]. pp.139).Thus, by Theorem 2.3 of [16], F is u.a.c.

Corollary 1. *If f is a weakly-continuous univocal mapping of a compact space X onto a T_4-space, then f is almost continuous ([8], Theorem 3).*

Theorem 2. *If the multifunction $F_1 : X \longrightarrow Y$ is u.s.c. and the multifunction $F_2 : Y \longrightarrow Z$ is u.a.c., then the multifunction $F = F_2 \circ F_1 : X \longrightarrow Z$ is u.a.c.*

Proof. Let G be any regularly open set of Y. Then $F^+(G) = F_1^+(F_2^+(G))$. According to Theorem 2.4, implication $(1) \Rightarrow (3)$ of [13], $F_2^+(G)$ is an open set of Y. Then, $F^+(G)$ is an open set of Y because F_1 is u.s.c. and F is u.a.c. from Theorem 2.4, implication $(3) \Rightarrow (1)$ of [13].

Theorem 3. *If the multifunction $F_1 : X \longrightarrow Y$ is l.s.c. and the multifunction $F_2 : Y \longrightarrow Z$ is l.a.c., then multifunction $F = F_2 \circ F_1 : X \longrightarrow Z$ is – a.c.*

The proof is similar to the proof of Theorem 2 and follows from Theorem 2.2 of [13].

Let X be a topological space. The family of regularly open sets of X forms a base for a semi-regular topology which is called the semi-regularization of the topology of X [19. In the present note we shall denote by X_S the set X endowed with such a topology.

Theorem 4. *If $F : X \longrightarrow Y$ is a multifunction so that:*

1) Y is an almost regular space,

2) F is punctually strictly paracompact,

then F is weakly continuous if and only if $F : X_S \longrightarrow Y$ is almost continuous.

Proof. Necessity. Suppose F is weakly continuous. Let V be any regularly open set of Y and $x \in F^+_{X_s}(V) = F^+(V)$, namely $F(x) \subset V$.

Let $F(x) = \bigcup_{i\in I}\{y_i\}$. Y being an almost regular space, for the regular open set V and for every y_i there is a regular open set V_i, so that $y_i \in V_i \subset ClV_i \subset V$ according to Theorem 2.2(a) of [21]. So, we have

$$F(x) \subset \bigcup_{i\in I} V_i \subset \bigcup_{i\in I} ClV_i \subset V.$$

$F(x)$ being a strictly paracompact set, there is a family $A = \{A_j : j \in J\}$ of open sets, so that $A_j \subset V_i$ for some $i \in I$ and A is a local finite covering of $F(x)$. We shall have

$$F(x) \subset \bigcup_{j\in J} A_j \subset \bigcup_{i\in I} V_i \subset \bigcup_{i\in I} ClV_i \subset V$$

and

$$F(x) \subset \bigcup_{j\in J} A_j \subset \bigcup_{j\in J} ClA_j \subset \bigcup_{i\in I} ClV_i \subset V.$$

Let $V_0 = \bigcup_{j\in J} A_j$. The family A being locally finite, then $ClV_0 = \bigcup_{j\in J} ClA_j$, so $F(x) \subset V_0 \subset ClV_0 \subset V$.

Since every u.w.c. multifunction, punctually strictly paracompact into an almost regular space, is u.a.c. from Theorem 2.3 of [16], thus by Theorem 2.4, implication (1) ⇒ (2) of [13] it follows that

$$x \in F^+(V_0) \subset IntF^+(IntClV_0) \subset F^+(ClV_0) \subset F^+(V).$$

Int $F^+(IntClV_0)$ is an open set and $F^+(ClV_0)$ is a closed set from Theorem 2.2, implication (1) ⇒ (6) of [13] because F being l.w.c. from Theorem 2.4 of [16] F is l.a.c. and ClV_0 being a regularly closed set.

We shall have

$$x \in F^+(V_0) \subset IntF^+(IntClV_0) \subset$$
$$\subset Int\ ClF^+(IntClV_0) \subset F^+(ClV_0) \subset F^+(V).$$

Let $U = IntClF^+(IntClV_0)$ be. Then, U is a regularly open set of X and

$$x \in F^+(V_0) \subset U \subset F^+(ClV_0) \subset F^+(V) = F^+_{X_S}(V).$$

This implies that $F^+(V)$ is an open set of X_S. Therefore, by Theorem 2.4, implication (3) ⇒ (1) of [13], F_{X_S} is u.a.c.

Then we can prove that F_{X_s} is l.a.c. Let V be any regularly open set of Y and $x \in F^-_{X_S}(V) = F^-(V)$, then $F(x \cap V \neq 0$. From Theorem 2.2 of [21], there is an open set $G \subset Y$ such that $F(x) \cap G \neq \phi$ and $ClG \subset V$. Thus,

$$x \in F^-(G) \subset F^-(ClG) \subset F^-(V).$$

The multifunction F being l.w.c., from Theorem 2.4 of [16], it follows that F is l.a.c.. Then, by Theorem 2.2 of [13], it follows that

$$x \in F^-(G) \subset IntF^-(IntClG) \subset F^-(ClG) \subset F^-(V).$$

Int $F^+(IntClG)$ is an open set and $F^+(ClG)$ is a closed set from Theorem 2.4, implication (1) ⇒ (6) of [13] because F being u.w.c. and punetually strictly paracompact from Theorem 2.3 of [16] is u.a.c. and $Cl\ G$ being a regularly closed set. We shall have

$$x \in F^-(G) \subset IntF^-(IntClG) \subset IntClF^-(IntClG) \subset F^-(ClG) \subset F^-(V)$$

Let $U = IntClF^-(IntClG)$ be.Thus U is a regularly open set and

$$x \in F^-(G) \subset U \subset F^-(ClG) \subset F^-(V) = F^-_{X_S}(V).$$

This implies that $F^-(V)$ is an open set of X_S. Therefore, by Theorem 2.2, impication (3) ⇒ (1) of [13], F_{X_S} is l.a.c.

Sufficiency. Suppose F_{X_S} is almost continuous. Since the identity mapping $i_X : X \longrightarrow X_S$ is a continuous function (multifunction) and $F = F_{X_S} \circ i_X$, then, by Theorem 2 and 3, F is almost continuous. Therefore, F is weakly continuous from (1) and (2).

Corollary 2. *Let Y be an almost regular space. Then, a univocal mapping $f : X \longrightarrow Y$ is weakly continuous if and only if $f_{X_S} : X_S \longrightarrow X$ is almost continuous ([9], Theorem 1)*

Theorem 5. *For a multifunction $F : X \longrightarrow Y$ the following are equivalent:*

1. *F is u.w.c.*
2. *For each open set $G \subset Y, ClF^-(G) \subset F^-(ClG)$ [18].*

Theorem 6. *For a multifunction $F : X \longrightarrow Y$ the following are equivalent:*

1. *F is l.w.c.*
2. *For each open set $G \subset Y, ClF^+(G) \subset F^+(ClG)$ [18].*

Now, we shall prove two similar theorems for almost continuous multifunctions.

Definition 5. *Let X be a topological space. A set $V \subset X$ is said to be semi-open if there exists an open set $G \subset X$ such that $G \subset V \subset ClG$ [6].*

Theorem 7. *For a multifunction $F : X \longrightarrow Y$ the following are equivalent:*

1. *F is u.a.c.*
2. *For each semi-open set $G \subset Y, ClF^-(G) \subset F^-(ClG)$.*

Proof. (1) ⇒ (2). Let G be a semi-open set of Y, then $Cl\,G$ is a closed set and according to theorem 2.4, implication (1) ⇒ (5) of [13], we have

$$F^-(ClG) \supset ClF^-(ClIntClG) \supset ClF^-(ClIntG).$$

The set G being semi-open, then from Theorem 1 of [6], $G \subset ClIntG$, therefore,

$$ClF^-(ClIntG) \supset ClF^-(G)$$

and

$$ClF^{-}(G) \subset F^{-}(ClG).$$

$(2) \Rightarrow (1)$. Since every regularly closed set is semi-open, therefore, for every regularly closed set G, we have

$$ClF^{-}(G) \subset F^{-}(ClG) = F^{-}(G).$$

Therefore, $F^{-}(G)$ is closed and F is u.a.c. from Theorem 2.4, implication $(6) \Rightarrow (1)$ of [13].

Theorem 8. *For a multifunction $F : X \longrightarrow Y$ the following are equivalent*

1. *F is l.a.c.*
2. *For each semi-open set $G \subset Y, ClF^{+}(G) \subset F^{+}(ClG)$.*

The proof is similar to the proof of Theorem 7 and follows from Theorem 2.2 of [13].

Corollary 3. *A univocal application $f : X \longrightarrow Y$ is almost continuous if and only if $Clf^{-1}(G) \subset f^{-1}(ClG)$ for each semi-open set $G \subset Y$. ([7], Theorem 2.1).*

References

[1] Joseph, J. E., Regularity, normality and multifunctions, Proc. Amer. Math. Soc., 70,2 (1978), 203-206.

[2] Joseph, J. E., Regularity, normality and weak continyity for multifunctions, Math. Japonica, 26(1981), 6, 647-651.

[3] Kessler, P., Multimi paracompacte, Anal. St. Mat. Fiz. Chim. Electrotehn. Univ. Craiova, I(1970), 39-44.

[4] Kovačević,I., On multifunctions and paracompactness, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., 12(1982), 61-68

[5] Levine, N., A decomposition of continuity in topological spaces, Amer. Math. Monthly, 68(1961), 44-46.

[6] Levine, N., Semi-open sets and semi-continuity in topological spaces Amer. Math. Monthly, 70(1963), 36-41.

[7] Mathur, A. and Deb, M., A. note on almost continuous mappings, Mathem. Student, XL, 2(1972), 173-184.

[8] Noiri, T., Between continuity and weak-continuity, Boll. U. M. I., (4), 9(1974), 647-654.

[9] Noiri, T., A note on semi-regularizations, Glasnik Math. -10(30), (1975), 141-143.

[10] Papić,P., Sur les espaces H-fermés, Glasnik Math. Fiz. Astr., 14(1959), 135-141.

[11] Popa, V., Weakly continuous multifunctions, Boll. U. M. I., (5), 15-A (1978), 379-388.

[12] Popa, V., On some properties of weakly continuous multifunctions, Studii si cercetări (Matematică), Bacău (1980), 121-126.

[13] Popa, V., Almost continuous multifunctions, Matemat. Vesnik, 34(1982), 75-84.

[14] Popa,V.,Fixed points theorems for multifunctions, Studia Univ. Babes-Bolyai, Mathematica, XXVII (1982), 21-27.

[15] Popa. V., Some properties of almost continuous multifunctions, Matemat. Vesnik, 35(1983), 425-432.

[16] Popa. V., On some weakened forms of continuity for multifunctions Matemat. Vesnik, 36(1984), 339-350.

[17] Popa. V., On almost continuous multifunctions, Review of Research, Faculty of Sience, University of Novi Sad, 15(1985), Mathematics Series, 145-154.

[18] Popa,V.,Sur les multifunctions faiblement continues et H-presque continues, Demonstratio Maathematica 22,2 (1989), 277-283.

[19] Porter, J. and Thomas, J., On H-closed and minimal Hausdorff spaces, Trans. Amer. Math. Soc., 138(1969), 159-170.

[20] Singal. M. K., and Singal, A. R., Almost continuous mappings,Yokohama Math. J., 16(1968), 63-73.

[21] Singal., M. K. and Arya, S. P., On almost regular spaces, Glasnik Math. 4(24), (1969), 89-99.

[22] Smithson, R. E., Almost and weak continuity for multifunctions, Bull. Calcutta Math. Soc., 70(1978), 383-390.

REZIME

SLABA NEPREKIDNOST I GOTOVO NEPREKIDNOST ZA MULTIFUNKCIJE

Svrha ove note je da se istraže neke osobine slabo neprekidnih multifunkcija i i skoro neprekidnih multifunkcija i da se dobiju nove karakterizacije skoro neprekidnih multifunkcija korišćenjem pojma poluotvorenog skupa, čime se uopštavaju rezultati iz [7]-[9].

Received by the editors August 13, 1987.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 195-212

Review of Research
Faculty of Science
Mathematics Series

A NOTE ON CONFORMALLY QUASI - RECURRENT MANIFOLDS

Mileva Prvanović
Institute of Mathematics, University of Novi Sad,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In §1 the condition for the conformally quasi-recurrent manifold (CQR-manifold for short) is found to be conformally recurrent.In §2 the conformal change of CQR- manifold is disscused and some results of [6] completed. In §3 some examples of CQR-manifolds are given. In §4 the umbilical hypersurface of the conformally quasi-recurrent space is studied.

AMS Mathematics Subject Classification (1980): 53B20, 53B21
Key words and phrases: Conformal change, conformal curvature tensor, conformally recurrent manifold

1. Introduction

Let (M, g) be an n-dimensional($n > 3$) Riemannian space and let C^h_{ijk} be its conformal curvature tensor, i.e.

$$C^h_{ijk} = R^h_{ijk} - \tfrac{1}{n-2}(R^h_k g_{ij} - R^h_j g_{ik} + \delta^h_k R_{ij} - \delta^h_j R_{ik}) + \tag{1.1}$$

$$+\frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}),$$

where R^h_{ijk} denotes the curvature tensor of (M,g), R_{ij} is the Ricci tensor, while R is the scalar curvature. It is well known that tensor (1.1) satisfies the relations

$$C_{hijk} = -C_{ihjk} = -C_{hikj} = C_{jkhi}, \tag{1.2}$$

$$C^h_{ijk} + C^h_{jki} + C^h_{kij} = 0, \tag{1.3}$$

$$C^p_{pjk} = C^p_{jpk} = C^p_{jkp} = 0. \tag{1.4}$$

(M,g) is said to be conformally recurrent (CR for short), if C^h_{ijk} satisfies the condition

$$\nabla_s C_{hijk} = a_s C_{hijk}, \tag{1.5}$$

where ∇ is the operator of the covariant derivative with respect to metric g and a_s is a vector field.

(M,g) is said to be conformally quasi-recurrent (CQR for short) if

$$\nabla_s C_{hijk} = 2a_s C_{hijk} + a_h C_{sijk} + a_i C_{hsjk} + a_j C_{hisk} + a_k C_{hijs}. \tag{1.6}$$

If in (1.5) or in (1.6) $a_i = 0$, we have

$$\nabla_s C_{hijk} = 0. \tag{1.7}$$

The Riemannian space satisfyng (1.7) is said to be conformally symmetric (CS for short).

Thus, the class of CR-manifolds as well as the class of CQR-manifolds contains all CS-manifolds. There arises the question: do there exist manifolds which are not CS-manifolds but are CR - as well as CQR-manifolds? And if such manifolds exist, do there exist CQR-manifolds which are not CR-manifolds and CR-manifolds which are not CQR-manifolds?

To answer these questions, we shall first mention the following

Lemma 1. *([7], Lemma 3) If c_j, p_j and B_{hijk} are numbers satisfying*

$$c_s B_{hijk} + p_h B_{sijk} + p_i B_{hsjk} + p_j B_{hisk} + p_k B_{hijs} = 0,$$

$$B_{hijk} = B_{jkhi} = -B_{hikj}, \quad B_{hijk} + B_{hjki} + B_{hkij} = 0,$$

then each $b_j = c_j + 2p_j$ is zero or each B_{hijk} is zero.

Now, we can easily prove

Theorem 1. *The necessary and sufficient condition for a CR-manifold to be a CQR-manifold and for a CQR-manifold be a CR-manifold is*

$$a_s C_{hijk} + a_j C_{hiks} + a_k C_{hisj} = 0. \tag{1.8}$$

Proof. - Let (M, g) be CR and CQR-manifold. Then, besides (1.6), the relation

$$\nabla_s C_{hijk} = k_s C_{hijk}$$

is satisfied, too. Therefore, we have

$$(2a_s - k_s)C_{hijk} + a_h C_{sijk} + a_i C_{hsjk} + a_j C_{hisk} + a_k C_{hijs} = 0. \tag{1.9}$$

In view of (1.2) and (1.3), the conditions of the preceding Lemma are satisfied. Supposing that (M, g) is not conformally flat, it follows that $k_s = 4a_s$. Hence, (1.9) takes the form

$$-2a_s C_{hijk} + a_h C_{sijk} + a_i C_{hsjk} + a_j C_{hisk} + a_k C_{hijs} = 0. \tag{1.10}$$

Permuting the indices s, j, k, cyclically, adding the obtained equations to the preceding one and using (1.2) and (1.3), we obtain (1.8). Now, let us suppose that (1.8) holds good. Then, taking into account (1.2), we also have.

$$a_s C_{hijk} + a_h C_{isjk} + a_i C_{shjk} = 0$$

Adding this to (1.8), we get (1.10). Therefore, (1.6) reduces to

$$C_{hijk} = 4a_s C_{hijk} \tag{1.11}$$

and (1.11) reduces to (1.6). This completes the proof of theorem.

As an immediate consequence of (1.8) and (1.4), we have

$$a_s C^s_{ijk} = 0. \tag{1.12}$$

The converse, of course, is not true, i.e. (1.8) does not follow from (1.12).

On the other hand, relation (1.12) is satisfied for any CQR-manifold [6]. Thus: a CQR-manifold which is not a CR-manifold is that satisfyng (1.6) (and consequently (1.12)), but not satisfyng (1.8). In §3 we give examples of such manifolds.

A CR-manifold which is not a CQR-manifold is that which does not satisfy (1.8). Example 2 given in [9] is an example of such a manifolds.

(M, g) which is a CR-manifold as well as a CQR-manifold is that which satisfies (1.5) (or (1.6)) and (1.8). We call such a manifold a CRQR (conformally recurrent and quasi-recurrent) manifold. In §2 we mention examles of CRQR-manifolds.

We have shown in [6] that if the vector field a_i in (1.6) is a gradient vector field, a CQR-manifold can be conformally related to a CS-one. In this paper, in §2, we discuss the conformal change of whichever CQR-manifold. Particularly, we investigate the conformal change of a CQR-manifold into a CQR-manifold as well as into a CR-one.

In §4 we study the umbilical hypersurface of a conformally quasi-recurrent manifold.

2. Conformal change of a CQR-manifold

Let us consider the conformal change

$$\tilde{g}_{ij} = e^{2\sigma} g_{ij}. \tag{2.1}$$

The Christoffel symbols of metrics $\tilde{g}$ and g are related as follows

$$\left\{ \begin{matrix} \tilde{h} \\ ij \end{matrix} \right\} = \left\{ \begin{matrix} h \\ ij \end{matrix} \right\} + \delta_i^h \sigma_j + \delta_j^h \sigma_i - g_{ij}\sigma^h, \ \ \sigma_i = \frac{\partial \sigma}{\partial x^i},$$

while the conformal curvature tensor is invariant

$$\tilde{C}^h_{ijk} = C^h_{ijk}. \tag{2.2}$$

Let $\tilde{\nabla}$ be the operator of the covariant derivative with respect to metric $\tilde{g}$. Applying it to (2.2), we get

$$\tilde{\nabla}_s \tilde{C}^h_{ijk} = \nabla_s C^h_{ijk} \tag{2.3}$$

$$- \quad 2\sigma_s C^h_{ijk} - \sigma^h C_{sijk} - \sigma_i C^h_{skj} - \sigma_j C^h_{isk} - \sigma_k C^h_{ijs} + $$
$$+ \quad \delta^h_s \sigma_r C^r_{ijk} + g_{is}\sigma^r C^h_{rjk} + g_{js}\sigma^r C^h_{irk} + g_{ks}\sigma^r C^h_{ijr}.$$

Now, let us suppose that both metrics g and $\tilde{g}$ are conformally quasi-recurrent. This means that beside (1.6) and (1.12), we have

$$\tilde{\nabla}_s \tilde{C}^h_{ijk} = 2b_s \tilde{C}^h_{ijk} + \tag{2.4}$$

$$+ b^h \tilde{C}_{sijk} + b_i \tilde{C}^h_{sjk} + b_j \tilde{C}^h_{isk} + b_k \tilde{C}^h_{ijs},$$

$$b_s \tilde{C}^s_{ijk} = 0. \tag{2.5}$$

Substituting (1.6) and (2.4) into (2.3) and taking into account (2.2), we find

$$2(b_s - a_s + \sigma_s)C^h_{ijk} + (b^h - a^h + \sigma^h)C_{sijk} + (b_i - a_i + \sigma_i)C^h_{sjk} + \tag{2.6}$$

$$+ (b_j - a_j + \sigma_j)C^h_{isk} + (b_k - a_k + \sigma_k)C^h_{ijs} =$$

$$= \delta^h_s \sigma_r C^r_{ijk} + g_{is}\sigma^r C^h_{rjk} + g_{js}\sigma^r C^h_{irk} + g_{ks}\sigma^r C^h_{ijr}.$$

Transvecting with respect to h and s and using (1.12), (2.5) and (2.2), we get

$$(n - s)\sigma_r C^r_{ijk} = 0$$

or

$$\sigma_r C^r_{ijk} = 0, \tag{2.7}$$

because of $n > 3$. Therefore, (2.6) reduces to

$$2(b_s - a_s + \sigma_s)C^h_{ijk} + (b^h - a^h + \sigma^h)C_{sijk} + (b_i - a_i + \sigma_i)C^h_{sjk} +$$

$$+(b_j - a_j + \sigma_j)C^h_{isk} + (b_k - a_k + \sigma_k)C^h_{ijs} = 0,$$

from wich, according to the Lemma of §1, there follows $b_s - a_s + \sigma_s = 0$. Thus, we have

Theorem 2. *The conformal change (2.1) transforms a CQR-manifold into a CQR-manifold if and only if function σ satisfies condition (2.7). The corresponding vector fields are related as follows*

$$b_i = a_i - \sigma_i.$$

If in (1.6) vector field a_i is a gradient vector field, then we can choose function σ in (2.1) such that $a_i = \sigma_i$, condition (2.7) being satisfied because of (1.12). But, then $b_i = 0$, and (M, $\tilde{g}$) is a CS-manifold. So,

Corollary 1. [6].- *A CQR-manifold (1.6) whose vector field a_i is a gradient vector field, can be conformally related to a CS-manifold.*

Roter proved in [7] facts which we can summarize as follows:

The conformal change (2.1) transforms a CR-manifold into a CR-manifold if and only if function σ satisfies the condition

$$\sigma_s C_{hijk} + \sigma_j C_{hiks} + \sigma_k C_{hisj} = 0. \tag{2.8}$$

The corresponding vector fields are related as follows

$$b_i = a_i - 4\sigma_i.$$

If in (1.5), vector field a_i is a gradient vector field satisfyng (1.8), we can choose function σ in (2.1) such that $\sigma_i = \frac{a_i}{4}$. Then $b_i = 0$ and (M,$\tilde{g}$) is a CS-manifold. Thus, we have

Corollary 2. *A CR-manifold (1.5) whose vector field a_i is a gradient and satisfies (1.8), can be conformally related to a CS-manifold.*

The CR-manifolds of Corollary 2 are called by Roter [8] SCR (special conformally recurrent) manifolds. According to the results obtained in §1, we have

Corollary 3. *Any SCR-manifold is a CRQR-manifold.*

Looking at Corollaries 1 and 2 on the part of a CS-manifold, we can say as follows:

Theorem 3. *If a CS-manifold allows function σ satisfyng (2.8), it can be conformally related to a CRQR-manifold.*

If a CS-manifold allows function σ satisfyng (2.7) but not satisfyng (2.8), it can be conformally related to an essentially CQR-manifold (that is, to a CQR-manifold which is not a CRQR-manifold).

So, to construct examples of essentially CQR-manifolds, it is sufficient to find a CS-manifold allowing function σ, such that the conditions of the second part of Theorem 3 are satisfied. We shall show in §3 that such SC-manifolds exist.

In the remaining part of this §, we shall investigate the conformal change of a CQR-manifold into a CR-manifold. In other words, we shall suppose that metric g satisfies (1.6), while metric $\tilde{g}$ satisfies

$$\tilde{\nabla}_s \tilde{C}^h_{ijk} = c_s \tilde{C}^h_{ijk}. \tag{2.9}$$

Substituting (1.6) and (2.9) into (2.3) and taking into account (2.2), we find

$$\begin{aligned} c_s C^h_{ijk} = {} & 2(a_s - \sigma_s) C^h_{ijk} + (a^h - \sigma^h) C_{sijk} + (a_i - \sigma_i) C^h_{sjk} + \\ & + (a_j - \sigma_j) C^h_{isk} + (a_k - \sigma_k) C^h_{ijs} + \\ & + \delta^h_s \sigma_r C^r_{ijk} + g_{is} \sigma^r C^h_{rjk} + g_{js} \sigma^r C^h_{irk} + g_{ks} C^h_{ijr} \sigma^r . \end{aligned} \tag{2.10}$$

Now, following step by step the way described in [7], we can conclude:

If the conformal change (2.1) transforms a CQR-manifold into a CR-manifold, function σ satisfies condition (2.7).

Substituting (2.7) into (2.10), we get

$$(2a_s - c_s - 2\sigma_s) C^h_{ijk} =$$

$$= (a^h - \sigma^h) C_{sijk} + (a_i - \sigma_i) C^h_{sjk} + (a_j - \sigma_j) C^h_{isk} + (a_k - \sigma_k) C^h_{ijs} = 0,$$

from which follows, according to the Lemma of §1,

$$c_s = 4(a_s - \sigma_s). \tag{2.11}$$

Therefore, the preceding relation reduces to

$$-2(a_s-\sigma_s)C_{hijk}+(a_h-\sigma_h)C_{sijk}+(a_i-\sigma_i)C_{hsjk}+$$
$$+(a_j-\sigma_j)C_{hisk}+(a_k-\sigma_k)C_{hijs}=0.$$

Permuting the indices j, s, k cyclically, adding the obtained equations to the preceding one and using (1.2) and (1.3), we get

$$(2.12)\qquad (a_j-\sigma_j)C_{hisk}+(a_k-\sigma_k)C_{hijs}+(a_s-\sigma_s)C_{hijk}=0.$$

Reversely, suppose that the vector field a_i and function σ satisfy the conditions (1.6) and (2.12). Then, the condition

$$(a_h-\sigma_h)C_{sijk}+(a_i-\sigma_i)C_{hsjk}+(a_s-\sigma_s)C_{ihjk}=0$$

is satisfied, too, so that, adding, we can find

$$(2.13)\qquad 4(a_s-\sigma_s)C_{hijk}=2(a_s-\sigma_s)C_{hijk}+$$

$$+(a_h-\sigma_h)C_{sijk}+(a_i-\sigma_i)C_{hsjk}+(a_j-\sigma_j)C_{hisk}+(a_k-\sigma_k)C_{hijs}.$$

Transvecting (2.12) with g^{hs}, we get

$$(a_s-\sigma_s)C^s_{ijk}=0,$$

which, because of (1.12), reduces to (2.7). Now, substituting (1.6) and (2.7) into (2.3), we have

$$\tilde\nabla_s\tilde C^h_{ijk}=2(a_s-\sigma_s)C^h_{ijk}+(a^h-\sigma^h)C_{sijk}+(a_i-\sigma_i)C^h_{sjk}+$$
$$+(a_j-\sigma_j)C^h_{isk}+(a_k-\sigma_k)C^h_{ijs}.$$

This can, taking into account (2.13) and (2.2) be rewritten in the form

$$\tilde\nabla_s\tilde C^h_{ijk}=4(a_s-\sigma_s)\tilde C^h_{ijk}.$$

Thus,

Theorem 4. *Let (M,g) be a CQR-manifold (1.6) and let (2.1) be a conformal change. Then $(M,\tilde g)$ is a CR-manifold if and only if a condition (2.12) is satisfied and (2.11) is the corresponding recurrence vector.*

The conditions (2.11) and (2.12) show that (M,$\tilde g$) is, in fact, CRQR-manifold.

3. Examples of CQR-manifolds

Investigating CS-manifolds, Derdzinski and Roter [4] proved that any essentially CS-manifold (i.e. a manifold which is neither conformally flat nor locally symetric), admits a unique function F (called the fundamental function) such that

$$FC_{hijk} = R_{ij}R_{hk} - R_{ik}R_{hj}.$$

Also, Derdzinski proved [1] that if (M,g) is an essentially CS-manifold with a non-recurrent Ricci tensor and whose fundamental function F is a constant, then one of the following cases hold.

1. $F \neq 0$, $\operatorname{rank} R_{ij} = 2$ and R_{ij} is semidefinite at each point of M;
2. $F \neq 0$, $\operatorname{rank} R_{ij} = 2$ everywhere and R_{ij} is semidefinite at no point of M;
3. $F = 0$; then rank $R_{ij} \leq 1$ everywhere.

These three cases are called elliptic, hyperbolic and parabolic, respectively. Derdzinski determined the metrics for all those manifolds ([1], [2], [3]). For example, he proved thet if (M,g) is an n-dimensional ($n \geq 4$) elliptic CS-manifold and $p \in M$ is a point at which $\nabla_k R_{ij} \neq 0$, there exists a coordinate system $(u^1, u^2, \ldots, u^n)$ in a neighbourthood of p such that $u^1(p) = \ldots = u^n(p) = 0$ and

$$(g_{ij}) = \left(\begin{array}{ccccc} g_{11} & g_{12} & 0\ldots0 & 0 & g_{1n} \\ g_{21} & g_{22} & 0\ldots0 & g_{2n-1} & 0 \\ 0 & 0 & & 0 & 0 \\ \vdots & \vdots & g_{zx} & \vdots & \vdots \\ 0 & 0 & & 0 & 0 \\ 0 & g_{n-12} & 0\ldots0 & 0 & 0 \\ g_{n1} & 0 & 0\ldots0 & 0 & 0 \end{array}\right) \tag{3.1}$$

where x, y run over $3, \ldots, n-2$;

$$\text{(3.2)}\quad \begin{cases} g_{11} = 2u^n e^{-T} + 2u^{n-1} e^T \frac{\partial T}{\partial u^2} - \frac{1}{n-2} \varepsilon e^{2T} \sum_x \varepsilon_z (u^x)^2 \\ \qquad -(n-2)\varepsilon F^{-1} e^{2T}, \\ g_{22} = 2u^n (e^T \frac{\partial T}{\partial u^1} - e^{-T}) - \frac{1}{n-2} \varepsilon e^{2T} \sum_x \varepsilon_x (u^x)^2 \\ \qquad -(n-2)\varepsilon T^{-1} e^{2T}, \\ g_{12} = -u^n e^T \frac{\partial T}{\partial u^2} - u^{n-1} (e^T \frac{\partial T}{\partial u^1} + 2e^{-T}), \\ g_{1n} = g_{2n-1} = e^T, \;\; g_{zx} = \varepsilon_x, \;\; |\varepsilon_x| = 1. \end{cases}$$

T is a function of the first two variables u^1, u^2 and is given by

$$T = -\frac{1}{2} \log S,$$

where

$$2S(p) = \max\{\nabla_k R_{ij} v^i v^j v^k | v^i \in T_p(M), R_{ij} v^i v^j = \varepsilon, \varepsilon = \pm\},$$

and satisfies the quasi-linear elliptic partial equation

$$\text{(3.3)}\quad \frac{\partial^2 T}{\partial u^1 \partial u^1} + \frac{\partial^2 T}{\partial u^2 \partial u^2} - 2e^{-4T} + \frac{1}{n-2} \varepsilon e^{2T} = 0.$$

Reversely, given the real number $F \neq 0, \varepsilon, \varepsilon_x$ with $|\varepsilon| = |\varepsilon_x| = 1$ and a function T of two real variables (u^1, u^2) satisfyng (3.3). Then, (3.1) and (3.2) define an essentially CS-Riemannian metric with a fundamental function equal to F, which is elliptic (namely, its Ricci tensor is ε-semidefinite).

With respect to this coordinate system, the only non-zero components of the conformal curvature tensor are C_{1212} and $C^n_{\;212}, C^{n-1}_{\;\;112}$. Now let us consider the function

$$\text{(3.4)}\quad \sigma = \sigma(u^1, \dots, u^{n-2}).$$

Then

$$\sigma_{n-1} = \frac{\partial \sigma}{\partial u^{n-1}} = 0, \sigma_n = \frac{\partial \sigma}{\partial u^n} = 0,$$

and

$$\sigma_x = \frac{\partial \sigma}{\partial u^x} \neq 0, x = 3, \dots, n-2.$$

Therefore,

$$\sigma_x C_{1212} + \sigma_1 C_{122x} + \sigma_2 C_{12x1} = \sigma_x C_{1212} \neq 0,$$

but

$$\sigma_r C^r_{212} = \sigma_n C^n_{212} = 0 \text{ and } \sigma_r C^r_{112} = \sigma_{n-1} C^{n-1}{}_{112} = 0.$$

So, condition (2.7) is satisfied while (2.8) is not. According to Theorem 3, the conformal change (2.1), (3.4) of the metric (3.1), (3.2), (3.3) is essentially CQR-metric. But if

$$\sigma = \sigma(u^1, u^2),$$

then condition (2.8) is satisfied too, and $(M, \tilde{g})$ is CRQR-manifold.

Proceeding in a similar manner in the case of a CS-manifold of the hyperbolic type [2] or of the parabolic type [3], we can obtain new examples of essentially CQR-manifolds.

4. Totally umbilical hypersurface of conformally quasi-recurrent space

Let $(\bar{M}, \bar{g})$ be an $(n+1)$-dimensional Riemannian space covered by a system of coordinate neighbourhoods $\{U, y^\alpha\}$. Let (M, g) be a hypersurface of $\bar{M}$, defined in a locally coordinate system by means of a system of parametric equations $y^\alpha = y^\alpha(x^i)$. Here and in the sequel, Greek indices take values $1, 2, \ldots, n+1$ and Latin indices - the values $1, 2, \ldots, n$. Let N^α be a local unit normal to (M, g). Then, we have

$$g_{ij} = \bar{g}_{\alpha\beta} B^\alpha_i B^\beta_j, \tag{4.1}$$

$$\bar{g}_{\alpha\beta} N^\alpha B^\beta_i = 0, \bar{g}_{\alpha\beta} N^\alpha N^\beta = \varepsilon, \varepsilon = \pm 1 \tag{4.2}$$

and

$$B^\alpha_i B^\beta_j g^{ij} = \bar{g}^{\alpha\beta} - \varepsilon N^\alpha N^\beta, \tag{4.3}$$

where

$$B^\alpha_i = \frac{\partial y^\alpha}{\partial x^i}.$$

The (M, g) is called a totally umbilical hypersurface of the $(\bar{M}, \bar{g})$ if its second fundamental form h_{ij} satisfies.

$$h_{ij} = H g_{ij},$$

where H is a scalar function. For a totally umbilical hypersurface (M,g) of $(\bar{M},\bar{g})$, the equations of Gauss and Codazzi, respectively, can be written in the forms

$$\bar{R}_{\alpha\beta\gamma\delta}B^{\alpha}_{i}B^{\beta}_{j}B^{\gamma}_{k}B^{\delta}_{l} = R_{ijkl} - H^2(g_{il}g_{jk} - g_{ik}g_{jl}), \tag{4.4}$$

$$\bar{R}_{\alpha\beta\gamma\delta}N^{\alpha}B^{\beta}_{j}B^{\gamma}_{k}B^{\delta}_{l} = g_{jk}H_l - g_{jl}H_k. \tag{4.5}$$

Here, $\bar{R}_{\alpha\beta\gamma\delta}$ are the components of the curvature tensor of $(\bar{M},\bar{g})$ and $H_l = \nabla_l H$.

Contracting (4.4) with g^{il} and taking into account (4.3), we find

$$\bar{R}_{\beta\gamma}B^{\beta}_{j}B^{\gamma}_{k} = \varepsilon\bar{R}_{\alpha\beta\gamma\delta}N^{\alpha}B^{\beta}_{j}B^{\gamma}_{k}N^{\delta} + R_{jk} - \varepsilon(n-1)H^2 g_{jk}. \tag{4.6}$$

Contracting (4.6) with g^{jk} and using (4.3), we obtain

$$\bar{R}_{\beta\gamma}N^{\beta}N^{\gamma} = \tfrac{\varepsilon}{2}(\bar{R} - R) + \tfrac{n(n-1)}{2}H^2. \tag{4.7}$$

On the other hand, contracting (4.5) with g^{jk}, we find

$$\bar{R}_{\alpha\delta}N^{\alpha}B^{\delta}_{l} = (n-1)H_l. \tag{4.8}$$

Now, let us consider the conformal curvature tensor of the space $(\bar{M},\bar{g})$:

$$\bar{C}_{\alpha\beta\gamma\delta} = \bar{R}_{\alpha\beta\gamma\delta} - \frac{1}{n-1}(\bar{R}_{\alpha\delta}\bar{g}_{\beta\gamma} - \bar{R}_{\alpha\gamma}\bar{g}_{\beta\delta} + \bar{R}_{\beta\gamma}\bar{g}_{\alpha\delta} - \bar{R}_{\beta\delta}\bar{g}_{\alpha\gamma}) + \frac{\bar{R}}{n(n-1)}(\bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma} - \bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta}).$$

Then we have [5]:

$$\bar{C}_{\alpha\beta\gamma\delta}N^{\alpha}B^{\beta}_{j}B^{\gamma}_{k}B^{\delta}_{l} = 0. \tag{4.9}$$

Also

$$\bar{C}_{\alpha\beta\gamma\delta}N^{\alpha}B^{\beta}_{j}B^{\gamma}_{k}N^{\delta} = \bar{R}_{\alpha\beta\gamma\delta}N^{\alpha}B^{\beta}_{j}B^{\gamma}_{k}N^{\delta} - - \frac{1}{n-1}(\bar{R}_{\alpha\delta}N^{\alpha}N^{\delta}g_{jk} + \varepsilon\bar{R}_{\beta\gamma}B^{\beta}_{j}B^{\gamma}_{k}) + \frac{\bar{R}}{n(n-1)}\varepsilon g_{jk}.$$

Substituting (4.6) and (4.7), we obtain

$$\bar{C}_{\alpha\beta\gamma\delta}N^{\alpha}B^{\beta}_{j}B^{\gamma}_{k}N^{\delta} = \varepsilon\frac{n-2}{n-1}\bar{R}_{\beta\gamma}B^{\beta}_{j}B^{\gamma}_{k} - \varepsilon R_{jk} +$$

$$+ [-\frac{n-2}{2n(n-1)}\varepsilon\bar{R} + \frac{1}{2(n-1)}\varepsilon R + \frac{n-2}{2}H^{2}]g_{jk}.$$

This can be rewritten in the form

$$\frac{1}{n-1}\bar{R}_{\beta\gamma}B^{\beta}_{j}B^{\gamma}_{k} = \frac{1}{n-2}Q_{jk} + \frac{1}{n-2}R_{jk} + \tag{4.10}$$

$$+ [\frac{1}{2n(n-1)}\bar{R} - \frac{1}{2(n-1)(n-2)}R - \frac{\varepsilon}{2}H^{2}]g_{jk},$$

where we have put

$$\bar{C}_{\alpha\beta\gamma\delta}N^{\alpha}B^{\beta}_{j}B^{\gamma}_{k}N^{\delta} = \varepsilon Q_{jk}. \tag{4.11}$$

Substituting (4.10) into

$$\bar{C}_{\alpha\beta\gamma\delta}B^{\alpha}_{i}B^{\beta}_{j}B^{\gamma}_{k}B^{\delta}_{l} = \bar{R}_{\alpha\beta\gamma\delta}B^{\alpha}_{i}B^{\beta}_{j}B^{\gamma}_{k}B^{\delta}_{l} -$$

$$- \frac{1}{n-1}(\bar{R}_{\alpha\delta}B^{\alpha}_{i}B^{\delta}_{l}g_{jk} - \bar{R}_{\alpha\gamma}B^{\alpha}_{i}B^{\gamma}_{k}g_{jl} + \bar{R}_{\beta\gamma}B^{\beta}_{j}B^{\gamma}_{k}g_{il} - \bar{R}_{\beta\delta}B^{\beta}_{j}B^{\delta}_{l}g_{ik}) +$$

$$+ \frac{\bar{R}}{n(n-1)}(g_{il}g_{jk} - g_{ik}g_{jl}),$$

and taking into account (4.4), we find

$$\bar{C}_{\alpha\beta\gamma\delta}B^{\alpha}_{i}B^{\beta}_{j}B^{\gamma}_{k}B^{\delta}_{l} = C_{ijkl} - \frac{1}{n-2}(Q_{il}g_{jk} - Q_{ik}g_{jl} + Q_{jk}g_{il} - Q_{jl}g_{ik}), \tag{4.12}$$

where $C_{ijkl}$is the conformal curvature tensor of the hypersurface (M, g).

We note that

$$Q_{jk} = Q_{kj} \text{ and } Q_{jk}g^{jk} = 0, \tag{4.13}$$

and

$$\bar{C}_{\alpha\beta\gamma\delta}B^{\alpha}_{i}B^{\beta}_{j}B^{\gamma}_{k}B^{\delta}_{l}g^{il} = -Q_{jk}. \tag{4.14}$$

Let ∇ be the operator of the van der Wearden-Bortolotti covariant derivative. Then

$$\nabla_r B_i^\beta = \varepsilon h_{rj} N^\beta .$$

Applying the operator ∇ to (4.12) and using (4.9), we have

$$\nabla_\rho \bar{C}_{\alpha\beta\gamma\delta} B_r^\rho B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta = \nabla_r C_{ijkl} -$$

$$- \tfrac{1}{n-2}(g_{jk}\nabla_r Q_{il} - g_{jl}\nabla_r Q_{ik} + g_{il}\nabla_r Q_{jk} - g_{ik}\nabla_r Q_{jl}). \tag{4.15}$$

Now, let us suppose that $(\bar{M}, \bar{g})$ is conformally quasi-recurrent. Then,

$$\bar{\nabla}_\rho \bar{C}_{\alpha\beta\gamma\delta} = 2a_\rho \bar{C}_{\alpha\beta\gamma\delta} + a_\alpha \bar{C}_{\rho\beta\gamma\delta} + a_\beta \bar{C}_{\alpha\rho\gamma\delta} + a_\gamma \bar{C}_{\alpha\beta\rho\delta} + a_\delta \bar{C}_{\alpha\beta\gamma\rho},$$

and we have

$$\bar{\nabla}_\rho \bar{C}_{\alpha\beta\gamma\delta} B_r^\rho B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta = 2a_\rho B_r^\rho \bar{C}_{\alpha\beta\gamma\delta} B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta +$$

$$+ a_\alpha B_i^\alpha \bar{C}_{\rho\beta\gamma\delta} B_r^\rho B_j^\beta B_k^\gamma B_l^\delta + a_\beta B_j^\beta \bar{C}_{\alpha\rho\gamma\delta} B_i^\alpha B_r^\rho B_k^\gamma B_l^\delta +$$

$$+ a_\gamma B_k^\gamma \bar{C}_{\alpha\beta\rho\delta} B_i^\alpha B_j^\beta B_r^\rho B_l^\delta + a_\delta B_l^\delta \bar{C}_{\alpha\beta\gamma\rho} B_i^\alpha B_j^\beta B_k^\gamma B_r^\rho .$$

As for vector field a_α, it can be decomposed, at the points of (M, g), as follows

$$a_\alpha = \bar{g}_{\alpha\mu} a^\mu = \bar{g}_{\alpha\tau}(B_t^\tau a^t + a N^\tau). \tag{4.16}$$

Substituting this, (4.15) and (4.12) into the preceding relation and putting $a_r = g_{rt} a^t$, we find

$$\nabla_r C_{ijkl} = 2a_r C_{ijkl} + a_i C_{rjkl} + a_j C_{irkl} + a_k C_{ijrl} + a_l C_{ijkr} + \tag{4.17}$$

$$+ \frac{1}{n-2}\{(g_{jk}\nabla_r Q_{il} - g_{jl}\nabla_r Q_{ik} + g_{il}\nabla_r Q_{jk} - g_{ik}\nabla_r Q_{jl}) -$$

$$- 2a_r(g_{jk}Q_{il} - g_{jl}Q_{ik} + g_{il}Q_{jk} - g_{ik}Q_{jl}) -$$

$$- a_i(g_{jk}Q_{rl} - g_{jl}Q_{rk} + g_{rl}Q_{jk} - g_{rk}Q_{jl}) -$$

$$- a_j(g_{rk}Q_{il} - g_{rl}Q_{ik} + g_{il}Q_{rk} - g_{ik}Q_{rl}) -$$
$$- a_k(g_{jr}Q_{il} - g_{jl}Q_{ir} + g_{il}Q_{jr} - g_{ir}Q_{jl}) -$$
$$- a_l(g_{jk}Q_{ir} - g_{jr}Q_{ik} + g_{ir}Q_{jk} - g_{ik}Q_{jr})\}.$$

The space $(\bar{M}, \bar{g})$ being conformally quasi-recurrent, the condition of the form (1.12) is satisfied, i.e. $a^\rho \bar{C}_{\rho\alpha\beta\gamma} = 0$. Thus,

$$a^\rho \bar{C}_{\rho\beta\gamma\delta} B^\beta_j B^\gamma_k B^\delta_l = 0$$

which can, taking into account (4.16) and (4.9), be rewritten in the form

$$a^r \bar{C}_{\rho\beta\gamma\delta} B^\rho_r B^\beta_j B^\gamma_k B^\delta_l = 0.$$

Substituting (4.12) into this equation, we find

$$a_r C^r_{jkl} = \tfrac{1}{n-2} a^r (g_{jk}Q_{rl} - g_{jl}Q_{rk} + g_{rl}Q_{jk} - g_{rk}Q_{jl}). \tag{4.18}$$

Contracting (4.18) with g^{jk} and using (4.13). we get

$$a^r Q_{rj} = 0. \tag{4.19}$$

Therefore, (4.18) reduces to

$$a^r C_{rjkl} = \tfrac{1}{n-2}(a_l Q_{jk} - a_k Q_{jl}). \tag{4.20}$$

On the other hand, contracting (4.17) with g^{jk}, and using (1.2). (4.13), (4.19) and (4.20), we have

$$\nabla_r Q_{il} = 2a_r Q_{il} + a_i Q_{rl} + a_l Q_{ir}.$$

Substituting this into (4.17), we find

$$\begin{aligned} \nabla_r C_{ijkl} = {} & 2a_r C_{ijkl} + a_i C_{rjkl} + a_j C_{irkl} + a_k C_{ijrl} + a_l C_{ijkr} + \\ (4.21) \qquad & + \tfrac{1}{n-2}[g_{rl}(a_j Q_{ik} - a_i Q_{jk}) + g_{rk}(a_i Q_{jl} - a_j Q_{il}) + \end{aligned}$$

$$+ g_{rj}(a_l Q_{ik} - a_k Q_{il}) + g_{ri}(a_k Q_{jl} - a_l Q_{jk})].$$

Thus, if

$$a_j Q_{ik} = a_i Q_{jk} \tag{4.22}$$

then (M,g) is conformally quasi-recurrent. Conversely, if (M,g) is conformally quasi-recurrent, then (4.21) reduces to

$$g_{rl}(a_j Q_{ik} - a_i Q_{jk}) + g_{rk}(a_i Q_{jl} - a_j Q_{il}) +$$

$$g_{rj}(a_l Q_{ik} - a_k Q_{il}) + g_{ri}(a_k Q_{jl} - a_l Q_{jk}) = 0,$$

from which, contracting with g^{rl}, we obtain (4.22). Thus, we have

Theorem 5. *Let $(\bar{M},\bar{g})$ be CQR-manifold and let (M,g) be its totally umbilical hypersurface. The (M,g) is CQR-manifold if and only if the vector field $a_i = a_\alpha B_i^\alpha$ satisfies (4.22). If $a_i = 0$, i.e. if at the points of (M,g), the vector field a^α is orthogonal to (M,g), (M,g) is CS-manifold.*

Now, let us suppose that $(\bar{M}\bar{g})$ is CRQR-manifold. Then, the condition

$$a_\delta \bar{C}_{\alpha\beta\gamma\delta} + a_\gamma \bar{C}_{\alpha\beta\delta\sigma} + a_\delta \bar{C}_{\alpha\beta\sigma\gamma} = 0 \tag{4.23}$$

is satisfied, too, Contracting it with $N^\sigma B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta$ and using (4.9), we find

$$a_\sigma N^\sigma \bar{C}_{\alpha\beta\gamma\delta} B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta = 0.$$

If, at the points of (M,g), vector filed a^α satisfies $a_\sigma N^\sigma \neq 0$ the preceding relation reduces to

$$\bar{C}_{\alpha\beta\gamma\delta} B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta = 0,$$

from which, using (4.14), we obtain $Q_{jk} = 0$. Therefore, (4.12) reduces to $C_{ijkl} = 0$.

On the other hand, contracting (4.23) with $B_s^\sigma B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta$ we get

$$a_s \bar{C}_{\alpha\beta\gamma\delta} B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta + a_k \bar{C}_{\alpha\beta\delta\sigma} B_i^\alpha B_j^\beta B_l^\delta B_s^\sigma +$$

$$+ a_l \bar{C}_{\alpha\beta\sigma\gamma} B_i^\alpha B_j^\beta B_s^\sigma B_k^\gamma = 0$$

which, in view, of (4.12), can be written in the form

$$a_s C_{ijkl} + a_k C_{ijls} + a_l C_{ijsk} + \tag{4.24}$$

$$+ \frac{1}{n-2}[g_{jl}(a_s Q_{ik} - a_k Q_{is}) + g_{js}(a_k Q_{il} - a_l Q_{ik}) + g_{jk}(a_l Q_{is} - a_s Q_{il}) + \\ + g_{ik}(a_s Q_{jl} - a_l Q_{js}) + g_{il}(a_k Q_{js} - a_s Q_{jk} + g_{is}(a_l Q_{jk} - a_k Q_{jl})] = 0$$

According to Theorem 5, the hypersurface is conformally quasi-recurrent if and only if condition (4.22) is satisfied. But in this case, (4.24) reduces to

$$a_s C_{ijkl} + a_k C_{ijls} + a_l C_{ijsk} = 0.$$

This means that the hypersurface is CRQR-manifold. Thus, we have

Theorem 6. *Let $(\bar{M}, \bar{g})$ be a CRQR-manifold and let (M, g) be its totally unbilical hypersurface. If at the points of (M, g) the vector field a^α is not tangetial to (M, g), (M, g) is conformally flat. If at the points of (M, g) the vector field a^α is tangential to (M, g) and (M, g) is a CQR-manifold, it is a CRQR-manifold, too.*

References

[1] A.Derdziński, The local structure of essentially conformally symmetric manifolds with constant fundamental function I, The elliptic case, Colloquium Mathematicum 42 (1979), pp. 59-81

[2] -, The local structure of essentially conformally symmetric manifolds with constant fundamental function II, The hyperbolic case, ibidem 44 (1981), pp.77-95

[3] -, The local structure of essentially conformally symmetric manifolds with constant fundamental function III, The parabolic case, ibidem 44 (1981), pp. 249-263

[4] - and W. Roter, Some properties of conformally symmetric manifolds which are not Ricci-recurrent, Tensor, 34 (1980), pp. 11-19

[5] Z. OLszak, Remarks on manifolds adnitting umbilical hypersurfaces, Demonstratio Math. 11 (1978), pp.695-702

[6] M. Prvanović. Some theorems on conformally quasi-recurrent manifolds, Univ. u Novom Sadu Zb. Rad. Prirod. - Mat. Fak. Ser. Mat. 19,2 (1989), 21-31.

[7] W. Roter, On conformally related conformally recurrent metrics I, Some general results, Colloquium Mathematicum 47 (1982), pp. 39-46

[8] -, On a class of conformally recurrent manifolds, Tensor, 39 (1982), pp. 207-217

[9] -, On the existence of certain conformally recurrent metrics, Colloquium Mathematicum, 51 (1987), pp. 315-327

REZIME

PRIMEDBA O KONFORMNO KVAZI REKURENTNIM MNOGOSTRUKOSTIMA

Ispitani su uslovi pod kojima se konformno kvazi rekurentna mnogostrukost svodi na konformno rekurentnu. Dati su primeri konformno kvazi rekurentnih prostora koji nisu ni konformno simetrični ni konformno rekurentni. Ispitane su totalno ombilične hiperpovrši konformno kvazi rekurentnih prostora.

Recived by the editors April 19, 1989.

UDK 51

YU ISSN 0352—0900

УНИВЕРЗИТЕТ У НОВОМ САДУ — UNIVERSITY OF NOVI SAD

ЗБОРНИК РАДОВА
ПРИРОДНО-МАТЕМАТИЧКОГ ФАКУЛТЕТА
СЕРИЈА ЗА МАТЕМАТИКУ

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

20

No. 2

НОВИ САД — NOVI SAD

1990.

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

INSTITUTE OF MATHEMATICS
TRG DOSITEJA OBRADOVIĆA 4
21000 NOVI SAD
YUGOSLAVIA

EDITORIAL BOARD

Editor in Chief: **O. Hadžić**, Novi Sad
Associate Editor in Chief: **D. Herceg**, Novi Sad
Editorial Secretary: **R. Vulanović**, Novi Sad

Editors:
E. Allgower, Fort Collins, CO
K.Gilezan, Novi Sad
A. Ivić, Belgrade
Z. Ivković, Belgrade
A. Kamiński, Katowice
V. Marić, Novi Sad
G. Milovanović, Niš
E. Pap, Novi Sad
S. Pilipović, Novi Sad
M. Prvanović, Novi Sad
B. Stanković, Novi Sad
Z. Stojaković, Novi Sad
B. Šešelja, Novi Sad
R. Tošić, Novi Sad
J. Ušan, Novi Sad
V. Vladimirov, Moscow

The journal publishes original research papers from all fields of pure and applied mathematics. When preparing the manuscript please consider "Instructions to Authors" at the end of each number. Manuscripts and inquiries should be directed to the associate editor in chief. Twenty offprints of each paper are supplied free of charge. Additional copies may be ordered at cost price at time when proofs are returned.

Annual subscription rate for foreign countries: US $ 60 (one volume of two numbers).

Orders, exchange journals, manuscripts and all corespondence concering offprints should be sent to the associate editor in chief:

D. Herceg
Institute of Mathematics
University of Novi Sad
Trg Dositeja Obradovića 4
21000 Novi Sad, Yugoslavia

Published by the Institute of Mathematics, Novi Sad, Yugoslavia
Printed by FTN, Štamparija za grafičku delatnost, Novi Sad
Circulation: 500

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 1-10

Review of Research
Faculty of Science
Mathematics Series

ON THE NUMERICAL SOLUTION OF A SINGULARLY PERTURBED NONLOCAL PROBLEM

Dragoslav Herceg
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

The numerical solution of a linear singularly pertubed nonlocal problem is considered. To approximate the differential equation, the Hermitian scheme on a special nonuniform mesh is used. The fourth order convergence uniform in the perturbation parameter is proved. The numerical results show the pointwise convergence, too.

AMS Mathematics Subject Classification (1980): 65L10, 65L50.
Key words and phrases: Singular perturbations, nonlocal problems, finite differences, uniform convergence.

1. Introduction

In this paper we shall consider the problem

$$\begin{aligned} &\varepsilon^2 u'' + b(x)u = f(x), \quad x \in I = [0,1], \\ &u(0) = 0 \\ &u(1) = \sum_{i=1}^{m} c_i u(s_i) + d, \\ &d, c_i \in \mathbf{R},\ i = 1,2,\dots m, \quad s_i \in (0,1),\ i = 1,2,\dots,m \end{aligned} \tag{1}$$

where, $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 << 1$ is a small perturbation parameter. We assume that the following conditions are satisfied:

$$b \in C^k(I),\ k \in \mathrm{N},\ 0 < \beta^2 \le b(x),\ x \in I \tag{2}$$

$$-\infty < \sum_{i=1}^{m} c_i \omega_0(s_i) < 1, \tag{3}$$

where

$$\omega_0(x) = \frac{(\exp(\alpha_0 x) - \exp(-\alpha_0 x))}{\exp(\alpha_0) - \exp(-\alpha_0))}, \quad \alpha_0 = \frac{\beta}{\varepsilon}.$$

Some problems of type (1)-(2) arise as a model problem for some physical phenomena, see [2], [9]. In [2] the numerical solution of problem (1) under conditions (2)-(3) was given. The finite elements on an equidistant mesh was applied there and second order uniform convergence was obtained. The case $\sum_{i=1}^{m} |c_i| < 1$ was considered in [8], too.

In this paper we use a uniform fourth order difference scheme developed in [5]. This sheme is constructed on a special nonequidistant meshes. The reason for using these meshes is our aim to obtain more mesh points in the region of boundary layers, whose width is $O(\varepsilon)$. It is known, see [2], that conditions (2)-(3) imply that problem (1) has a uniqe solution u_ε for which one can estimate the derivates, see [3]. In general, the solution u_ε has boundary

layers at $x = 0$ and $x = 1$. So, knowing the behaviour of the exact solution of our problem, we use the mesh generating function $\lambda(t)$ which is suitable for problem (1).

We end our paper with some numerical results, which show that theoretical order of convergence is also established numerically.

The constans M will be independent of the discretization mesh and ε.

2. The numerical method

We shall consider discretization on discretization mesh $I_s = I_h \, U\{s_1, s_2, \cdots \ldots, s_m\}$ where

$$I_h = \{x_i = \lambda(t_i) : i = 0, 1, \ldots, n\},\ t_i = ih,\ h = \frac{1}{n},\ n = 2n_0, n_0 \in \mathrm{N},$$

with mesh generating function [10], [11], [5]:

$$\lambda(t) = \begin{cases} \omega(t) = \frac{a\varepsilon t}{q-t}, & t \in [0,\alpha], \\ \omega(\lambda) + \omega'(\alpha)(t-\alpha), & t \in [\alpha, 0.5], \\ 1 - \lambda(1-t), & t \in [0.5, 1], \end{cases}$$

and

$$q \in (0, 0.5), \ a\varepsilon_0 \le q.$$

The parameter α is

$$\alpha = \frac{q - \sqrt{aq\varepsilon(1 - 2q + 2a\varepsilon)}}{1 + 2a\varepsilon}.$$

It is easy to see that $\lambda(t)$ is a monotone increasing function on I, and we can consider the points s_i as values of $\lambda(t_{si})$, for some $t_{si} \in (0,1)$, $i = 1, 2, \ldots, m$. From now on we denote the points of the mesh I_s as x_i, $i = 0, 1, \ldots N$, $n \le N \le n + m$.

The other properties of mesh generating function $\lambda(t)$, which are important for our analysis, are given in [5], [10], [11], [12].

Let

$$Q = 2(1 + \frac{\sqrt{3}}{3})$$

$$I'_h = \{x_i \in I_h : q - Qh < t_{i-1} < \alpha \ \text{or} \ 1 - \alpha < 1 - Qh\}.$$

Let us note that the set I'_h can be empty.

In order to obtain the numerical solution of problem (1) we shall consider as in [3] the following two boundary value problems

(4)
$$-\varepsilon^2 z'' + b(x)z = 0,$$
$$z(0) = 0, \ z(1) = 1,$$

(5)
$$-\varepsilon^2 y'' + b(x)y = f(x),$$
$$y(0) = 0, y(1) = 0.$$

In [3] it is proved that

$$u_\varepsilon(x) = y_\varepsilon(x) + \lambda_\varepsilon z(x), \quad \lambda_\varepsilon = \frac{\sum_{i=1}^m c_i y_\varepsilon(s_i) + d}{1 - \sum_{i=1}^m c_i z_\varepsilon(s_i)}$$

is the solution of problem (1), where z_ε and y_ε are the solution to problems (4) and (5) respectively.

Since, see [3], $0 \le z(x) \ \le \omega_0(x)$, λ_ε is well defined and bounded.

If we have numerical solutions of the problems (4) and (5), say z_h and y_h, we can form in the same way the numerical solution u_h of (1):

$$u_h = y_h + \lambda_h z_h, \quad \lambda_h = \frac{\sum_{i=1}^{m} c_i y_h(s_i) + d}{1 - \sum_{i=1}^{m} c_i z_h(s_i)}. \tag{6}$$

Let us consider now problem (4). Problem (5) can be treated in the same way. We discretize problem (4) on the mesh I_s by using the same scheme as in [5], [6], [7]:

$$\begin{gathered} z_h(0) = 0 \\ \varepsilon^2(a_1(i)z_h(x_{i-1}) + a_0(i)z_h(x_i) + a_2(i)z_h(x_{i+1}))+ \\ +B_1(i)b(x_{i-1}) + B_0 b(x_i) + B_2 b(x_{i+1}) = 0, \quad i = 1, 2, \ldots, N-1, \\ z_h = 1, \end{gathered} \tag{7}$$

where

$$a_1(i) = \frac{-2}{h_i(h_i + h_{i+1})}, \quad a_0(i) = \frac{2}{h_i h_{i+1}}, \quad a_2(i) = \frac{-2}{h_{i+1}(h_i + h_{i+1})},$$

$$h_i = x_i - x_{i-1}, \quad i = 1, 2, \ldots, n,$$

and, if $x_i \in I_s \ / \ I'_h$,

$$B_1(i) = \frac{-a_1(i)}{12}(h_i^2 - h_{i+1}^2 + h_i h_{i+1}),$$

$$B_0(i) = \frac{-a_0(i)}{12}(h_i^2 + h_{i+1}^2 + 3h_i h_{i+1}),$$

$$B_2(i) = \frac{-a_2(i)}{12}(h_{i+1}^2 - h_i^2 + h_i h_{i+1}),$$

or

$$B_0(i) = 1, \quad B_1(i) = B_2(i), \quad \text{if } x_i \in I'_h.$$

Now, in the same way as in [5] we can prove the following theorem.

Theorem 1. *Suppose that condition (2) is satisfied with* $k = 8$. *Let* $|b'(x)| \leq L,\ x \in I$ *and let* z_ε *be the solution of (4) and* $z_{\varepsilon,h} = [z_\varepsilon(0),\ z_\varepsilon(x_1),\ \dots, z_\varepsilon(1)]^T$. *If*

$$n > \max\{\frac{2L}{3\beta^2(1-2q)K},\ \frac{Q}{q}\},\ 0 < K < 1, \tag{8}$$

then there exists a unique solution $z_h = [z_h(0),\ z_h(x_1), \dots, z_h(1)]^T$ *to (7) and it holds*

$$\|z_{\varepsilon,h} - z_h\|_\infty \leq Mh^4. \tag{9}$$

Analogously, under the same assumptions by using same scheme, we obtain that there exists a uniqe numerical solution $y_h = [y_h(0), y_h(x_1), \dots$ $\dots, y_h(1)]^T$ *and that it holds that*

$$\|y_{\varepsilon,h} - y_h\|_\infty \leq Mh^4, \tag{10}$$

where y_ε *is the solution to problem (5) and* $y_{\varepsilon,h} = [y_\varepsilon(0), y_\varepsilon(x_1), \dots, y_\varepsilon(1)]^T$.

Now, we can prove the main result.

Theorem 2. *Suppose that conditions (2), (3) with* $k = 8$ *are satisfied. Let* $|b'(x)| \leq L,\ x \in I$ *and let* z_h *and* y_h *be the numerical solutions to (4) and (5) respectively on discretization mesh* I_s *with (8). Then, if*

$$1 - \sum_{i=1}^{m} c_i z_h(s_i) \neq 0 \tag{11}$$

it holds that

$$|\ u_{\varepsilon,h} - u_h\|_\infty \leq Mh^4, \tag{12}$$

where u_h *is given in (6),* $u_{\varepsilon,h} = [u_\varepsilon(0),\ u_\varepsilon(x_1), \dots, u_\varepsilon(1)]^T$ *and* u_ε *is the solution to (1).*

Proof. In [3] it is proved that under considered conditions there exists a unique solution u_ε to problem (1). Also, it is shown that $|\lambda|$ is bounded.

From [1], [4] it follows that z_ε and y_ε are bounded. Also, from (9) and (10) we conculde that z_h and y_h are bounded. Obviously

$$\|u_{\varepsilon,h} - u_h\|_\infty \le \|y_{\varepsilon,h} - y_h\|_\infty + \tag{13}$$

$$|\lambda_\varepsilon|\,\|z_{\varepsilon,h} - z_h\|_\infty + |\lambda_\varepsilon - \lambda_h|\,\|z_h\|_\infty.$$

From the definitiones of λ_ε and λ_h we obtain

$$\lambda_\varepsilon - \lambda_h = \sum_{i=1}^{m} c_i(y_\varepsilon(s_i) - y_h(s_i)) + \sum_{i=1}^{m} c_i(\lambda_\varepsilon z_h(s_i) - \lambda_h z_h(s_i))$$

and thus

$$(\lambda_\varepsilon - \lambda_h)(1 - \sum_{i=1}^{m} c_i z_\varepsilon(s_i)) = \sum_{i=1}^{m} c_i[y_\varepsilon(s_i) - y_h(s_i) + \lambda_h(z_\varepsilon(s_i) - z_h(s_i))],$$

$$|\lambda_\varepsilon - \lambda_h| \le M(\|y_{\varepsilon,h} - y_h\|_\infty + \|z_{\varepsilon,h} - z_h\|_\infty) \le Mh^4.$$

Now, from (13) follows (12), that is, we have a numerical solution to problem (1) which is of the fourth order accuarcy uniform in ε. □

Remark. Condition (11) is an artificial one. In effect, z_h is the fourth order approximation of z and for a sufficiencly small h from (3) follows (11).

3. Numerical examples

As illustrative test problems to verify our method we consider well-known problems, [5-7], [10-12]:

$$-\varepsilon u'' + u = 1 \tag{14}$$

$$-\varepsilon^2 u'' + u + \cos^2(\pi x) = -2(\varepsilon\pi)^2\cos(2\pi x) \tag{15}$$

with the conditions

$$u(0) = 0, u(1) = \sum_{i=1}^{m} c_i u(s_i) + d. \tag{16}$$

The exact solutions of these problems in case $u(0) = u(1) = 0$, are given in [5-6]. Now, for given $m, c_i,\ i = 1, 2, \ldots, m$, we determine d so that $u(1) = 0$.

We denote by E_N the maximum of $|u_\varepsilon(x) - u_h(x)|$, $x \in I_s$, i.e.

$$E_N = \|u_{\varepsilon,h} - u_h\|_\infty.$$

Also, we define in the usual way the order of convergence Ord for two successive values of n with respective errors E_N and E_{N_2}:

$$\mathrm{Ord} = \frac{\log E_N - \log E_{N_2}}{\log N - \log N_2}.$$

We expect that Ord = 4. Table 1 presents the results for the numerical solution obtained by our method for example (14), (16), where $a = 1$, $q = 0.48$, $m = 3$ and

i	s_i	c_i
1	0.99999	0.0300
2	0.10000	0.2000
3	0.20000	5.0000

Table 1.

$N(n)\backslash\varepsilon$	2^{-10}	2^{-15}	2^{-20}	$2^{-30} - 2^{-50}$	
7(2)	3.1959E-01	3.3668E-01	3.3722E-01	3.3724E-01	E_N
	-	-	-	-	Ord
11(4)	5.9696E-02	5.9980E-02	5.9989E-02	5.9989E-02	E_N
	3.7120	3.8168	3.8200	3.8201	Ord
19(8)	9.4931E-03	9.4948E-03	9.4949E-03	9.4949E-03	E_N
	3.3642	3.3726	3.3729	3.3729	Ord
35(16)	2.9128E-03	2.9128	2.9128E-03	2.9128E-03	E_N
	1.9339	1.9342	1.9342	1.9342	Ord
67(32)	1.5315E-04	1.5315E-04	1.5315E-04	1.5315E-04	E_N
	4.5360	4.5360	4.5360	4.5360	Ord
131(64)	9.1652E-06	9.1652E-06	9.1652E-06	9.1652E-06	E_N
	4.1999	4.1999	4.1999	4.1999	Ord
259(128)	5.6859E-07	5.6859E-07	5.6859E-07	5.6859E-07	E_N
	4.0785	4.0785	4.0785	4.0785	Ord
515(256)	3.5390E-08	3.5390E-08	3.5390E-08	3.5390E-08	E_N
	4.0398	4.0398	4.0398	4.0398	Ord

In Table 2 are given the results for example (15), (16) where $\varepsilon = 2^{-25}$, $a = 1$, $q = 0.4$ and s_i, c_i, $i = 1,2,3$, are as above. Here T is the number of mesh points in $(0,\varepsilon) \cup (1-\varepsilon,\ 1)$.

Table 2.

n	N	T	E_N	Ord	λ_h
2	7	1	1.8888E-01	8.5650E-01	3.546631E-09
4	11	3	1.2686E-01	8.8056E-01	-1.715722E+00
8	19	7	1.4140E-02	4.0145E+00	4.865624E-01
16	35	13	2.9822E-03	2.5476E+00	2.129330E+00
32	67	25	3.3565E-04	3.3639E+00	2.844466E+00
64	131	51	1.9299E-05	4.2595E+00	3.173170E+00
128	259	103	1.1801E-06	4.0996E+00	3.330554E+00
265	515	205	7.3429E-08	4.0403E+00	3.407553E+00
512	1027	409	4.5861E-09	4.0179E+00	3.445636E+00

References

[1] Bahvalov, N. S.: K optimizacii metodov resheniya zadach pri nalichii pogrannichnogo sloya, Zhurn. vychislit. mat. i mat. fiz. 9(1969)4, 841-859.

[2] Bicadze, A.B., Samarskii, A.A.: O nekotoryh prosteĭshih obobshcheniyah lineĭnyah éllipticheskih kraevyah zadach, DAN SSSR, 185(1969)4, 739-740.

[3] Chegis, R. Yu.: Chislennoe reshenie zadach s malym parametrom pri starsheĭ proizvodnoĭ i nelokal'nym usloviem, Liet. matem. rink., 28(1988)1, 144-152.

[4] Doolan, E. P., Miller, J. J. H., Schilders, W. H. A.: Uniform numerical methods for problems with initial and boundary layers, Dublin, Boole Press, 1980.

[5] Herceg, D.: Uniform fourth order difference scheme for a singular perturbation problem, Numer. Math. 56(1990), 675-693.

[6] Herceg, D., Petrović, N.: On numerical solution of a nonlocal problem. Z. angew. Math. Mech. Bd. 70(1990) 649-552.

[7] Herceg, D., Vulanović, R., Petrović, N.: Higher order schemes and Richardson extrapolation for singular perturbation problems, Bull. Aust. Math. Soc., 39(1989) 129-139.

[8] Petrović, N.: On a uniform numerical method for a nonlocal problem. Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. (in print).

[9] Smit, R.: Poluprovodniki, Moskva, Mir, 1982.

[10] Vulanović, R.: On a numerical solution of a type of singularly perturbed boundary value problem by using a special discretization mesh, Univ. u Novim Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., 13(1983) 187-201.

[11] Vulanović, R.: Mesh construction for the discretization of singularly pertubed boundary value problems, Ph. D. Thesis, University of Novi Sad, 1986.

[12] Vulanović, R., Herceg, D., Petrović, N.: On the extrapolation for a singulary perturbed boundary value problem, Computing, 36(1986) 69-79.

REZIME

O NUMERIČKOM REŠAVANJU SINGULARNO PERTURBOVANOG NELOKALNOG PROBLEMA

Posmatra se numeričko rešavanje linearnog singularno perturbovanog nelokalnog problema. Koristi se specijalna mreža diskretizacije i diferencna šema zasnovana na Hermitovoj aproksimaciji diferencijalne jednačine. Dokazana je uniforma po malom parametru konvergencija četvrtog reda. Numerički rezultati takođe pokazuju tačkastu konvergenciju.

Received by the editors February 20, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 11-19

Review of Research
Faculty of Science
Mathematics Series

SOME CLASSES OF DIHEDRAL n-QUASIGROUPS

Zoran Stojaković[1]
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

An n-quasigroup (Q, f) is called dihedral iff $f(x_1, \ldots, x_n) = x_{n+1} \Leftrightarrow f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = x_{\sigma(n+1)}$ for every permutation $\sigma \in D_{n+1}$, where D_{n+1} is the dihedral subgroup of the symmetric group S_{n+1} of degree $n+1$. Dihedral n-quasigroups (D-n-quasigroups) represent a generalization of totally symmetric binary quasigroups. In the paper several classes of D-n-quasigroups are considered: (i,j)-associative D-n-quasigroups, D-n-groups, medial D-n-quasigroups, self-orthogonal D-n-quasigroups and D-n-quasigroups satisfying Menger identities. Their properties are described and some characterizations given.

AMS Mathematics Subject Classification (1980): 20N15
Key words and phrases: n-quasigroup, n-group, parastrophe

1. Introduction and Definitions

First we give some basic definitions and notations. Other notions from the theory of n-quasigroups can be found in [1].

The sequence $x_m, x_{m+1}, \ldots, x_n$ we shall denote by x_m^n or $\{x_i\}_{i=m}^n$. If $m > n$, then x_m^n will be considered empty.

---

[1] This research was supported by Science Fund of Serbia, grant number 0401A, throught Matematički institut.

An n-ary groupoid (n-groupoid) (Q, f) is called an n-quasigroup iff the equation $f(a_1^{i-1}, x, a_{i+1}^n) = b$ has a unique solution x for every $a_1^n, b \in Q$ and every $i \in \{1, \ldots, n\} = N_n$.

An n-quasigroup (Q, f) is called (i,j)-associative iff the following identity holds

$$f(x_1^{i-1}, f(x_i^{i+n-1}), x_{i+n}^{2n-1}) = f(x_1^{j-1}, f(x_j^{j+n-1}), x_{j+n}^{2n-1}). \tag{1}$$

An n-quasigroup which is (i,j)-associative for all $i, j \in N_n$ is called an n-group.

An n-quasigroup (Q, f) is called m-associative iff for every $i, j \in N_n$ and every sequence x_1^{2n-1} of elements from Q which contains at most m different elements (1) holds.

An n-quasigroup (Q, f) is medial iff $f(y_1^n) = f(z_1^n)$, where $y_i = f(\{x_{ij}\}_{j=1}^n)$, $z_j = f(\{x_{ij}\}_{i=1}^n)$ for all $x_{ij} \in Q$, $i, j \in N_n$.

By S_n we denote the symmetric group of degree n, by A_n its alternating subgroup, and by D_n its dihedral subgroup.

If (Q, f) is an n-quasigroup and $\sigma \in S_{n+1}$, then the n-quasigroup f^σ defined by

$$f^\sigma(\{x_{\sigma(i)}\}_{i=1}^n) = x_{\sigma(n+1)} \Longleftrightarrow f(x_1^n) = x_{n+1}$$

is called a σ-parastrophe (or simply parastrophe) of f. If $f = f^\sigma$, then σ is called an autoparastrophism of f. The set of all autoparastrophisms of f is a subgroup of S_{n+1} which is denoted by $\Pi(f)$.

An n-quasigroup $(Q.f)$ is called

a) totally symmetric (TS) if $f = f^\sigma$ for all $\sigma \in S_{n+1}$,

b) alternating symmetric (AS) iff $f = f^\sigma$ for all $\sigma \in A_{n+1}$ ([5]),

c) cyclic iff $f = f^\sigma$ for all $\sigma \in C$, where C is a subgroup of S_{n+1} generated by the cycle $(12 \ldots n+1)$ ([4]).

In [7] a class of n-quasigroups called dihedral n-quasigroups was defined and considered. An n-quasigroup (Q, f) is called dihedral iff $f = f^\sigma$ for all $\sigma \in D_{n+1}$. Such n-quasigroups represent a generalization of TS binary quasigroups, different from other generalizations such as TS-n-quasigroups or AS-n-quasigroups. In [6] a class of D-3-quasigroups which is equivalent to a new class of quadruple systems which lies between Steiner and Mendelsohn quadruple systems was cosidered.

An n-quasigroup (Q,f) is dihedral iff $f = f^{\phi} = f^{\psi}$ where $\phi = (12\ldots n+1)$, $\psi = \prod_{2\leq i<n+3-i}(i \quad n+3-i)$ are generators of D_{n+1}. In the sequel ϕ and ψ will always denote these two permutations. Every D-n-quasigroup can be defined by a system of idetities ([7]). The existence of an n-quasigroup (Q,f) of order mp for every $m > n$, $p \geq 2$ such that $\Pi(f) = D_{n+1}$, follows from [2].

Here we shall continue investigation of D-n-quasigroups and consider several classes of D-n-quasigroups satisfying some additional conditions.

2. Dihedral n-groups

Since every D-n-quasigroup is cyclic, by the results from [8] we get the following theorems.

Theorem 1. *If (Q,f) is an (i,j)-associative D-n-quasigroup, then for every integer m f is $(i+m,j+m)$-associative n-quasigroup, (where $i+m, j+m$ are reduced modulo n).*

Theorem 2. *Let (Q,f) be an (i,j)-associative D-n-quasigroup, where $j-i$ is relatively prime to n. Then f is an n-group.*

Theorem 3. *Let (Q,f) be an n-group, where $n = 2k$, $k \in N$. The n-group (Q,f) is a D-n-group iff there exists a commutative group $(Q,+)$ such that $x = -x$ for all $x \in Q$ and*

$$f(x_1^n) = \sum_{1}^{n} x_i + c$$

where c is a fixed element from Q.

Theorem 4. *There exists a nontrivial finite D-n-group (Q,f) of order q, where n is even, iff $q = 2^t$, $t \in N$. (An n-groupoid (Q,f) is called trivial iff $|Q| = 1$.)*

Theorem 5. *Let (Q,f) be an n-group, where $n = 2k+1$, $k \in N$. The group (Q,f) is a D-n-group iff there exists a commutative group $(Q,+)$ such that*

$$f(x_1^n) = x_1 - x_2 + x_3 - \ldots + x_n + c, \tag{2}$$

where $c = -c$ is an element from Q.

Theorem 6. *A nontrivial finite D-n-group of order q, where $n = 2k+1$, $k \in N$ exists for every $Q \in N$, and every such n-group is represented by (2).*

When $n = 2k$, $k \in N$, an n-group described in Theorem 3 is an n-group with unity. A unit of that n-group is the element $c \in Q$, and there are no other units.

When $n = 2k+1$, $k \in N$, then an n-group described in Theorem 5 in the case $c = 0$ is an n-group with unity and every element of that n-group is a unity, and in the case $c \neq 0$ it is an n-group without unity.

In [3] it is proved that the Hosszú-Gluskin theorem is valid for m-associative n-quasigroups, where $m \geq n+2$, which means that every such n-quasigroup is necessarily an n-group. Hence the theorems analogous to Theorems 3 and 5 of the present paper can be proved for dihedral m-associative n-quasigroups, $m \geq n+2$.

3. Medial D-n-quasigroups

Theorem 7. *Let (Q,f) be a medial n-quasigroup. (Q,f) is dihedral iff there exists a commutative group $(Q,+)$ such that*

a) when $n = 2k$, $k \in N$,

$$f(x_1^n) = -\sum_{i=1}^{n} x_i + b,$$

b) when $n = 2k+1$, $k \in N$,

$$f(x_1^n) = \theta x_1 - x_2 + \theta x_3 - x_4 + \ldots + \theta x_n + b,$$

where θ is an automorphism of the group $(Q,+)$ such that $\theta^2 = \epsilon$ (identity mapping) and b is a fixed element of Q such that $\theta b = -b$.

Proof. In [1] it is proved that for every medial n-quasigroup (Q,f) there exist a commutative group $(Q,+)$ such that

$$f(x_1^n) = \sum_{i=1}^{n} \theta_i x_i + b,$$

where θ_i, $i = 1, \ldots, n$ are automorphisms of the group $(Q, +)$, $\theta_i\theta_j = \theta_j\theta_i$ for all $i, j \in N_n$ and b is a fixed element from Q.

If $f(x_1^n) = x_{n+1}$, then

$$\sum_{i=1}^{n} \theta_i x_i + b = x_{n+1}, \tag{3}$$

and since f is dihedral, $f = f^{\psi}$, hence

$$\theta_1 x_1 + \theta_2 x_{n+1} + \theta_3 x_n + \ldots + \theta_n x_3 + b = x_2 \tag{4}$$

Putting in (3) and (4) $x_3 = \ldots = x_{n+1} = 0$, where 0 is the neutral element of $(Q, +)$, we get

$$\theta_1 x_1 + \theta_2 x_2 + b = 0 \tag{5}$$

and

$$\theta_1 x_1 + b = x_2. \tag{6}$$

Subtracting (6) from (5) it follows that $\theta_2 x_2 = -x_2$. Since in (3) any n elements can be arbitrarily chosen, it follows that $\theta_2 x = -x$ for all $x \in Q$.

Putting now in (3) and (4) $x_2 = x_4 = \ldots = x_{n+1} = 0$, we get

$$\theta_1 x_1 + \theta_3 x_3 + b = 0$$

and

$$\theta_1 x_1 + \theta_n x_3 + b = 0,$$

which implies $\theta_3 = \theta_n$. By a similar procedure we get that $\theta_i = \theta_{n+3-i}$ for all $i = 3, \ldots, n$.

Since f is dihedral the following identity is valid

$$f(x_n, f(x_1^n), x_1^{n-2}) = x_{n-1},$$

that is,

$$\begin{aligned}\theta_1 x_n - (\theta_1 x_1 - x_2 + \theta_3 x_3 + \ldots + \theta_n x_n + b)\\ +\theta_3 x_1 + \ldots + \theta_n x_{n-2} + b = x_{n-1},\end{aligned} \tag{7}$$

where $\theta_i = \theta_{n+3-i}$ for all $i = 3, \ldots, n$.

Putting in (7) $x_2 = \ldots = x_n = 0$, we get $\theta_1 = \theta_3$, then putting $x_1 = x_2 = x_4 = \ldots = x_n = 0$ we get $\theta_3 = \theta_5$ and similarly we obtain that all automorphisms θ_i with odd indices are equal.

Further, putting in (7) $x_1 = x_3 = \ldots = x_n = 0$, it follows that $\theta_4 x = -x$, putting $x_1 = x_2 = x_3 = x_5 = \ldots = x_n = 0$, we get $\theta_6 x = -x$ and similarly we get that for all even indexes i, $\theta_i x = -x$.

From the dihedrality of f it follows that the identity

$$f(f(x_1^n), x_1^{n-1}) = x_n$$

holds, that is,

$$\theta_1(\theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_n x_n + b) + \theta_2 x_1 + \ldots + \theta_n x_{n-1} + b = x_n.$$

Putting in the preceding identity $x_1 = \ldots = x_n = 0$, we get that $\theta_1 b = -b$, and putting $x_2 = \ldots = x_n = 0$ we get that $\theta_1^2 x = x$ for all $x \in Q$.

When n is even, from $\theta_3 = \theta_n$ it follows that $\theta_i x = -x$ for all $i \in N_n$.

Hence, if we denote $\theta_{2k+1} = \theta$ for all k, we have proved that when n is even

$$f(x_1^n) = -\sum_{i=1}^{n} x_i + b,$$

and when n is odd

$$f(x_1^n) = \theta x_1 - x_2 + \theta x_3 - x_4 + \ldots + \theta x_n + b,$$

where $\theta^2 = \epsilon$ and $\theta b = -b$.

The converse part of the theorem is straightforward.

4. Self-orthogonal D-n-quasigroups

The set $\{(Q, f_1), \ldots, (Q, f_n)\}$ of n-quasigroups is said to be orthogonal iff for each $(a_1^n) \in Q^n$, there exists a unique $(b_1^n) \in Q^n$ such that

$$f_i(b_1^n) = a_i, \quad i = 1, \ldots, n.$$

If (Q, f) is an n-quasigroup such that the set $\{f, f_1, \ldots, f_{n-1}\}$ is otrhogonal, where f_i, $i = 1, \ldots, n-1$ are parastrophes of f defined by $f_i(x_1^n) = f(x_{i+1}^n, x_1^i)$, $i = 1, \ldots, n-1$, then (Q, f) is called a self-orthogonal n-quasigroup. Self-orthogonal cyclic n-quasigroups were considered in [9].

Theorem 8. *If (Q, f) is a commutative group, $n > 2$ odd integer, such that the mapping $x \mapsto 2x$ is a bijection,then by*

$$f(x_1^n) = \sum_{i=1}^{n}(-1)^{i+1}x_i \tag{8}$$

a self-orthogonal D-n-quasigroup (Q, f) is defined.

Proof. It is obvious that $f = f^{\phi} = f^{\psi}$, hence (Q, f) is a D-n-quasigroup.

Let a_1^n be arbitrary elements from Q and consider the system

$$\begin{cases} f(x_1^n) = a_1 \\ f_i(x_1^n) = a_{i+1}, \quad i = 1, \ldots, n-1 \end{cases} \tag{9}$$

where the parastrophes f_i, $i = 1, \ldots, n-1$ of the n-quasigroup f are defined by $f_i(x_1^n) = f(x_{i+1}^n, x_1^i)$. Adding the i-th and (i+1)-th equation from (9) for $i = 1, \ldots, n-1$ and the first and the last equation, we get that $x_i = (a_i + a_{i+1})/2, \quad i = 1, \ldots, n-1, \quad x_n = (a_1 + a_n)/2$ which is a unique solution of (9).

Theorem 9. *If $n, q > 2$ are odd integers, then there exists a self-orthogonal D-n-quasigroup of order q.*

Proof. In the ring Z_q of integers modulo q, the mapping $x \mapsto 2x$ is a bijection, hence using the group $(Z_q, +)$ by (8) we get a self-orthogonal D-n-quasigroup of order q.

If $(Q_1, \bigoplus_1)$ and $(Q_2, \bigoplus_2)$ are two commutative groups having the property that $x \mapsto x \bigoplus_i x, \quad i = 1, 2$, are bijections, then their direct product $Q_1 \times Q_2$ has the same property. This means that if by (8) two self-orthogonal D-n-quasigroups of orders q_1 and q_2 respectively, are constructed, then using a direct product a self-orthogonal D-n-quasigroup of order $q_1 q_2$ can be constructed.

5. Menger D-n-quasigroups

If (Q, f) is an n-groupoid, the identity

$$f(x_1^{i-1}, f(y_1^n), x_{i+1}^n) = f(z_1^{i-1}, y_i, z_{i+1}^n), \tag{10}$$

where $z_j = f(x_1^{i-1}, y_j, x_{i+1}^n)$, $j = 1, \ldots, i-1, i+1, \ldots, n$ is called the i-th Menger identity ([1]). Identity (10) for $i = 1$ is called the Menger identity and the corresponding n-groupoid satisfying it is called a Menger n-groupoid.

Theorem 10. *If $n = 2k+1$, $k \in N$, then a D-n-quasigroup (Q, f) satisfying the i-th Menger identity for all $i \in N_n$ is trivial.*

If $n = 2k$, $k \in N$, then there exists a nontrivial D-n-quasigroup (Q, f) of order q satisfying the i-th Menger identity for all $i \in N_n$ iff $q = 2^t$, $t \in N$. This n-quasigroup has the form $f(x_1^n) = x_1 \ldots x_n$, where $(Q, \cdot)$ is a boolean group.

Proof. If (Q, f) is an n-quasigroup satisfying the i-th Menger identity for all $i \in N_n$, then ther exists a commutative group $(G, \cdot)$ of degree $n-2$ such that $f(x_1^n) = x_1 \ldots x_n$ ([1])(a group $(Q, \cdot)$ is of degree m iff $x^m = e$ for all $x \in Q$, where e is the unit of the group).

Let (Q, f) be a D-n-quasigroup satisfying the i-th Menger identity for all $i \in N_n$. If $f(x_1^n) = x_{n+1}$, then

$$x_1 \ldots x_n = x_{n+1},$$

and since $f = f^{\phi}$, we have

$$x_2 \ldots x_{n+1} = x_1.$$

From the last two equations it follows that

$$(x_1 x_{n+1}^{-1})^2 = e.$$

This means that for every $x \in Q$, $x^2 = e$. Hence if n is odd from the fact that for all $x \in Q$, $x^{n-2} = e$ and $x^2 = e$ it follows that $x = e$ for all $x \in Q$, that is, $|Q| = 1$.

If n is even, since all nonidentity elements from $(Q, \cdot)$ are of order 2, $(Q, \cdot)$ is a boolean group, hence the order of $(Q, \cdot)$ is 2^t, $t \in N$.

References

[1] Belousov, V.D.: n-arnye kvazigruppy. Shtiintsa, Kishinev, 1972.

[2] Hoffman, D.G.: On the spectrum of n-quasigroups with given conjugate invariant subgroup, J. Combin. Theory Ser. A, 35(1983), 98-99.

[3] Sokolov, E.I.: O teoreme Gluskina-Hossu dlya n-grupp Dërnte, Mat. Issled. 39(1976), 187-189.

[4] Stojaković, Z.: Cyclic n-quasigroups, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser.Mat. 12(1982), 407-415.

[5] Stojaković, Z.: Alternating symmetric n-quasigroups, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser.Mat. 13(1983), 259-272.

[6] Stojaković, Z.: Dihedral quadruple systems (to appear).

[7] Stojaković, Z.: Dihedral n-quasigroups, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser.Mat., 20,1(1990), 53-61.

[8] Stojaković, Z., Paunić, Dj.: On a class of n-groups, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser.Mat. 14,2(1984), 147-153.

[9] Stojaković, Z., Paunić, Dj.: Self-orthogonal n-quasigroups, Aequationes Math., 30(1986), 252-257.

REZIME

NEKE KLASE DIEDARSKIH n-KVAZIGRUPA

n-kvazigrupa (Q, f) se naziva diedarska ako je $f(x_1, ..., x_n) = x_{n+1} \Leftrightarrow f(x_{\sigma(1)}, ..., x_{\sigma(n)}) = x_{\sigma(n+1)}$ za svaku permutaciju $\sigma \in D_{n+1}$, gde je D_{n+1} diedarska podgrupa simetrične grupe S_{n+1} stepena $n+1$. Diedarske n-kvazigrupe (D-n-kvazigrupe) predstavljaju generalizaciju totalno simetričnih binarnih kvazigrupa.U ovom radu razmatrane su sledeće klase D-n-kvazigrupa: (i,j)-asocijativne D-n-kvazigrupe, D-n-grupe, medijalne D-n-kvazigrupe, samoortogonalne D-n-kvazigrupe i D-n-kvazigrupe koje zadovoljavaju Mengerove identitete. Odredjene su neke njihove osobine i date neke karakterizacije.

Received by the editors September 20, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 21-29

Review of Research
Faculty of Science
Mathematics Series

SPECIAL ELEMENTS OF THE LATTICE AND LATTICE IDENTITIES

Branimir Šešelja and **Andreja Tepavčević**
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

Special elements of a lattice L (distributive, codistributive, neutral etc.) induce a congruence relation on L. Here we consider the following problem: If the lattice identity is satisfied on a class of that congruence (a sublattice of L), under which conditions this identity holds on the lattice itself?

Several algebraic results are deduced from the obtained lattice properties: characterizations of the congruence extension and of the congruence intersection properties, and some general properties of the weak congruence lattice of an algebra.

AMS Mathematics Subject Classification (1980): 06B10, 08A30
Key words and phrases: Lattice identites.

1. Preliminaries

In the paper [3], it was proved that the weak congruence lattice Cw$\mathcal{A}$ of an algebra $\mathcal{A}$ (i.e. the lattice of all the congruences on all the subalgebras of $\mathcal{A}$ is modular if and only if both Con$\mathcal{A}$ and Sub$\mathcal{A}$ are modular lattices, and $\mathcal{A}$ has the CEP and the CIP (the congruence extension, and the congruence intersection property). Con$\mathcal{A}$ and Sub$\mathcal{A}$ are two sublattices of Cw$\mathcal{A}$. Thus, under some conditions (the CEP and the CIP), a lattice identity (modularity) holds on Cw$\mathcal{A}$ if and only if it holds on two special sublattices of that lattice. To consider the general problem, concerning any lattice identity on any algebraic lattice L, provided that this identity holds on the special sublattices of L, we need the following definitions and propositions, which are mainly from [1], [2], and from the references given there.

An element a of the lattice L is **codistributive**, if for all $x, y \in L$

$$a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y),$$

or, equivalently, if and only if the mapping $m_a : x \rightarrow x \wedge a$ is a homomorphism of L into the ideal (a].

Note that for a codistributive element a of L, the filter [a) is the class of the congruence induced by m_a.

A **distributive** element a of L, is defined dually (the corresponding homomorphism from L into the filter [a) is obviosly $x \rightarrow x \vee a$, and the ideal (a] is the class of the congruence induced by that mapping).

An element a of L is said to be **comodular**, if $x \leq y$ implies $x \vee (a \wedge y) = (x \vee a) \wedge y$.

The lattice Cw$\mathcal{A}$ of weak congruences of an algebra $\mathcal{A} = (A, F)$ is the lattice of all the symmetric and transitive subalgebras of $\mathcal{A}^2$ under the set inclusion. The diagonal relation $\Delta = \{(x, x) \mid x \in A\}$ is a codistributive element of that lattice , $[\Delta)$ is Con$\mathcal{A}$, and Sub$\mathcal{A}$ is isomorphic with $(\Delta]$ under $\rho \rightarrow \{x \mid x\rho x\}$.

Recall that $\mathcal{A}$ has the **congruence extension property** (CEP), if every congruence on a subalgebra of $\mathcal{A}$ is a restriction of the congruence on $\mathcal{A}$.

$\mathcal{A}$ has the CEP if and only if Δ is a comodular element of Cw$\mathcal{A}$ ([2]).

$\mathcal{A}$ is said to have the **congruence intersection property** (CIP), if for $\rho \in Con\mathcal{B}$, $\theta \in Con\mathcal{C}$, $\mathcal{B}, \mathcal{C} \in Sub\mathcal{A}$,

$$(\rho \cap \theta)_A = \rho_A \cap \theta_A,$$

where ρ_A stands for the least congruence on $\mathcal{A}$ extending ρ. Since $\rho_A = \rho \vee \Delta$ in Cw$\mathcal{A}$, $\mathcal{A}$ has the CIP if and only if Δ is a distributive element of Cw$\mathcal{A}$.

2. Identities on lattices

In this section we shall describe the conditions under which a lattice identity holds on L, provided that it holds on (a] and (or) on [a), where a is a special element of L (distributive, codistributive, modular, etc.).

Lemma 1. *If a is a distributive element of the lattice L, and $f(x_1,\dots,x_n)$ is an arbitrary lattice therm, then for all $x_1,\dots,x_n \in L$,*

$$f(x_1,\dots,x_n) \vee a = f(x_1 \vee a,\dots,x_n \vee a).$$

Proof. Straightforward, by induction of the number of operational symbols.

□

A dual proposition is the following.

Lemma 2. *If a is a codistributive element of the lattice L, and $f(x_1,\dots,x_n)$ is an arbitrary lattice therm, then for all $x_1,\dots,x_n \in L$,*

$$f(x_1,\dots,x_n) \wedge a = f(x_1 \wedge a,\dots,x_n \wedge a).$$

The following three lemmas will be necessary in the algebraic applications of the results in this section. We shall assume that the classes of the congruence induced by the mapping $x \to x \wedge a$, where a is a codistributive element of the lattice L, always have the greatest elements. For the class to which $x \in L$ belongs, the greatest element will be denoted by $\bar{x}$.

Lemma 3. *If a is a codistributive element of the lattice L, then the following are equivalent, for all $x, y \in L$:*

(i) *if $x \wedge a = y \wedge a$ and $x \vee a = y \vee a$, then $x = y$;*

(ii) *if $x \leq y$, then $x \vee (a \wedge y) = (x \vee a) \wedge y$;*

(iii) *if $x \leq \bar{y}$, then $x \vee (a \wedge \bar{y}) = (x \vee a) \wedge \bar{y}$;*

(iv) *$x \vee (a \wedge y) = (x \vee a) \wedge (x \vee y)$.*

Proof. $(i) \iff (ii) \iff (iii)$, proved in[2].
$(iv) \Rightarrow (ii)$, obvious.
$(ii) \Rightarrow (iv)$. Since $x \leq x \vee y$, it follows that

$$x \vee (a \wedge (x \vee y)) = (x \vee a) \wedge (x \vee y).$$

Moreover, a is codistributive, and

$$x \vee (a \wedge (x \vee y)) = x \vee (a \wedge x) \vee (a \wedge y) = x \vee (a \wedge y).$$

□

Lemma 4. *If a is a distributive element of the lattice L, then the following are equivalent, for all $x, y \in L$:*

(i) if $x \leq y$, then $x \vee (a \wedge y) = (x \vee a) \wedge y$;

(ii) $x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$.

Proof. $(ii) \Rightarrow (i)$, obvious.
$(i) \Rightarrow (ii)$. Since $x \wedge y \leq x$, it follows that

$$(x \wedge y) \vee (a \wedge y) = ((x \wedge y) \vee a) \wedge y, \text{ and}$$

$$((x \wedge y) \vee a) \wedge y = (x \vee a) \wedge (y \vee a) \wedge y = (x \vee a) \wedge y.$$

□

Lemma 5. *If a is a codistributive element of the lattice L, satisfying any of the conditions from Lemma 3. then the following are equivalent :*

(i) a is a distributive element of L ;

(ii) for all $x, y \in L, (x \wedge a) \vee (x \wedge y) = ((x \wedge a) \vee y) \wedge x$.

Proof. $(i) \Rightarrow (ii)$

$$a \wedge ((x \wedge a) \vee (x \wedge y)) = (a \wedge x \wedge a) \vee (a \wedge x \wedge y) = a \wedge x.$$

$$a \wedge (((x \wedge a) \vee y) \wedge x) = a \wedge x.$$

$$a \vee ((x \wedge a) \vee (x \wedge y)) = a \vee (x \wedge y).$$

$$a \vee (((x \wedge a) \vee y) \wedge x) = (a \vee (x \wedge a) \vee y) \wedge (a \vee x) = (a \vee y) \wedge (a \vee x) = a \vee (x \wedge y).$$

Now, by (i) in Lemma 3, it follows that

$$(x \wedge a) \vee (x \wedge y) = ((x \wedge a) \vee y) \wedge x.$$

$(ii) \Rightarrow (i)$. We shall use condition (iii) from Lemma 3. Therefore, let $\bar{z} = \overline{x \vee y}$. Hence, $\bar{z} \wedge a = (x \vee y) \wedge a$, and thus

$$\bar{z} \wedge a = (x \wedge a) \vee (y \wedge a), \text{ and } \bar{z} \wedge a \geq x \wedge a, \ \bar{z} \wedge a \geq y \wedge a, \ \bar{z} \geq x, \ \bar{z} \geq y.$$

Now, we have

$$a \vee (x \wedge y) = a \vee (y \wedge a) \vee (x \wedge y) = a \vee (((x \wedge a) \vee y) \wedge x) =$$
$$= a \vee (((x \wedge a) \vee (y \wedge a) \vee y) \wedge x) = a \vee ((((x \vee y) \wedge a) \vee y) \wedge x) =$$
$$= a \vee (((\bar{z} \wedge a) \vee y) \wedge x) = a \vee (\bar{z} \wedge (a \vee y) \wedge x) = a \vee (x \wedge (a \vee y)) =$$
$$((a \vee y) \wedge a) \vee (x \wedge (a \vee y)) = (a \vee x) \wedge (a \vee y).$$

□

Theorem 1. *If a is a distributive, codistributive and comodular element of the lattice L, then an arbitrary lattice identity is satisfied on L, if and only if this identity holds on (a] and on [a).*

Proof.
(⇒). Obvious.
(⇐). Let the identity

$$f(x_1 \dots, x_n) = g(x_1 \dots, x_n)$$

hold on $(a]$, and on $[a)$. Then, for $y_1, \dots, y_n \in L$

$$f(y_1 \wedge a, \dots, y_n \wedge a) = g(y_1 \wedge a, \dots, y_n \wedge a), \text{ and}$$
$$f(y_1 \vee a, \dots, y_n \vee a) = g(y_1 \vee a, \dots, y_n \vee a).$$

By Lemma 1 and Lemma 2, it follows that

$$f(y_1, \dots, y_n) \wedge a = g(y_1, \dots, y_n) \wedge a, \text{ and}$$
$$f(y_1, \dots, y_n) \vee a = g(y_1, \dots, y_n) \vee a.$$

Now, by Lemma 3 it follows that

$$f(y_1, \dots, y_n) = g(y_1, \dots, y_n).$$

□

Lemma 6. *Let 1 be the greatest element of the lattice L,* $f(x_1, \dots, x_n) = g(x_1, \dots, x_n)$ *an arbitrary lattice identity, and* $b \in L$.
Now, if for all $x_1, \dots, x_{n-1} \in L$

$$f(x_1, \dots, x_{n-1}, 1) = g(x_1, \dots, x_{n-1}, 1),$$

then for all $x_1, \dots, x_{n-1} \in (b]$

$$f(x_1, \dots, x_{n-1}, b) = g(x_1, \dots, x_{n-1}, b).$$

Proof. Follows by the fact that 1 is the greatest element of L, and b the greatest in $(b]$. It is a straightforward proof by induction that any equality of the form

$$f(x_1,\ldots,x_{n-1},b)=g(x_1,\ldots,x_{n-1},b),$$

where $x_1,\ldots,x_{n-1}\in(b]$, reduces to $b=b$, or to an equality not containing b.

□

Dually, we have the following.

Lemma 7. *Let 0 be the smallest element of the lattice L,* $f(x_1,\ldots,x_n)=g(x_1,\ldots,x_n)$ *an arbitrary lattice identity, and* $b\in L$.
If for all $x_1,\ldots,x_{n-1}\in L$

$$f(x_1,\ldots,x_{n-1},0)=g(x_1,\ldots,x_{n-1},0),$$

then for all $x_1,\ldots,x_{n-1}\in[b)$

$$f(x_1,\ldots,x_{n-1},b)=g(x_1,\ldots,x_{n-1},b).$$

□

Theorem 2. *Let a be a distributive, codistributive and comodular element of the lattice L, and let* $f(x_1,\ldots,x_n)=g(x_1,\ldots,x_n)$ *be an arbitrary lattice identity. Let also* $b\in(a]$, *and let* $\bar{b}$ *be the greatest element of the class to which b belongs, under* $x\to x\wedge a$.
Now, if the equality

$$f(x_1,\ldots,x_{n-1},a)=g(x_1,\ldots,x_{n-1},a)$$

is satisfied on $(a]$ *and on* $[a)$, *then*

$$f(x_1,\ldots,x_{n-1},b)=g(x_1,\ldots,x_{n-1},b)$$

holds on the ideal $(\bar{b}]$ *(i.e. for all* $x_1,\ldots,x_{n-1}\in(\bar{b}]$ *).*

Proof. Let $x_1,\ldots,x_{n-1}\in(\bar{b}]$. Then, by Lemma 2 and Lemma 6

$$f(x_1,\ldots,x_{n-1},b)\wedge a=f(x_1\wedge a,\ldots,x_{n-1}\wedge a,b)=g(x_1\wedge a,\ldots,x_{n-1}\wedge a,b)=$$

$$=g(x_1,\ldots,x_{n-1},b)\wedge a,$$

since $x_1 \wedge a, \ldots, x_{n-1} \wedge a \in (b]$, and the corresponding equality holds on $(a]$ by assumption. Similarly, by Lemma 1,

$$f(x_1, \ldots, x_{n-1}, b) \vee a = f(x_1 \vee a, \ldots, x_{n-1} \vee a, b) = g(x_1 \vee a, \ldots, x_{n-1} \vee a, b) =$$
$$= g(x_1, \ldots, x_{n-1}, b) \vee a,$$

since the corresponding equality holds on the filter $[a)$.

Finally, by Lemma 3, it follows that for all $x_1, \ldots, x_{n-1} \in (\bar{b}]$

$$f(x_1, \ldots, x_{n-1}, b) = g(x_1, \ldots, x_{n-1}, b)$$

□

3. Application to algebra

Lemma 8. *The following are equivalent for an algebra $\mathcal{A}$:*

(0) $\mathcal{A}$ has the CEP;

(i) if for $\rho, \theta \in Con\mathcal{B}$, $\mathcal{B} \in Sub\mathcal{A}$, $\rho \vee \Delta = \theta \vee \Delta$ in $Cw\mathcal{A}$, then, $\rho = \theta$;

(ii) if for $\rho, \theta \in Cw\mathcal{A}$, $\rho \leq \theta$, then $\rho \vee (\Delta \wedge \theta^2) = (\rho \vee \Delta) \wedge \theta^2$;

(iii) if for $\rho \in Cw\mathcal{A}$ and $\mathcal{B} \in Sub\mathcal{A}$, $\rho \leq B^2$, then $\rho \vee (\Delta \wedge B^2) = (\rho \vee \Delta) \wedge B^2$;

(iv) for $\rho, \theta \in Cw\mathcal{A}$, $\rho \vee (\Delta \wedge \theta) = (\rho \vee \Delta) \wedge (\rho \vee \theta)$.

Proof. Algebraic reformulation of Lemma 3.

□

Proposition 1. *If an algebra $\mathcal{A}$ has the CIP, then $\mathcal{A}$ has the CEP if and only if for all $\rho, \theta \in Cw\mathcal{A}$, $\rho \wedge (\Delta \vee \theta) = (\rho \wedge \Delta) \vee (\rho \wedge \theta)$.*

Proof. By Lemma 4.

□

Proposition 2. *If an algebra $\mathcal{A}$ has the CEP, then $\mathcal{A}$ has the CIP if and only if for $\rho, \theta \in Cw\mathcal{A}$, $\theta \in Con\mathcal{B}$, $\mathcal{B} \in Sub\mathcal{A}$,*

$$\Delta_B \vee (\rho \wedge \theta) = (\Delta_B \vee \rho) \wedge \theta.$$

(Δ_B stands for $B^2 \cap \Delta$.)

Proof. By Lemma 5.

□

Theorem 3. *If an algebra $\mathcal{A}$ has the CEP and the CIP, then an arbitrary lattice identity is satisfied on $Cw\mathcal{A}$ if and only if this identity holds on $Sub\mathcal{A}$ and on $Con\mathcal{A}$.*

Proof. Immediately by Theorem 1, since Δ is a codistributive element of $Cw\mathcal{A}$, and since $\mathcal{A}$ has the CEP and the CIP if and only if Δ is distributive and comodular as well (the latter because of Lemma 8, (ii)).

□

The following two propositions were proved in [3]. We give them as the consequences of Theorem 3, and of the lattice interpretation of the CEP and the CIP (the preceding lemmas).

Corollary 1. *An algebra $\mathcal{A}$ has a modular lattice of weak congruences if and only if $\mathcal{A}$ has the CEP and the CIP, and $Sub\mathcal{A}$ and $Con\mathcal{A}$ are modular lattices.*

Corollary 2. *An algebra $\mathcal{A}$ has a distributive lattice of weak congruences if and only if $\mathcal{A}$ has the CEP and the CIP, and $Sub\mathcal{A}$ and $Con\mathcal{A}$ are distributive lattices.*

Theorem 4. *Let $\mathcal{A}$ be an algebra which has the CEP and the CIP, and let $f(x_1,\ldots,x_n) = g(x_1,\ldots,x_n)$ be an arbitrary lattice identity. If the equality*

$$f(x_1,\ldots,x_{n-1},A) = g(x_1,\ldots,x_{n-1},A)$$

is satisfied on $Sub\mathcal{A}$, and the equality

$$f(x_1,\ldots,x_{n-1},\Delta) = g(x_1,\ldots,x_{n-1},\Delta)$$

holds on $Con\mathcal{A}$, then for every subalgebra $\mathcal{B}$ of $\mathcal{A}$

$$f(x_1,\ldots,x_{n-1},\Delta_B) = g(x_1,\ldots,x_{n-1},\Delta_B)$$

holds on $Cw\mathcal{B}$.

Proof. By Theorem 2.

□

Corollary 3. *If an algebra $\mathcal{A}$ has the CEP and the CIP, then every subalgebra of $\mathcal{A}$ has the CIP.*

□

References

[1] Šešelja, B., Tepavčević, A.: Infinitely distributive elements in the lattice of weak congruences, General Algebra 6, Genaral Algebra 1988, R. Mlitz (editor), Elsevier Science Publ. B. V. (North-Holland), 1990, 241-253.

[2] Šešelja, B., Tepavčević, A.: On CEP and semimodularity in the lattice of weak congruence, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. (to appear).

[3] Vojvodić, G., Šešelja, B.: On the lattice of weak congruence relations, Algebra Universalis, 25(1988), 121-130.

REZIME

SPECIJALNI ELEMENTI MREŽE I MREŽNI IDENTITETI

Daju se uslovi pod kojima proizvoljan mrežni identitet važi na mreži, ako je on zadovoljen na podmrežama (nekim) te mreže, indukovanim specijalnim kongruencijama te mreže.

Navedene su algebarske primene tih rezultata na mrežama kongruencija, podalgebri i slabih kongruencija proizvoljne algebre.

Received by the editors December 6, 1989.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 31-41

Review of Research
Faculty of Science
Mathematics Series

GROUP THEORETICAL METHODS IN GRAY-SCALE MATHEMATICAL MORPHOLOGY

G. Crombez
State University of Ghent, Seminar of Higher Analysis Krijgslaan 281,
B-9000 GENT (Belgium)

Abstract

We show that for the morphological operations involving a finite number of bounded functions a mathematical foundation in group-theoretical terms can be given which unifies the notions for subsets and for gray-scale images, and this as well in the Euclidean as in the digital case.

AMS Mathematics Subject Classification: 68U10, 68U05, 52A22
Key words and phrases: Mathematical morphology, image processing,image analysis

1. Introduction

For some years, the operations of standard mathematical morphology for subsets of R^n or Z^n (dilation, erosion, opening, closing) have been generalized to the case of gray-scale images; we refer to [1] and [2] for a complete description of that method.

There seem to be, however, some draw-backs in that method, among which we mention:
(i) The use of the umbra transforms of the functions, instead of the functions themselves (which is more natural).

(ii) A separate construction has to be made for the morphology for subsets of R^2(or Z^2) and for gray-scale images.

It would be valuable if a uniform theory could be constructed in which no distinction has to be made either between the morphological operations for subsets and for gray-scale images, or between the Euclidean and the digital case, and without using the umbra transform.

In what follows we shall show that such a construction is indeed possible when we start by defining the morphological operations immediately for functions defined on subsets of a group G (e.g., G is $(R^2,+)$or$(Z^2,+)$) and with values in a suitable group V (V is either $(R,+)$ or $(Z,+)$ or $\{0\}$; for$V=\{0\}$ we obtain the notions for subsets). In that manner the construction of a dilation, ..., does not suffer from the draw-backs mentioned above.

In section 3 we shall define the r-dilation and r-erosion of two functions; the resulting quantities are in general not functions, but, rather, general subsets of $G \times V$. We shall then show in section 4 that, for bounded functions, the needed morphological operations dilation and erosion are obtained by taking the "surface" of the corresponding r-quantities in section 3. The other notions such as opening, closing, ... are then derived from these two. Of the multitude of properties only a few are quoted at each step, just to give a flavour of them.

We finally mention that we only stressed the group theoretical facts; vector space or module concepts like convexity have not been considered here.

An algebraic basis for mathematical morphology in terms of complete lattices has also been constructed recently; for this we refer to [3] and [4].

2. Classical definitions of dilation and erosion

For selfcontainment, we state in this section the classical definitions of dilation and erosion, first for subsets of R^n, and then for functions defined on a subset of R^{n-1}; these definitions may be found in [1] and [2].

Given two subsets A and B of R^n, the dilation (or Minkowski addition) of A by B is the set $\mathcal{D}(A,B)$ defined by

$$\mathcal{D}(A,B) = \bigcup_{b\in B}(A+b),$$

while the erosion of A by B is the set $\mathcal{E}(A,B)$ defined by

$$\mathcal{E}(A,B) = \{x \in R^n : B + x \subset A\}$$

(see figure 1, figure 2).

For a gray-scale image, defined as a real-valued function f on a subset A of R^{n-1}, the umbra $U[f]$ of f is the set consisting of all the function values $f(x)$ and everything below it; this means $U[f] = \{(x,y) \in A \times R : y \leq f(x)\}$. Hence, the umbra of f is a subset of R^n. Given two real valued functions f and g defined on subsets A and B of R^{n-1} respectively, the dilation $\mathcal{D}(f,g)$ of f by g is defined as the "top" of the set-dilation of $U[f]$ and $U[g]$, i.e., for $x \in \mathcal{D}(A,B)$

$\mathcal{D}(f,g)(x) = \sup\{y : (x,y) \in \mathcal{D}(U[f],U[g])\}$, while the erosion $\mathcal{E}(f,g)$ of f by g is given as the top of the set-erosion of $U[f]$ and $U[g]$ (see figure 3, ..., figure 6).

3. Relational morphological operations

Let G be a (not necessarily commutative) group with a multiplicative group operation and group identity e. For $a \in G$ en $B \subset G$ we denote the left multiplication of B by a as aB, where $aB = \{ab : b \in B\}$; analogously, the right multiplication of B by a is $Ba = \{ba : b \in B\}$. The inverse of an element $a \in G$ is a^{-1}, and $B^{-1} = \{b^{-1} : b \in B\}$.

By V we denote either the additive group $(R, +)$ of real numbers with zero element 0, or the additive group $(Z, +)$ of integers again with zero 0, or the group $\{0\}$ containing only the zero 0.

$G \times V$ is also a group for the following operation: for $x \in G$, $y \in G$, $r \in V$, $s \in V$ we define

$$(x,r).(y,s) = (xy, r+s).$$

For $V = \{0\}, G \times V$ can be identified in the usual manner with G.

Let $f : D_f \to V$ an $g : D_g \to V$ be functions defined on subsets D_f and D_g of G. Clearly, any function like f is a subset of $G \times V$. We write $g << f$ when the following two conditions are satisfied:

(i) $D_g \subset D_f$

(ii) $g(y) \leq f(y)$ for all $y \in D_g$.

When $V = \{0\}$, the condition $g << f$ reduces to the ordinary inclusion relation $D_g \subset D_f$.

Definition 1. *The r-dilation* $\mathcal{D}r(f,g)$ *of two functions* $f : D_f \to V$ *and* $g : D_g \to V$ *is the subset of* $G \times V$ *defined by*

$$\begin{aligned} \mathcal{D}_r(f,g) &= \{(x,f(x)).(y,g(y)) : x \in D_f, y \in D_g\} \\ &= \{(xy, f(x)+g(y)) : x \in D_f, y \in D_g\}, \end{aligned}$$

Hence $\mathcal{D}_r(f,g)$ *is just another notation for the result of applying the group operation in* $G \times V$ *on its subsets* f *and* g.

For the special case when $V = \{0\}$, we identify f with D_f, g with $D_g, G \times V$ with G, and we obtain the usual definition of the dilation of two subsets D_f and D_g of G. From the definition it readily follows that for $V = \{0\}$ we have (with $h : D_h \to V$):

$$\mathcal{D}_r(D_f, D_g) = \bigcup_{y \in D_g} D_f y = \bigcup_{x \in D_f} x D_g,$$

$$\mathcal{D}_r(\mathcal{D}_r(D_f, D_g), D_h) = \mathcal{D}_r(D_f, \mathcal{D}_r(D_g, D_h));$$

$$\text{for } f << h : \mathcal{D}_r(D_f, D_g) \subset \mathcal{D}_r(D_h, D_g).$$

We just give a proof for one property in case $V = \{0\}$.

Proposition 1.

$$\mathcal{D}_r(D_f, D_g) = \{z \in G : zD_g^{-1} \cap D_f \neq \emptyset\}.$$

Proof.

$$\begin{aligned} \mathcal{D}_r(D_f, D_g) &= \{z \in G : z \in U_{x \in D_f} x D_g\} \\ &= \{z \in G : \exists x \in D_f \text{ such that } x^{-1} z \in D_g\} \\ &= \{z \in G : \exists x \in D_f \text{ such that } x \in z D_g^{-1}\} \\ &= \{z \in G : zD_g^{-1} \cap D_f \neq \emptyset\}. \ \square \end{aligned}$$

For general V and $(z,t) \in G \times V$, the set $g.(z,t)$ is defined as the group product in $G \times V$ of the subset $\{(y, g(y)) : y \in D_g\}$ with the element (z,t); hence $g.(z,t) = \{(yz, g(y)+t) : y \in D_g\}$. We remark that $g.(z,t)$ is in fact just the right translate $g_{(z,t)}$ of the function g, defined by $g_{(z,t)}(u) = g(uz^{-1}) + t$ for all u such that $uz^{-1} \in D_g$, i.e.,

$u \in D_g z$. Analogously, the left translate ${}_{(z,t)}g$ of g is defined by ${}_{(z,t)}g(u) = g(z^{-1}u) + t$ for $u \in zD_g$, and this is precisely the subset $(z,t).g$. The following result then follows immediately

Proposition 2.

$$\mathcal{D}_r(f, g.(z,t)) = \mathcal{D}_r(f,g).(z,t)$$

$$\mathcal{D}_r((z,t).f, g) = (z,t).\mathcal{D}_r(f,g).\Box$$

Definition 2. *The r-erosion* $\mathcal{E}_r(f,g)$ *of two functions* $f : D_f \to V$ *and* $g : D_g \to V$ *is the subset of* $G \times V$ *defined by*

$$\mathcal{E}_r(f,g) = \{(z,t) \in G \times V : (z,t).g << f\}.$$

Since

$$\begin{aligned} (z,t).g &= \{(zy, g(y)+t : y \in D_g\} \\ &= \{(u, g(z^{-1}u)+t : u \in zD_g\}, \end{aligned}$$

we also obtain

$$\mathcal{E}_r(f,g) = \{(z,t) \in G \times V : zD_g \subset D_f, g(z^{-1}u) + t \leq f(u)\}$$

for all u such that $z^{-1}u \in D_g$.

For fixed g it follows from $f << h$ that $\mathcal{E}_r(f,g) << \mathcal{E}_r(h,g)$, while for fixed f and $g << h$ we have that $\mathcal{E}_r(f,h) << \mathcal{E}_r(f,g)$. When $(y,s) \in G \times V$ it may also be verified that $\mathcal{E}_r(f,(y,s).g) = \mathcal{E}_r(f,g).(y,s)^{-1}$, and analogously $\mathcal{E}_r((y,s).f,g) = (y,s).\mathcal{E}_r(f,g)$.

For $V = \{0\}$ we obtain

$$\begin{aligned} \mathcal{E}_r(D_f, D_g) &= \{z \in G : zD_g \subset D_f\} \\ &= \bigcap_{y \in D_g} \{z \in G : zy \in D_f\} \\ &= \bigcap_{y \in D_g} D_f y^{-1}. \end{aligned}$$

Moreover, when $e \in D_g$ then $\mathcal{E}_r(D_f, D_g) \subset D_f$.

4. True gray-scale morphological operations

The relational morphologic operations of section 3 associate with two gray-scale images f and g a new subset of $G \times V$ which, except for the trivial case when $V = \{0\}$, is in general not a function, and hence is not a new image. In order to obtain from the r-dilation and r-erosion real gray-scale images (which will then be called the dilation and erosion), it is sufficient to take a suitable subset of those relational morphologic structures. This is easily obtained by taking the surface of such a subset, according to the following definition.

Definition 3. *When* $\mathcal{A} \subset G \times V$, *we call surface of* $\mathcal{A}$ *(denoted by* $S(\mathcal{A})$*) the set*

$$S(\mathcal{A}) = \{(x,t) \in G \times V : x \in proj_G\mathcal{A}, t = \sup\{s \in V : (x,s) \in \mathcal{A}\}\},$$

where $proj_G\mathcal{A}$ *denotes the projection of* $\mathcal{A}$ *on* G, *and where sup possibly might be replaced by maximum; we always suppose that sup exists in* V. *Hence,* $S(\mathcal{A})$ *is a function defined on* $proj_G\mathcal{A}$.

Proposition 3. *Let* $\mathcal{A} \subset G \times V, (z,t) \in G \times V$. *Then*

$$S(\mathcal{A}.(z,t)) = S(\mathcal{A}).(z,t); S((z,t).\mathcal{A}) = (z,t).S(\mathcal{A}).$$

Proof. We shall just prove the first equality. Since $(z,t) = (z,0).(e,t)$, it is sufficient to prove that $S(\mathcal{A}.(e,t)) = S(\mathcal{A}).(e,t)$ and that $S(\mathcal{A}.(z,0)) = S(\mathcal{A}).(z,0)$.

That $S(\mathcal{A}.(e,t)) = S(\mathcal{A}).(e,t)$ follows from the fact that $\mathcal{A}.(e,t) = \{(u,s+t) : (u,s) \in \mathcal{A}\}$, which leads to $proj_G(\mathcal{A}.(e,t)) = proj_G\mathcal{A}$, and from the fact that $S(\mathcal{A}.(e,t)) = \{(x,r) \in G \times V : x \in proj_G\mathcal{A}, r = \sup\{s+t : (x,s) \in \mathcal{A}\}\}$, and hence $r = \sup\{s : (x,s) \in \mathcal{A}\}+t$. In an analogous manner we obtain $proj_G(\mathcal{A}.(z,0)) = (proj_G\mathcal{A})z$, and when $(x,s) \in S(\mathcal{A}.(z,0))$, then $s = \sup\{r \in V : (x,r) \in \mathcal{A}\}$. □

Now we are able to introduce the gray-scale morphological notions for functions $f : D_f \to V$ and $g : D_g \to V$, which we suppose to be bounded.

Definition 4. *The dilation* $\mathcal{D}(f,g)$ *of* f *and* g *is defined by*

$$\mathcal{D}(f,g) \quad = \quad S(\mathcal{D}_r(f,g))$$

$$\begin{aligned} &= \{(z,t) \in G \times V : z \in proj_G \mathcal{D}_r(f,g), t \\ &= \sup\{s \in V : (z,s) \in \mathcal{D}_r(f,g)\}\}. \end{aligned}$$

In view of the definition of $\mathcal{D}_r(f,g)$, z belongs to $D_f D_g$, and hence it can be written in different manners in the form $z = xy$ with $x \in D_f, y \in D_g$; the value of $\mathcal{D}(f,g)$ in such a point z is then given by

$$\begin{aligned} \mathcal{D}(f,g)(z) &= \sup\{f(x) + g(x^{-1}z) : x \in D_f, x^{-1}z \in D_g\} \\ &= \sup\{f(zy^{-1}) + g(y) : y \in D_g, zy^{-1} \in D_f\}. \end{aligned}$$

According to our definition, dilation of two bounded functions is an operation which is always defined and is associative. For $V = \{0\}$ it leads to the usual definition of dilation of two subsets, and in that case $\mathcal{D}(f,g)$ and $\mathcal{D}_r(f,g)$ coincide.

In view of the results given before, a lot of properties about dilation may be proved. We shall just mention a few of them and give only a proof of the first one.

Proposition 4. *For* $(z,r) \in G \times V$ *we have*

$$\mathcal{D}(f, g.(z,r)) = \mathcal{D}(f,g).(z,r); \mathcal{D}((z,r).f, g) = (z,r).\mathcal{D}(f,g).$$

Proof.

$$\begin{aligned} \mathcal{D}(f, g.(z,r)) &= S(\mathcal{D}_r(f, g.(z,r))) = S(\mathcal{D}_r(f,g).(z,r)) = \\ &= S(\mathcal{D}_r(f,g)).(z,r) = \mathcal{D}(f,g).(z,r). \quad \square \end{aligned}$$

Proposition 5. *If* $f_1 << f_2$ *then* $\mathcal{D}(f_1,g) << \mathcal{D}(f_2,g)$. □

Proposition 6. *If* $D_g = D_h$, *then* $\mathcal{D}(f, \max(g,h)) = \max(\mathcal{D}(f,g), \mathcal{D}(f,h))$. □

Definition 5. *The erosion* $\mathcal{E}(f,g)$ *of* f *and* g *is defined by*

$$\begin{aligned} \mathcal{E}(f,g) &= S(\mathcal{E}_r(f,g)) \\ &= \{(x,t) \in G \times V : xD_g \subset D_f, t \\ &= \sup\{s \in V : g(x^{-1}u) + s \le f(u)\}\} \end{aligned}$$

for all u *such that* $x^{-1}u \in D_g$.

Hence the domain of $\mathcal{E}(f,g)$ is the set of those points $x \in G$ such that $xD_g \subset D_f$; the value t of $\mathcal{E}(f,g)(x)$ may also be given by one of the following expressions:

$$\begin{aligned}\mathcal{E}(f,g)(x) &= \sup_{y\in D_g}\{s \in V : g(y) + s \leq f(xy)\}\\ &= \sup_{y\in D_g}\{s \in V : s \leq f(xy) - g(y)\}\\ &= \inf_{y\in D_g}\{f(xy) - g(y)\}.\end{aligned}$$

For $V = \{0\}, \mathcal{E}(f,g)$ and $\mathcal{E}_r(f,g)$ coincide, giving the usual definition of erosion of two subsets.

Proposition 7. *For* $(z,r) \in G \times V$ *we have*

$$\mathcal{E}(f,(b,r).g) = \mathcal{E}(f,g).(b,r)^{-1}$$

Proof.

$$\begin{aligned}\mathcal{E}(f,(b,r).g) &= S(\mathcal{E}_r(f(b,r).g) = S(\mathcal{E}_r(f,g).(b,r)^{-1}) =\\ &= \mathcal{E}(f,g).(b,r)^{-1}.\square\end{aligned}$$

Proposition 8. *For* $f_1 << f_2$ *we have* $\mathcal{E}(f_1,g) << \mathcal{E}(f_2,g)$. □

Proposition 9. *For* $g_1 << g_2$ *we have* $\mathcal{E}(f,g_2) << \mathcal{E}(f,g_1)$. □

The other morphologic notions can be derived from the two basic ones, dilation and erosion. We restrict ourselves to the definition of opening.

Definition 6. *The opening* $0(f,g)$ *of two functions* f *and* g *is defined by*

$$0(f,g) = \mathcal{D}(\mathcal{E}(f,g),g);$$

hence,

$$0(f,g)(z) = \sup\{\mathcal{E}(f,g)(x) + g(x^{-1}z) : x \in D_{\mathcal{E}(f,g)}, x^{-1}z \in D_g\},$$

which leads to one of the following alternative expressions

$$\begin{aligned}0(f,g)(z) &= \sup\{\inf_{y\in D_g}\{f(xy) - g(y)\} + g(x^{-1}z) : xD_g \subset D_f, z \in xD_g\}\\ &= \sup\{\sup_{y\in D_g}\{t : t \leq f(xy) - g(y)\} +\\ &\quad g(x^{-1}z) : xD_g \subset D_f, z \in xD_g\}.\end{aligned}$$

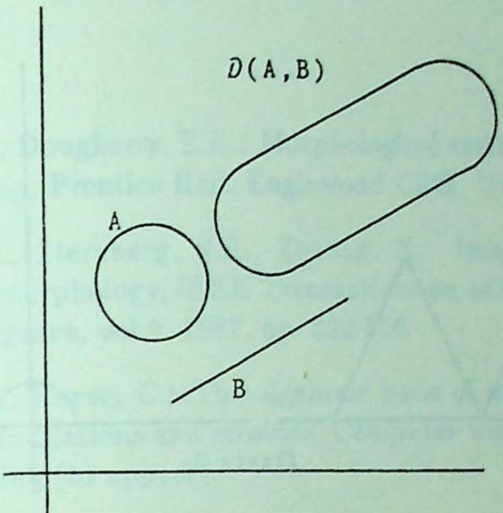


figure 1

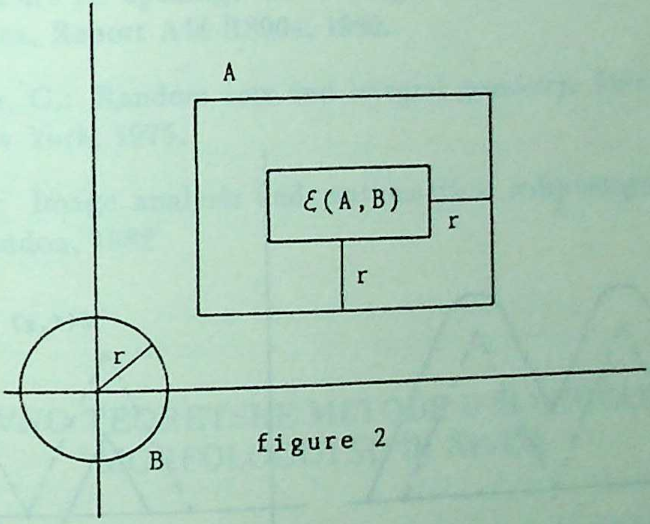


figure 2

It will be clear from the foregoing that, based on the notion of functions defined on some subset of a group and with values in a suitable subgroup of R, all the necessary notions pertaining to the morphology of subsets or grayscale images, whether digital or not, may be derived in a unifying manner using only the functions themselves.
In any case, it is no longer necessary to make a distinction in the definition between the cases where subsets are used, or functions defined on subsets; nor do we have to distinguish between continuous or digital images. One definition is sufficient, since we only have to adapt the set V. Morever, the umbra transform has not been used. This leads to a very economic way of treating the morphologic machinery.

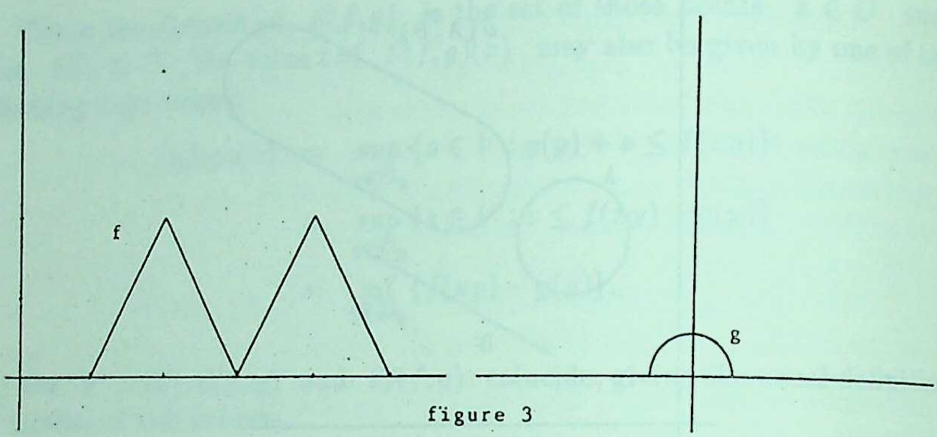


figure 3

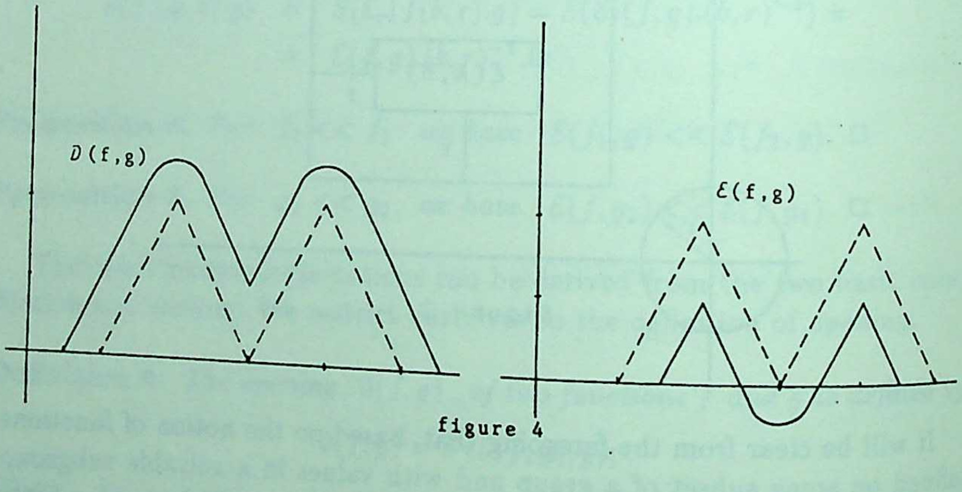


figure 4

References

[1] Giardina, C.R., Dougherty, E.R.: Morphological methods in image and signal processing, Prentice Hall, Englewood Cliffs, New Jersey, 1988.

[2] Haralick, R.M., Sternberg, S.R., Zhuang, X.: Image analysis using mathematical morphology, IEEE Transactions on pattern analysis and machine intelligence, vol.9, 1987, pp. 532-550

[3] Heijmans, H.J., Ronse, C.: The algebraic basis of mathematical morphology, Part I: dilations and erosions, Computer Vision, Graphics and Image Processing (to appear).

[4] Ronse, C., Heijmans, H.J.: The algebraic basis of mathematical morphology. Part II: openings and closings, Centrum voor Wiskunde en Informatica, Report AM-R8904, 1989.

[5] Matheron, G.: Random sets and integral geometry, John Wiley and Sons, New York, 1975.

[6] Serra, J.: Image analysis and mathematical morphology, Academic Press, London, 1982.

REZIME

GRUPOVNO TEORETSKE METODE U MATEMATIČKOJ MORFOLOGIJI SIVIH NIVOA

Pokazano je da za morfološke operacije koje uključuju konačan broj ograničenih funkcija, može biti dato matematičko zasnivanje u grupovno teoretskim terminima, što unificira pojmove podskupova i sivih nivoa i to i u Euklidovom i digitalnom slučaju.

Received by the editors December 5, 1988.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20,2 (1990), 43-52

Review of Research
Faculty of Science
Mathematics Series

SOME RESULTS ON THE REDUCED ENERGY OF GRAPHS, I

Mirko Lepović
Industrija "Filip Kljajić", Djure Djakovića 49
34000 Kragujevac

Abstract

In a recent paper [6], A.Torgašev described all the finite connected graphs whose energy (i. e. the sum of all the positive eigenvalues including also their multiplicities) does not exceed 3. In this paper are described all the connected graphs whose reduced energy, i. e. the sum of absolute values of all the eigenvalues except the largest one, does not exceed 5.

AMS *Mathematics Subject Classification* (1980): 05C50.
Key words and phrases: Graph, spectrum of graph, energy of graph.

In this paper we shall consider only finite connected graphs having no loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$, and its order (number of vertices) by $|G|$. The spectrum of such a graph is the set $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ of eigenvalues of its $0-1$ adjacency matrix.

The sum of eigenvalues $|\lambda_2| + \cdots + |\lambda_n|$ is denoted by $R_1(G)$ and called the **reduced** energy of G. We note that $|\lambda_n| \geq 1$, hence $R_1(G) \geq 1$ for any graph G. For any real $a \geq 1$, we can consider the class of graphs

$$C_1(a) = \{G | \ R_1(G) \leq a\}.$$

In this paper we describe completely the class $C_1(5)$.

Briefly, any graph $G \in C_1(5)$ is called **admissible**, and any other graph - **impossible** (or forbidden) for this class.

If H is any connected (induced) subgraph of a graph G **we denote it** by $H \subseteq G$. Making use of the known interlacing theorem [l;p.19] we have

$R_1(H) \leq R_1(G)$, whence we have that any connected subgraph of an admissible graph is also admissible. This implies that the method of forbidden subgraphs can be consistently applied.

We shall firstly prove an important property of the general class

$$C_1(a)\ (a \geq 1).$$

Theorem 1. *The class $C_1(a)$ is finite for every $a \geq 1$.*

Proof. Let G be an arbitrary graph from the class $C_1(a)$. Then we have

$$a \geq \sum_{i=2}^{n} |\lambda_i| \geq \sum_{\lambda_i<0} |\lambda_i| = \sum_{\lambda_i>0} |\lambda_i|,$$

thus $G \in S(a)$, where

$$S(a) = \{G|\ \sum_{\lambda_i>0} |\lambda_i| \leq a\}$$

is the class treated in paper [6]. Hence, $C_1(a) \subseteq S(a)$. Since by Theorem 2 in [6] , the class $S(a)$ is finite for every $a \geq 1$, our Theorem is proved.

□

Next, let $K_{n_1 n_2 \dots n_m}$, P_n and C_n be the complete m–partite graph, the path and the cycle with n vertices, respectively. Since the complete m-partite graph $K_{n_1 n_2 \dots n_m}$ has just one positive eigenvalue $r(G)$, it will belong to the class $C_1(a)$ if and only if $r(G) \leq a$.

We shall firstly determine the exact values of parameters $n_1, n_2, \dots, n_m$ for which the graph $K_{n_1 n_2 \dots n_m} (n_1 \leq n_2 \leq \dots \leq n_m)$ is admissible.

Proposition 1. *The graph $K_{n,m}$ $(n \leq m)$ is admissible exactly for the following values of parameters n,m:*

1. $n = 1,\ m = 1, 2, \dots, 25.$
2. $n = 2,\ m = 2, 3, \dots, 12.$
3. $n = 3,\ m = 3, 4, \dots, 8.$
4. $n = 4,\ m = 4, 5, 6.$
5. $n = 5,\ m = 5.$

Proof. Since the graph $K_{n,m}$ is the complete bipartite graph, we have that $r(G) = \sqrt{nm}$. Therefore $G \in C_1(5)$ if and only if $nm \leq 25$, which easily gives the statement.

□

Proposition 2. *The graph* $K_{n,m,k}(n \leq m \leq k)$ *is admissible exactly for the following values of parameters* n, m, k*:*

1. $n = 1,\ m = 1,\ k = 1, 2, \ldots, 10.$
2. $n = 1,\ m = 2,\ k = 2, 3, \ldots, 6.$
3. $n = 1,\ m = 3,\ k = 3, 4.$
4. $n = 2,\ m = 2,\ k = 2, 3.$

Proof. The characteristic polynomial of this graph is

$$P(\lambda) = \lambda^{n+m+k-3}(\lambda^3 - (nm + nk + mk)\lambda - 2nmk).$$

Hence, it is a matter of routine to see that $r(G) \leq 5$ exactly for the mentioned values of parameters n, m, k.

□

Proposition 3. *The graph* $K_{n,m,k,l}(n \leq m \leq k \leq l)$ *is admissible exactly for the following values of parameters* n, m, k, l*:*

$$(n,m,k,l) = \begin{array}{l}(1,1,1,1),(1,1,1,2),(1,1,1,3),(1,1,1,4),\\ \qquad (1,1,1,5),(1,1,2,2),(1,1,2,3).\end{array}$$

Proof. It is easy to check that all the above graphs are admissible. Besides, since the graph $K_{n,m,k,l}$ is forbidden for the values of parameters $(n,m,k,l) = (1,1,1.6),(1,1,2,4),(1,1,3,3)$, the interlacing theorem provides the statement.

□

Proposition 4. *The graph* $K_{n,m,k,l,p}(n \leq m \leq k \leq l \leq p)$ *is admissible exactly for the following values of parameters* n, m, k, l, p*:*

$$(n,m,k,l,p) = (1,1,1,1,1),(1,1,1,1,2)$$

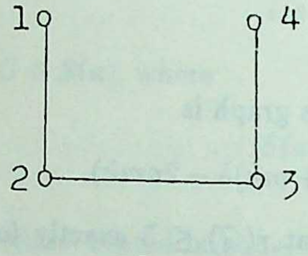
Proposition 5. *The graph $G = K_{n_1 n_2 \dots n_m}(m \geq 6)$ is admissible if and only if $m = 6$ and G is the graph K_6.*

Proof. The graph K_6 is obviosly an admissible graph. Besides, since the graphs $K_{1,1,1,1,1,2}$ and K_7 are forbidden, the statement is immediate.

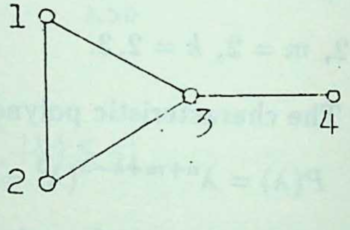
□

Next, we shall determine all the admissible graphs from the class $C_1(5)$ wich have at least two positive eigenvalues. By a result of Smith [5] we have

Theorem A *Graph G has at least two positive eigenvalues if and only if it has one of the following graphs as an (induced) subgraph.*



G_1



G_2

Note that the mentioned graphs G_1 and G_2 are admissible. Denote $\mathcal{G}_1 = \{G_1, G_2\}$. If $G \in \mathcal{G}_1$ and S is any nonempty subset from the vertex set $V(G)$, let G_x be the graph obtained from G by adding a new vertex x adjacent just to the vertices from S. Denote by $\mathcal{G}_2$ the set of all the nonisomorphic admissible graphs G_x, where $G \in \mathcal{G}_1$, and $S \subseteq V(G) \setminus \{\emptyset\}$. If the class $\mathcal{G}_i$ is constructed, we define $\mathcal{G}_{i+1}$ by the class obtained by $\mathcal{G}_i$ in a similar way as $\mathcal{G}_2$ has been obtained by $\mathcal{G}_1$. Denote

$$\mathcal{G} = \cup_{i \in N} \mathcal{G}_i$$

By theorem 1, class $\mathcal{G}$ is finite, i.e. we have $\mathcal{G}_i = \emptyset$ for all i large enough. All the graphs $G \in \mathcal{G}$ are, by definition admissible graphs, and $\mathcal{G}$ contains all such graphs with more than one positive eigenvalue. This follows from the fact that for any connected graph G and any of its connected induced subgraph H, there is a sequence of connected induced subgraphs $H_i \subseteq G$ $(i = 0, 1, \dots, r)$ such that

$$H = H_O \subseteq H_1 \subseteq \dots \subseteq H_r = G$$

and
$$|H_{i+1}| = |H_i| + 1 \ \ (i = 0, 1, \ldots, r-1).$$

Using a computer device, we can generate the set of all the nonisomorphic admissible graphs with at least two positive eigenvalues.

In this way we get the following theorem.

Theorem 2. *The class* $C_1(5)$ *contains exactly* 137 *nonisomorphic graphs which have at least two positive eigenvalues. The orders of these graphs run over the set* $\{4, 5, \ldots, 11\}$. *All these graphs are represented in List 1.*

Proposition 1 – 5 and Theorem 3 describe completely the class $C_1(5)$. In particular, we get that class $C_1(5)$ contains exactly $75 + 137 = 212$ nonisomorphic graphs.

We should notice that all the graphs in this list are represented in the form

$$n_1 \ n_2 \ n_3 \ \ a_{12} \ a_{13} \ a_{23} \ldots a_{1n} \ a_{2n} \ldots a_{n-1,n},$$

where n_1 is the ordering number of a corresponding graph, n_2 is the number of its vertices, n_3 is the number of its edges and $a_{12} \ a_{13} \ a_{23} \ldots a_{1n} \ a_{2n} \ldots a_{n-1,n}$ is the upper diagonal part of its adjacency matrix.

LIST 1. ADMISSIBLE GRAPHS WITH AT LEAST TWO POSITIVE EIGENVALUES

001 04 03 1 10 001
002 04 04 1 10 110

003 05 04 1 10 001 1000
004 05 04 1 10 001 0100
005 05 05 1 10 001 0101
006 05 05 1 10 011 1000
007 05 05 1 10 001 1100
008 05 05 1 10 110 1000
009 05 05 1 10 001 1010
010 05 06 1 10 001 1110
011 05 06 1 10 110 1001
012 05 06 1 10 001 1101
013 05 06 1 10 110 1010
014 05 07 1 11 111 1000
015 05 07 1 10 011 1101
016 05 07 1 10 111 0011
017 05 08 1 11 111 1001

018 06 05 1 10 001 1000 10000
019 06 05 1 10 001 1000 00010
020 06 05 1 10 001 1000 01000
021 06 05 1 10 001 1000 00100
022 06 06 1 10 001 1010 00100
023 06 06 1 10 001 0100 01100
024 06 06 1 10 001 1010 00001
025 06 06 1 10 011 1000 00100
026 06 06 1 10 011 1000 10000
027 06 06 1 10 011 1000 00001
028 06 06 1 10 110 1000 10000
029 06 06 1 10 110 1000 00100
030 06 06 1 10 001 1000 00110
031 06 07 1 10 001 1000 11100
032 06 07 1 10 001 1000 10110
033 06 07 1 10 001 1010 10100

034 06 07 1 10 110 1001 00100
035 06 07 1 10 110 1001 10000
036 06 07 1 10 100 1000 01101
037 06 07 1 10 001 1101 01000
038 06 07 1 10 001 1000 01101
039 06 08 1 10 100 1111 00010
040 06 08 1 10 100 1111 00001
041 06 08 1 10 111 0011 00010
042 06 08 1 10 011 1101 00001
043 06 08 1 10 011 1101 00100
044 06 08 1 10 110 1001 00110
045 06 08 1 10 011 1001 00011
046 06 08 1 10 011 1001 00101
047 06 08 1 10 110 1000 11010
048 06 08 1 10 001 1010 10101
049 06 09 1 11 111 1001 10000
050 06 09 1 11 111 1001 01000
051 06 09 1 10 110 1000 01111
052 06 09 1 10 011 1000 01111
053 06 09 1 10 110 1000 10111
054 06 09 1 10 011 1101 01001
055 06 09 1 10 001 1101 01110
056 06 09 1 10 011 1000 11110
057 06 09 1 10 110 1001 11001
058 06 10 1 10 111 0111 00011
059 06 10 1 10 111 0111 10001
060 06 10 1 11 111 1001 10010
061 06 10 1 10 110 1001 11011
062 06 10 1 10 011 1001 01111
063 06 11 1 11 111 1000 01111
064 06 11 1 10 011 1101 11110
065 06 11 1 10 111 1111 00011
066 06 12 1 10 111 0111 11011

067 07 06 1 10 100 1000 10000 000001
068 07 06 1 10 001 1000 00100 001000
069 07 06 1 10 001 1000 00010 100000
070 07 06 1 10 001 1000 00010 000100

071 07 07 1 10 011 1000 10000 000100
072 07 07 1 10 001 1000 10000 101000
073 07 07 1 10 100 1000 10000 010010
074 07 07 1 10 110 1000 10000 100000
075 07 07 1 10 001 1000 00100 101000
076 07 08 1 10 001 1000 01101 001000
077 07 08 1 10 100 1000 10000 001101
078 07 08 1 10 110 1001 10000 000100
079 07 08 1 10 100 1000 10000 100011
080 07 08 1 10 011 1000 00001 100100
081 07 08 1 10 100 1000 01101 000001
082 07 09 1 10 100 1000 01101 100001
083 07 09 1 10 001 1010 00100 011010
084 07 09 1 10 100 1000 10000 101110
085 07 09 1 10 110 1000 11010 100000
086 07 09 1 10 001 1000 10000 011011
087 07 09 1 10 001 1010 10100 101000
088 07 10 1 10 100 1000 01111 100010
089 07 10 1 10 011 1001 01101 001000
090 07 10 1 10 100 1000 01111 001100
091 07 10 1 10 100 1111 00001 100001
092 07 10 1 10 110 1001 11001 100000
093 07 10 1 10 100 1000 11111 000001
094 07 10 1 10 011 1101 00001 100100
095 07 11 1 10 011 1001 01111 000001
096 07 11 1 10 011 1001 00101 011010
097 07 11 1 10 110 1000 10000 011111
098 07 11 1 10 111 0111 10001 000100
099 07 12 1 10 100 1111 00001 011110
100 07 12 1 10 110 1001 10000 011111
101 07 13 1 10 100 1111 00001 111101
102 07 13 1 10 100 1000 01111 011111
103 07 13 1 10 110 1000 01111 011110
104 07 14 1 10 111 0111 10001 011101
105 07 14 1 10 111 0111 11101 011000
106 08 07 1 10 001 1000 00100 001000 1000000
107 08 07 1 10 001 1000 00100 001000 0010000
108 08 07 1 10 100 1000 10000 100000 0000010

```
109 08 07 1 10 001 1000 00010 100000 1000000
110 08 08 1 10 100 1000 10000 100000 0000101
111 08 08 1 10 001 1000 00100 101000 1000000
112 08 08 1 10 110 1000 10000 100000 1000000
113 08 08 1 10 110 1000 10000 100000 0100000
114 08 09 1 10 100 1000 10000 100000 0010101
115 08 09 1 10 100 1000 10000 100000 1100010
116 08 09 1 10 110 1001 10000 000100 0001000
117 08 09 1 10 001 1000 10000 101000 1010000
118 08 10 1 10 100 1000 10000 100000 1011100
119 08 10 1 10 100 1000 01101 100001 1000000
120 08 10 1 10 001 1010 10100 101000 1000000
121 08 10 1 10 100 1000 01101 100001 0000010
122 08 11 1 10 100 1000 10000 100000 0101111
123 08 11 1 10 100 1000 10000 100000 1101110
124 08 11 1 10 100 1000 11111 000001 1000000
125 08 11 1 10 100 1000 01111 100001 0000001
126 08 12 1 10 100 1000 10000 100000 1011111
127 08 13 1 10 110 1000 10000 100000 0111111
128 08 16 1 10 100 1000 01111 100001 0111111

129 09 08 1 10 100 1000 10000 100000 1000000 01000000
130 09 09 1 10 110 1000 10000 100000 1000000 10000000
131 09 10 1 10 100 1000 10000 100000 1100010 10000000
132 09 13 1 10 100 1000 10000 100000 0111111 00000001
133 09 14 1 10 100 1000 10000 100000 1000000 11101111

134 10 09 1 10 100 1000 10000 100000 1000000 01000000 100000000
135 10 10 1 10 100 1000 10000 100000 1000000 10000000 100000001

136 11 10 1 10 100 1000 10000 100000 1000000 01000000 100000000 1000000000
137 11 11 1 10 100 1000 10000 100000 1000000 10000000 100000001 1000000000
```

References

[1] Cvetković, D., Doob, M., Sachs, H.: Spectra of Graphs – Theory and Application. VEB Deutsch. Verl. Wiss., Berlin, 1980.

[2] Cvetković, D., Doob, M., Gutman, I., Torgašev, A.: Recent Results in the Theory of Graph Spectra. North–Holland, Amsterdam, 1988.

[3] Petrović M. M.: On graphs whose spectral spread does not exceed 4, Publ. Inst. Math. (Beograd), 34(48)(1983), 169–174.

[4] Petrović M. M.: Graphs with bounded reduced positive energy , In Proc. Fourth Yugoslav Sem. Appl. Math., Split, May 28-30, 1984, 147-153.

[5] Smith, J.H.: Some properties of the spectrum of a graph. In Combinatorial Structures and Their Applications, Gordon and Breach, New York, 1970, 403–406.

[6] Torgašev, A.: Graphs whose energy does not exceed 3, Czech. Math. Journal 36 (111)(1986), No. 2, 167–171.

REZIME

NEKI REZULTATI O PRVOJ REDUKOVANOJ ENERGIJI GRAFA

U nedavno objavljenom radu [6], A. Torgašev je opisao sve konačne povezane grafove čija energija (tj. suma svih pozitivnih sopstvenih vrednosti uključujući takodje njihove višestrukosti) nije veća od 3. U ovom radu opisujemo sve povezane grafove čija prva redukovana energija, tj. suma apsolutnih vrednosti svih sopstvenih vrednosti bez maksimalne, nije veća od 5.

Received by the editors December 8, 1989.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 53-59

Review of Research
Faculty of Science
Mathematics Series

NORMAL FUZZY SUBGROUPS

A. S. Mashour, M. H. Ghanim and **F. I. Sidky**
Department of Mathematics, Faculty of Science
Zagazig University, Zagazig - Egypt

Abstract

Rosenfeld in [8] has introduced the concept of a fuzzy subgroup. His definition has been generalised by Anthony and Sherwood in [2]. In this paper we introduce the concept of a normal fuzzy subgroup and prove some properties of this new concept.

AMS Mathematics Subject Classification (1980): 20N25
Key words and phrases: Normal fuzzy subgroups

Throughout this paper G will denote a group whose operation is suppressed and indicated by juxtaposition. We use the concept of a fuzzy subgroup as defined in [2]. For the definition of a t - norm we refer the reader to [1, 2]. We do not distinguish in notion between a fuzzy set [11] and its membership function.

Definition 1. *Let* A *be a fuzzy subset of* G *and let* ${}_xf : G \longrightarrow G$ $(f_x : G \longrightarrow G)$ *be a function defined by* ${}_xf(g) = xg$ $(f_x(g) = gx)$. *A fuzzy left (right) coset* $xA(Ax)$ *is defined to be* ${}_xf(A)\,(f_x(A))$.

If is easily seen that $(xA)(y) = A(x^{-1}y)$ and $(Ax)(y) = A(yx^{-1})$, for every y in G.

Proposition 1. *Let* A *be a fuzzy subset of* G. *Then the following conditions are equivalent for each* x, y *in* G.

(i) $A(xyx^{-1}) \geq A(y)$.
(ii) $A(xyx^{-1}) = A(y)$.
(iii) $A(xy) = A(yx)$.
(iv) $xA = Ax$.
(v) $xAx^{-1} = A$.

Proof. Straightforward.

Proposition 2. *If* A *is a fuzzy subgroup of* G, *then* gAg^{-1} *is also a fuzzy subgroup of* G *for all* g *in* G.

Proof. Let A be a fuzzy subgroup of G, under a t - norm T. Then $gAg^{-1}(e) = A(e) = 1$, $gAg^{-1}(x) = A(g^{-1}xg) = A(g^{-1}xg) = gAg^{-1}\,(x^{-1})$ for all x, g in G and $gAg^{-1}\,(xy) = A(g^{-1}(xy)g) = Ag^{-1}(xgg^{-1}y)g) = A((g^{-1}xg)(g^{-1}yg)) \leq T(A(g^{-1}xg),\ Ag^{-1}yg)) = T(gAg^{-1}\,(x),\ gAg^{-1}\,(y))$ for all x, y in G. Hence gAg^{-1} is a fuzzy subgroup of G under T.

Definition 2. *A fuzzy subgroup* A *of* G *is said to be normal if it satisfies one of the equivalent conditions in Proposition 1.*

Proposition 3. *If* A *is a fuzzy subgroup of* G, *then* $\bigcap_{g \in G} gAg^{-1}$ *is a normal fuzzy subgroup of* G *(under a t - norm as* A*).*

Proof. Similar to the crisp case.

Proposition 4. *Let* A *and* B *be two fuzzy subgroups of* G *under the t - norms* T_1 *and* T_2, *respectively. Then* $A \cap B$ *is a fuzzy subgroups under any t- norm* T *such that* $T_1, T_2 \geq T$.

Proof. Let $X = \{T | T_1, T_2 \geq T\}$ and let Z be a t - norm defined by:

$$Z(a, b) = 0 \text{ if } a \neq 1, b \neq 1 \text{ and } Z(a, b) = a (= b) \text{ if } b = 1\ (a = 1).$$

Since $Min \geq T' \geq Z$, for any t - norm T' (see [6, 10]), then $Z \in X$, i. e., $X \neq \emptyset$. Now let x, y in G. Since $Min \gg T' \gg T' \gg Z$ (see [10]); then

$A \cap B\,(xy^{-1}) = Min\,(A\,(xy^{-1}), B\,(xy^{-1})) \geq Min(T_1\,(A\,(x), A\,(y)),$
$T_2\,(B\,(x), B\,(y)) \geq Min\,(T\,(A\,(x), A(y)), T\,(B\,(x), B\,(y))) =$
$= T\,(Min\,(A\,(x), B\,(x)), Min\,(A\,(y), B\,(y))) =$
$= T\,(A \cap B\,(x), A \cap B\,(y)).$
Hence $A\,B$ is a fuzzy subgroups of G under T.

Proposition 5. *The intersection of any two normal fuzzy subgroups of* G *is also a normal fuzzy subgroup of* G *under any t - norm weaker than the t - norms of the two fuzzy subgroups.*

Proof. Let A and B be two normal fuzzy subgroups of G under t - norms T and T', respectively. According to Proposition 4, $A \cap B$ is a fuzzy subgroup of G under any t - norm weaker than T and T'. Now, for all x, y in G we have $A \cap B(xyx^{-1}) = Min(A(xyx^{-1}), B(xyx^{-1})) = Min(A(y), B(y)) = A \cap B(y)$. Hence $A \cap B$ is a normal fuzzy subgroup of G.

Proposition 6. *Let* $f : G \longrightarrow H$ *be a group homomorphism.*

(i) If A *is a normal fuzzy subgroups of* H*, then* $f^{-1}(A)$ *is a normal fuzzy subgroup of* G *(under the t - norm of* A*).*

(ii) If f *is an epimorphism and* A *is a normal fuzzy subgroups of* G*, then* $f(A)$ *is a normal fuzzy subgroups of* H *(under the t - norm of* A*, which is continious).*

Proof.

(i) Let A be a fuzzy subgroup of H under T. Then $f^{-1}(A)$ is a fuzzy subgroup of G under T (cf. [2,3]). Now, for all x, y in G, we have $f^{-1}(A)(xyx^{-1}) = A(f(xyx^{-1})) = A(f(x)f(y)f(x)^{-1}) = A(f(y)) = f^{-1}(A)(y)$. Hence $f^{-1}(A)$ is a normal fuzzy subgroup of G.

(ii) Let A be a fuzzy subgroup of G under a continous t - norm T. Then $f(A)$ is a fuzzy subgroup of H under T (cf. [2]). Now, for all u, v in H, we have

$$f(A)(uvu^{-1}) = \sup_{F(y)=uvu^{-1}} A(y) = \sup_{F(x)=u, f(y)=v} A(xyx^{-1})$$

$$= \sup_{F(y)=v} A(y) = f(A)(v), \text{ since } f \text{ is epimorphism.}$$

Hence $f(A)$ is a normal fuzzy subgroup of H.

Definition 3. [9]. *Let A and B be two fuzzy subsets of G. The T - product of A and B is defined to be the fuzzy subset AB of G given by:*

$$AB(x) = \sup_{yz=x} T(A(y),\ A(z)), \quad x \in G.$$

Note that a fuzzy coset xA may be also defined as the T -product of $\{x\}$ and A.

Proposition 7. *Let $f : G \longrightarrow . H$ be a group homomorphism and let A and B a fuzzy subsets of G. Then $f(AB) = f(A)f(B)$, where the products are under the same t - norm.*

Proof. Similar to the proof of Proposition 2.2 in [5].

Note that Proposition 6(ii) may also be proved, using Proposition 7 as follows: Since $xAx^{-1} = A$ for all $x \in G$; then $f(xAx^{-1}) = f(A)$, i. e., $f(x)f(A)f(x)^{-1} = f(A)$. But since f is an epimorphism, then $f(A)$ is a normal fuzzy subgroup of H.

Proposition 8. [3]. *Let A be a fuzzy subgroup of G under Min. Then the level subset $G^t = \{x \in G \mid A(x) \geq t\}$, for every t in [0, 1] such that $G^t_A \neq \emptyset$, is a subgroup of G.*

In the above proposition, Min can not be replaced by an arbitrary t - norm. The following example indicates this fact.

Example 1. *Let $G = \{e, b, c, d\}$ be a Klein's four group and let A be a fuzzy subset of G defined by $A(e) = 1$, $A(b) = t_1$, $A(c) = t_2$ and $A(d) = t_3$ such that $t_3 = t_1 t_2$ and $1 \geq t_1 \geq t_2 \geq t_3$. Then it is easily verified that A is a fuzzy subgroup of G under Prod. But $G^{t_2}_A$ is not a subgroup of G; since $bc = d \notin G^{t_2}_A = \{a, b, c\}$.*

Proposition 9. .

(i) If A is a normal fuzzy subgroup of G under Min, then every G^t_A $(\neq 0)$, t in [0, 1], is a normal subgroup of G.

(ii) Let A be a fuzzy subset of G such that, for every t in $Im(A)$, G is a normal subgroup of G. Then A is a normal fuzzy subgroup of G under Min.

Proof. From Proposition 8, it follows that G_A^t, for every t in $[0, 1]$, is a subgroup of G. Let y in G. Then $A(xyx^{-1}) = A(y) \geq t$. Thus $xyx^{-1} \in G_A^t$, i. e., $y \in x^{-1}G_A^t x$. This implies that G is a normal subgroup of G.

(ii) Let G_A^t, t in Im (A), be a normal subgroup of G. Then $xG_A^t\ x^{-1} \subseteq G_A^t$, for all $x \in G$. Let $y \in G$ with $A(y) = t$. Then $A(xyx^{-1}) \geq t = A(y)$. Since A is a fuzzy subgroup (under Min); then A is a normal subgroup of G.

One may easily verify the following.

Proposition 10. *If A and B are normal fuzzy subgroups of G, then the T - product AB is a normal fuzzy subgroup of G under T.*

Definition 4. *Let A be a fuzzy subgroup of G and M be a subset of G such that $mAm^{-1} = A$, for all $m \in M$. We call M the normaliser of A.*

Proposition 11. *The normaliser of a fuzzy subgroup of G is a subgroup of G.*

Proof. Similar to the crisp case.

Note that, if A is a fuzzy subgroup of G and M its normaliser, then $G_A^1 \subseteq M$. Note also that a fuzzy subgroup A of G is a normal iff its normaliser $M = G$.

Example 2. *Let $G = \{a, b, c, d, e, f\}$, where $a = (1)$, $b = (12)\,(36)\,(45)$, $c = (13)\,(25)\,(46)$, $d = (14)\,(26)\,(35)$, $e = (156)\,(234)$ and $f = (165)\,(243)$.*

1. If A is a fuzzy subset of G defined by : $A(a) = 1$, $A(b) = A(c) = A(d) = t_1$ and $A(e) = A(f) = t_2$, such that $1 \geq t_2 \geq t_1$, where t_1 and t_2 in [0,1]. Then A is a fuzzy subgroup of G under Min and $G = M$ (M is the normaliser of A). Hence A is normal.

2. If A is a fuzzy subset of G defined by : $A(a) = 1$, $A(b) = t_1$, $A(c) = A(d) = A(e) = A(f) = t_2$, such that $1 \geq t_2 \geq t_1$, where t_1 and t_2 in [0, 1]. Then A is a fuzzy subgroup of G under Min and its normaliser $M = \{a, b\}$. Note that if $t_2 \neq 0$, then $A \nsubseteq M$.

References

[1] Abu Osman, M.T.: One the direct product of fuzzy subgroups, Fuzzy Sets and Systems 12 (1984), 87 - 91.

[2] Anthony, J.M., Sherwood, H.: Fuzzy groups redefined, J. Math. Anal. Appl., 69 (1979), 124 - 130.

[3] Das,P.S.: Fuzzy groups and level subgroups, J. Math. Anal. Appl. 84(1981), 264 - 269.

[4] Foster, D.A.: Fuzzy topological groups, J. Math. Anal. Appl. 67(1979), 549 - 564.

[5] Katsaras, A.K., Liu, D.B.: Fuzzy vector spaces and fuzzy topological vector spaces, J. Math. Anal. Appl. 58(1977), 135 - 146.

[6] Wang-jin Liu: Fuzzy invariant subgroups and fuzzy ideals, Fuzzy Sets and Systems, 8(1982), 133 - 139.

[7] Negoita, C.V., Ralescu, D.A.: Applications of fuzzy sets to system analysis, John Wiley and Sons, New York - Toronto, 1975.

[8] Rosenfeld, A.: Fuzzy groups, J. Math. Anal. Appl. 35(1971), 512 - 517.

[9] Sessa, S.: On fuzzy subgroups and fuzzy ideals under triangular norms, Fuzzy Sets and Systems 13(1984), 95 - 100.

[10] Sherwood, H.: Product of fuzzy subgroups, Fuzzy Sets and Systems 11(1983), 79 - 89.

[11] Zadeh, L.A.: Fuzzy sets, Information and Control 8(1965), 338 - 353.

REZIME

NORMALNE RASPLINUTE PODGRUPE

Rosenfeld je u [8] uveo pojam rasplinute podgrupe. Njegovu definiciju su uopštili Anthony i Sherwood u [2]. U ovom radu je uveden pojam normalne rasplinute podgrupe i ispitane su neke osobine ovog novog pojma.

Received by the editors October 10, 1986.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 61-69

Review of Research
Faculty of Science
Mathematics Series

WEAK CONGRUENCES AND HOMOMORPHISMS

Branimir Šešelja and **Andreja Tepavčević**
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

The necessary and sufficient conditions under which a weak congruence lattice of a factor algebra is a sublattice of a weak congruence lattice are given.

Some classes of algebras satisfying this condition are described (any variety of coherent algebras , ∪-algebras, etc.).

AMS Mathematics Subject Classification (1980) : 08A30, 06B10
Key words and phrases: Lattices, coherent algebras.

1. Preliminaries

It is well known that for a congruence θ on an algebra $\mathcal{A}$, $Con\mathcal{A}/\theta$ is (up to the isomorphism) a sublattice $[\ \theta)$ of $Con\mathcal{A}$. For the weak congruence lattice $Cw\mathcal{A}$ (i.e. for the lattice of all congruences on all the subalgebras of $\mathcal{A}$, see for example [3]) it is not always the case. As an example, consider the four element groupoid $(G, *)$, given by its Cayley table.

*	a	b	c	d
a	a	c	b	b
b	c	b	d	d
c	b	d	a	a
d	b	d	a	a

$\mathcal{G} = (\{a,b,c,d\},*)$, $\mathcal{A} = (\{a\},*)$, $\mathcal{B} = (\{b\},*)$
$Con\mathcal{G} = (\{\Delta,\theta,G^2\},\leq)$, $\theta = \{\{a\},\{b\},\{c,d\}\}$,
$Sub\mathcal{G} = (\{\emptyset,A,B,G\},\leq)$.

If θ is the congruence determined by the partition $\{\{a\},\{b\},\{c,d\}\}$, then the corresponding weak congruence lattice $Cw\mathcal{A}/\theta$ is not a sublattice of $Cw\mathcal{A}$ (see Fig.1). Note that $\mathcal{G}$ has two subgroupoids, both one-element.

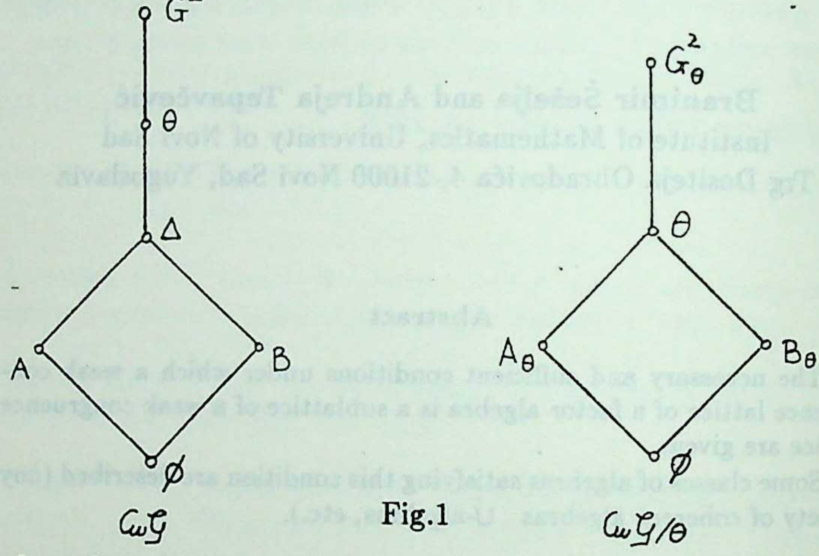


Fig.1

Algebras for which $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$ ($\theta \in Con\mathcal{A}$) have to fulfill some lattice conditions (concerning the weak-congruence lattice), but also algebraic ones. To describe the former, we shall prove some pure lattice-theoretical propositions. For the latter, we shall use the well known facts about the construction of factor-algebras.

The notations, definitions, and some necessary results are from [2], [3], and from the references given there.

An element a of lattice L is **codistributive**, if for all $x, y \in L$

$$a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y),$$

or, equivalently, if and only if the mapping $m_a : x \to x \wedge a$ is a homomorphism of L into the ideal $(\ a]$.

A **distributive** element a of L is defined dually (the corresponding homomorphism from L into the filter $[\ a)$ is obvioisly $x \to x \vee a$).

An element a of L is said to be **modular**, if for all $x, y \in L$, $a \leq y$ implies $a \vee (x \wedge y) = (a \vee x) \wedge y$.

An element a of L is said to be **comodular**, if $a \leq y$ implies $x \vee (a \wedge y) = (x \vee a) \wedge y$.

The lattice CwA of weak congruences of an algebra $\mathcal{A} = (A, F)$ is the lattice of all the symmetric and transitive subalgebras of $\mathcal{A}^2$ under the set inclusion. The diagonal relation $\Delta = \{(x,x) \mid x \in A\}$ is a codistributive element of that lattice, $[\ \Delta\ )$ is $Con\mathcal{A}$, and $Sub\mathcal{A}$ is isomorphic with $(\ \Delta]$ under $\rho \to \{x \mid x\rho x\}$.

Recall that $\mathcal{A}$ has the **congruence extension property** (CEP), if every congruence on a subalgebra of $\mathcal{A}$ is a restriction of the congruence on $\mathcal{A}$. $\mathcal{A}$ has the CEP if and only if Δ is a comodular element of $Cw\mathcal{A}$ ([2]).

$\mathcal{A}$ is said to have the **congruence intersection property** (CIP), if for $\rho \in Con\mathcal{B}$, $\theta \in Con\mathcal{C}, \mathcal{B}, \mathcal{C} \in Sub\mathcal{A}$,

$$(\rho \cap \theta)_A = \rho_A \cap \theta_A,$$

where ρ_A stands for the least congruence on $\mathcal{A}$ extending ρ. Since $\rho_A = \rho \vee \Delta$ in $Cw\mathcal{A}$, $\mathcal{A}$ has the CIP if and only if Δ is a distributive element of $Cw\mathcal{A}$.

A is said to have the **weak congruence intersection property** (wCIP), if for $\rho \in Con\mathcal{B}$, $\theta \in Con\mathcal{A}$, $\mathcal{B} \in Sub\mathcal{A}$

$$(\rho \cap \theta)_A = \rho_A \cap \theta.$$

Obviously, $\mathcal{A}$ has the wCIP if and only if Δ is a modular element of $Cw\mathcal{A}$.

2. Sublattices determined by special elements

In the following, a will be a fixed codistributive element of the lattice L, such that the classes of the congruence induced by $m_a : x \to x \wedge a$ have maximum elements, the collection of which is denote by M_a. For $x \in L$, $\bar{x} \in M_a$, and $m_a(x) = m_a(\bar{x})$.

Since a is a codistributive element of L, we have that

$$L = \cup([a \wedge \bar{x},\ \bar{x}] \mid x \in L),$$

where the interval sublattice $[a \wedge \bar{x},\ \bar{x}]$ is a class of the congruence induced by m_a.

For $b \in [a)$, let $L_b = \cup([b \wedge \bar{x},\ \bar{x}] \mid x \in L)$. L_b is not necessarily a sublattice of L, and in the following proposition we give some necessary conditions under which it is.

Proposition 1. *Let a be a modular and comodular element of the lattice L, and let M_a be a sublattice of L. If b is a distributive element of the filter $[a)$, then L_b is a sublattice of L.*

Proof. Let $u, v \in \bigcup([b \wedge \bar{x},\ \bar{x}] \mid x \in L)$. Then $u \in [b \wedge \bar{y},\ \bar{y}]$, $v \in [b \wedge \bar{z},\ \bar{z}]$, for some $y, z \in L$, i.e. $b \wedge \bar{y} \leq u \leq \bar{y}$, and $b \wedge \bar{z} \leq v \leq \bar{z}$. Now

$$b \wedge \overline{y \wedge z} = b \wedge \bar{y} \wedge \bar{z} \leq u \wedge v \leq \bar{y} \wedge \bar{z} = \overline{y \wedge z},$$

and $u \wedge v \in [b \wedge \overline{y \wedge z}, \overline{y \wedge z}]$.

Moreover, $(b \wedge \bar{y}) \vee (b \wedge \bar{z}) \leq u \vee v \leq \bar{y} \wedge \bar{z} = \overline{y \vee z}$, since M_a is a sublattice of L.

By asumption a is modular in L and b distributive in $[\ a)$. Thus, $((b \wedge \bar{y}) \vee (b \wedge \bar{z})) \vee a = ((b \wedge \bar{y}) \vee a) \vee ((b \wedge \bar{z}) \vee a) = (b \wedge (\bar{y} \vee a)) \vee (b \wedge (\bar{z} \vee a)) = b \wedge (\bar{y} \vee \bar{z} \vee a) = (b \wedge (\bar{y} \vee \bar{z})) \vee a$.

Now, since a is comodular, it follows that

$(b \wedge \bar{y}) \vee (b \wedge \bar{z}) = b \wedge (\bar{y} \vee \bar{z}) = b \wedge (\overline{y \vee z})$, and $u \vee v \in [b \wedge (\overline{y \vee z}), (\overline{y \vee z}]$.

□

Proposition 2. *Let M_a be a sublattice of L and take $b \in [a)$. If for $y, z \in L_b$*

$$b \wedge (y \vee z) = (b \wedge y) \vee (b \wedge z),$$

then L_b is a sublattice of L.

Proof. As in the previous proposition, if $u, v \in L_b$, then obviously $u \wedge v \in L_b$. If $b \wedge \bar{y} \leq u \leq \bar{y}$ and $b \wedge \bar{z} \leq \bar{z}$, then

$$b \wedge (\overline{y \vee z}) = b \wedge (\bar{y} \vee \bar{z}) = (b \wedge \bar{y}) \vee (b \wedge \bar{z}) \leq u \vee v \leq \overline{y \vee z},$$

which proves that L_b is a sublattice of L.

□

3. Weak congruence lattices of $\mathcal{A}$ and of $\mathcal{A}/\theta$

Let $\mathcal{A}$ be an algebra and $\theta \in Con\mathcal{A}$. If $\mathcal{B}_\theta \in Sub\mathcal{A}/\theta$, then $\mathcal{B} \in Sub\mathcal{A}$, and $B = B[\theta]$, where $B[\theta] = \{x \in A \mid x\theta b, \text{ for some } b \in B\}$.

For the congruence lattice of $\mathcal{B}/\theta$ we have that

$$Con\mathcal{B}/\theta \cong [B^2 \wedge \theta, B^2],$$

where $[B^2 \wedge \theta, B^2]$ is the interval sublattice of $Cw\mathcal{A}$. Consequently, $Cw\mathcal{A}/\theta$ is (up to the isomorphism) a sublattice of $Cw\mathcal{A}$, provided that

$$\cup([B^2 \wedge \theta, B^2] \mid \mathcal{B} \in Sub\mathcal{A},\ B = B[\theta])$$

is a sublattice of $Cw\mathcal{A}$.

Theorem 1. *Let $\mathcal{A}$ be an algebra, and $\theta \in Con\mathcal{A}$. Then, $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$ if and only if the following conditions are satisfied for all $\mathcal{B},\mathcal{C} \in Sub\mathcal{A}$, such that $B = B[\theta]$, and $C = C[\theta]$:*

(i) $B \vee C = B \vee C[\theta]$;

(ii) $\theta \wedge (B^2 \vee C^2) = (\theta \wedge B^2) \vee (\theta \wedge C^2)$;

(iii) $\theta \wedge (B^2 \vee C^2) = \theta \wedge (B \vee C)^2$.

Proof. ($\Leftarrow$)
Let $B = B[\theta]$, and $C = C[\theta]$. If $\rho \in [B^2 \wedge \theta, B^2]$, $\phi \in [C^2 \wedge \theta, C^2]$, then $B^2 \wedge \theta \leq \rho \leq B^2$, $C^2 \wedge \theta \leq \phi \leq C^2$, and $(B \wedge C)^2 \wedge \theta \leq \rho \wedge \phi \leq B^2 \wedge C^2 = (B \wedge C)^2$. For the suprema, we have that $(B^2 \wedge \theta) \vee (C^2 \wedge \theta) \leq \rho \vee \phi$, and by (i), (ii). (iii) and since $B^2 \vee C^2 \leq (B \vee C)^2$, it follows that $(B \vee C)^2 \wedge \theta \leq \rho \vee \phi \leq (B \vee C)^2$, where $B \vee C = B \vee C[\theta]$.

($\Rightarrow$)
If $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$, then from $B^2 \wedge \theta \leq \rho \leq B^2$ and $C^2 \wedge \theta \leq \phi \leq C^2$ for $B = B[\theta]$ and $C = C[\theta]$, it follows that $D^2 \wedge \theta \leq \rho \vee \phi \leq D^2$ where $\mathcal{D} \in Sub\mathcal{A}$, $D = D[\theta]$. Obviously, $\rho \vee \phi \in Con(\mathcal{B} \vee \mathcal{C})$, and thus $\mathcal{D} = \mathcal{B} \vee \mathcal{C}$. Hence, $B \vee C = B \vee C[\theta]$. Now, since $\rho = B^2 \wedge \theta$, $\phi = C^2 \wedge \theta$, it follows that

$$(B \vee C)^2 \wedge \theta \leq (B^2 \wedge \theta) \vee (C^2 \wedge \theta).$$

Finally, since

$$(\theta \wedge B^2) \vee (\theta \wedge C^2) \leq \theta \wedge (B^2 \vee C^2) \leq \theta \wedge (B \vee C)^2,$$

the proof is complete.

□

It is obvious that the collection

$$\{B^2 \wedge \theta \mid \mathcal{B} \in Sub\mathcal{A},\ B = B[\theta]\}$$

represents, (up to the isomorphism) all the subalgebras of $\mathcal{A}/\theta$ ($\theta \in Con\mathcal{A}$). Hence, the previous proposition can be formulated in the following way.

Corollary 1. *For an algebra $\mathcal{A}$, and $\theta \in Con\mathcal{A}$, $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$ if and only if $Sub\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$.*

Proposition 3. *Let $\mathcal{A}$ be an algebra for which $B_m \neq \emptyset$, where*

$$B_m = \cap(B \mid \mathcal{B} \in Sub\mathcal{A}),$$

and with the property that for $\mathcal{B}, \mathcal{C} \in Sub\mathcal{A}$, if $B = B[\theta]$ and $C = C[\theta]$, then $B \vee C = B \vee C[\theta]$. Now, if θ is a codistributive element of $Cw\mathcal{A}$, then $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$.

Proof. Immediately by Proposition 2.

□

Proposition 4. *Let $\mathcal{A}$ be such that $B_m \neq \emptyset$, and that for $\mathcal{B}, \mathcal{C} \in Sub\mathcal{A}$, $B = B[\theta]$, $C = C[\theta]$, $B \vee C = B \vee C[\theta]$. If $\mathcal{A}$ satisfies the CEP and the wCIP, and if θ is a codistributive element of $Con\mathcal{A}$, then $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$.*

Proof. By Proposition 1. and by Theorem 1.

□

Proposition 4 gives some sufficient conditions under which $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$ (for $\theta \in Con\mathcal{A}$). However, these conditions are not necessary. In the following example ([3]), $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$, but the groupoid $\mathcal{A}$ does not satisfy any of the required conditions (namely, $\mathcal{A}$ has neither the CEP, nor the wCIP, and $B_m = \emptyset$, see Fig.2 and 3).

$$\mathcal{A} = (\{a, b, c, d\}, *)$$

*	a	b	c	d
a	c	c	c	c
b	c	c	c	d
c	c	c	b	d
d	d	d	d	d

$\theta : \{\{a\}, \{b, c\}, \{d\}\}$

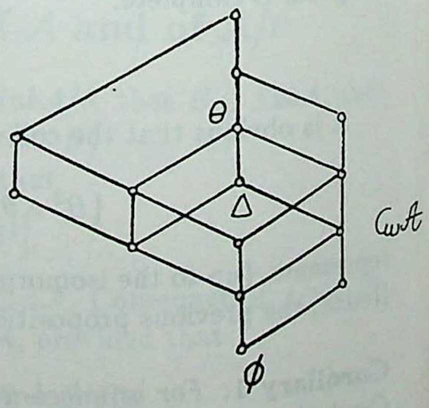


Fig.2

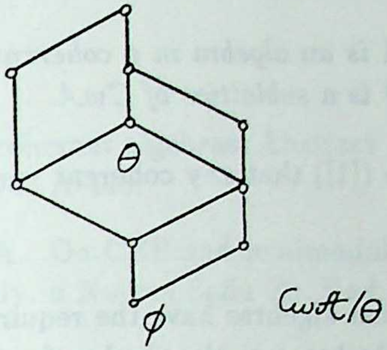


Fig.3

Which known classes of algebras have a property that $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$?

Recall that a **coherent** algebra is the one having the property that if any subalgebra of it contains a class of some congruence, then that subalgebra is a union of classes of that congruence. Obviously, for a coherent algebra $\mathcal{A}$, if $\mathcal{B}, \mathcal{C} \in Sub\mathcal{A}$ and $B = B[\theta]$, $C = C[\theta]$, then $B \vee C = B \vee C[\theta]$.

An algebra $\mathcal{A}$ is **regular** if every congruence on $\mathcal{A}$ is uniquely determined by any of its classes.

Theorem 2. *If $\mathcal{A}$ is a coherent algebra and if all its subalgebras are regular, then, for any $\theta \in Con\mathcal{A}$, $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$.*

Proof. Since $\mathcal{A}$ is coherent, all we have to prove is that for $\theta \in Con\mathcal{A}$, and for $\mathcal{B}, \mathcal{C} \in Sub\mathcal{A}$, $B = B[\theta]$, $C = C[\theta]$,

$$\theta \wedge (B^2 \vee C^2) = (\theta \wedge B^2) \vee (\theta \wedge C^2).$$

Let $\phi = \theta \wedge (B \vee C)^2$. Obviously, $\phi \in Con(\mathcal{B} \vee \mathcal{C})$, and ϕ is a union of the classes of θ. On the other hand, the congruence $\psi = (\theta \wedge B^2) \vee (\theta \wedge C^2)$ also belongs to $Con(\mathcal{B} \vee \mathcal{C})$, and contains the union of the classes from $\theta \wedge B^2$ and $\theta \wedge C^2$. The same classes are contained in ϕ. But $\psi \leq \phi$, and since $\mathcal{B} \vee \mathcal{C}$ is regular, $\psi = \phi$.

□

Corollary 2. *For every group G, and $N \triangleleft G$, $C\omega G/N$ is a sublattice of $C\omega G$.*

Proof. Obvious, since every group is coherent and regular.

□

Corollary 3. *If $\mathcal{A}$ is an algebra in a coherent variety, then for every $\theta \in Con\mathcal{A}$ $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$.*

Proof.. It is known ([1]) that any coherent variety is congruence regular.

□

Not only coherent algebras have the required property.

Let $\mathcal{A}$ be a $\cup$-algebra, i.e. the algebra for which the set-theoretic union of any two subalgebras is again a subalgebra of $\mathcal{A}$.

Theorem 3. *If $\mathcal{A}$ is a $\cup$-algebra, then for $\theta \in Con\mathcal{A}$, $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$.*

Proof. If $\theta \in Con\mathcal{A}$, $\mathcal{B}, \mathcal{C} \in Sub\mathcal{A}$, $B = B[\theta]$ and $C = C[\theta]$, then

$$B \vee C = B \cup C = B \vee C[\theta].$$

Moreover, if $[x]_\theta$ is a θ-congruence class to which $x \in A$ belongs, then

$$\cup([x]_\theta |\ x \in B \cup C) = \cup([x]_\theta |\ x \in B) \cup\ \cup([x]_\theta |\ x \in C)\text{i.e.}$$

$$\theta \cap (B \cup C)^2 = (\theta \cap B^2) \cup (\theta \cap C^2).$$

Now, $\theta \cap (B \cup C)^2 \in Con(B \cup C)$ and thus

$$(\theta \cap B^2) \cup (\theta \cap C^2) = (\theta \wedge B^2) \vee (\theta \wedge C^2)$$

in $Cw\mathcal{A}$. Hence,

$$(\theta \wedge (B \vee C)^2) = (\theta \wedge B^2) \vee (\theta \wedge C^2),$$

and the proof is complete by Theorem 1.

□

Corollary 4. *If $\mathcal{A}$ is a unary algebra, then for $\theta \in Con\mathcal{A}$, $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$.*

Proof. Obvious, since $\mathcal{A}$ is a $\cup$ -algebra.

□

References

[1] Geiger, D.: Congruence coherent algebras, Abstract 74T-A130, Notices Amer. Math. Soc. 21(1974) A-436.

[2] Šešelja, B., Tepavčević, A.: On CEP and semimodularity in the lattice of weak congruences, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. (to appear).

[3] Vojvodić, G., Šešelja, B.: On the lattice of weak congruence relations, Algebra Universalis, 25(1988), 121–130.

REZIME

SLABE KONGRUENCIJE I HOMOMORFIZMI

U radu se daju potrebni i dovoljni uslovi u odnosu na koje je mreža slabih kongruencija homomorfne slike neke algebre podmreža (do na izomorfizam) mreže slabih kongruencija same algebre. Opisane su neke klase algebri koje taj uslov zadovoljavaju (varijeteti koherentnih algebri, U-algebre itd.).

Received by the editors December 6, 1989.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 71-80

Review of Research
Faculty of Science
Mathematics Series

ON A CONSTRUCTION OF CODES BY P–FUZZY SETS

Branimir Šešelja and **Andreja Tepavčević**
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

P–fuzzy sets are considered as mappings from an arbitrary nonempty set S into a partially ordered set P. The necessary and sufficient conditions are given under which a family P of subsets of S represents a collection of level subsets for a fuzzy set $\overline{A} : S \to P$. Thus the conditions are obtained under which a binary block-code V can be ordered, so that it uniquely determines a P–fuzzy set and vice-versa. An explicit description of a Hamming distance for such codes is given, and it is shown that some well known binary block-codes (BCD, Gray's codes) can be represented by P–fuzzy sets.

AMS Subject Classification (1980): 03E72, 94D05
Key words and phrases: Decomposition of fuzzy sets, codes

1. P–fuzzy sets

Let S be an arbitrary set which is not empty, and $(P, \leq)$ a partially ordered set. Any function $\overline{A} : S \to P$ is a *P–fuzzy set on S*. Let also for $p \in P$, $\overline{A_p} : S \to \{0,1\}$, so that for $x \in S$, $\overline{A_p}(x) = 1$ iff $\overline{A}(x) \geq p$. Obviously, $\overline{A_p}$ is a characteristic function of a p–level subset (or, a p–cut)

$$A_p = \{x \in S \mid \overline{A_p}(x) = 1\}.$$

Let $\overline{A} : S \to P$ be a P-fuzzy set on S, and $\sim$ a binary relation on P, such that for $p, q \in P$

$$p \sim q \text{ iff } A_p = A_q.$$

$\sim$ is obviously an equivalence relation on P. Let

$$F = \overline{A}(S) = \{p \in P \mid p = \overline{A}(x), \text{ for some } x \in S\},$$

and for $p \in P$, let

$$[p) = \{q \in P \mid p \leq q\}.$$

Lemma 1. *If $\overline{A} : S \to P$ is a P-fuzzy set on S, then for $p, q \in P$*

$$p \sim q \text{ iff } [p) \cap F = [q) \cap F.$$

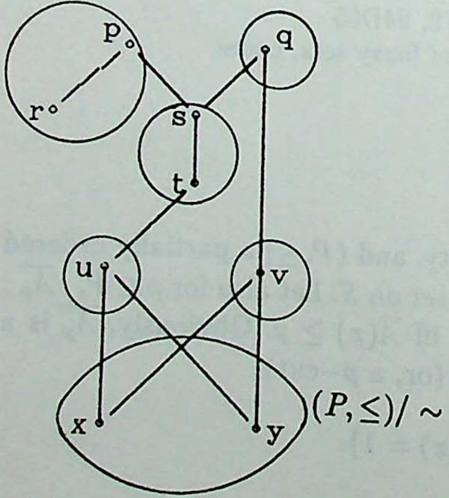
Proof.

$$\begin{aligned} p \sim q \ &\text{iff} \ A_p = A_q \ \text{iff (for } x \in S)(\overline{A}(x) \geq p \text{ iff } \overline{A}(x) \geq q) \\ &\text{iff} \ \{x \in S \mid \overline{A}(x) \in [p)\} = \{x \in S \mid \overline{A}(x) \in [q)\} \\ &\text{iff} \ [p) \cap F = [q) \cap F. \qquad \Box \end{aligned}$$

Example 1.

$$S = \{a, b, c, d, e\} \quad P = \{p, q, r, s, t, u, v, x, y\}$$

$$\overline{A} = \begin{pmatrix} a & b & c & d & e \\ s & u & v & p & q \end{pmatrix}$$



	a	b	c	d	e
	s	u	v	p	q
$\overline{A_p}$	0	0	0	1	0
$\overline{A_q}$	0	0	0	0	1
$\overline{A_r}$	0	0	0	1	0
$\overline{A_s}$	1	0	0	1	1
$\overline{A_t}$	1	0	0	1	1
$\overline{A_u}$	1	1	0	1	1
$\overline{A_v}$	0	0	1	0	1
$\overline{A_x}$	1	1	1	1	1
$\overline{A_y}$	1	1	1	1	1

$$A_p = A_r = \{d\};\ A_q = \{e\};\ A_s = A_t = \{a,d,e\};$$

$$A_u = \{a,b,d,e\};\ A_v = \{c,e\};\ A_x = A_y = \{a,b,c,d,e\}$$

Lemma 2. *Let $\overline{A} : S \to P$ be a fuzzy set. Now for every $x \in S$, if $\overline{A}(x) = p$, then p is a supremum of the class to which it belongs, i.e. $p = \bigvee[p]_\sim$.*

Proof. If $q \in [p]_\sim$, then $p = \overline{A}(x) \geq q$. Hence, $p = \bigvee[p]_\sim$. □

The following statement is a Theorem of decomposition for P–fuzzy sets.

Theorem 1. *If $\overline{A} : S \to P$ is a P–fuzzy set on S, then for $x \in S$,*

$$\overline{A}(x) = \bigvee(p \in P | \overline{A_p}(x) = 1)$$

(i.e. the supremum on the right exists in $(P, \leq)$ for every $x \in S$, and is equal to $\overline{A}(x)$).

Proof. Let $\overline{A}(x) = r \in P$. Then, $\overline{A_r}(x) = 1$. Now, if for any $p \in P$ $\overline{A_p}(x) = 1$, then $\overline{A}(x) \geq p$, i.e. $r \geq p$. On the other hand, $r \in \{p \in P | \overline{A_p}(x) = 1\}$, and thus r is the greatest element of that family. Thus,

$$\overline{A}(x) = r = \bigvee(p | \overline{A_p}(x) = 1). \quad \square$$

Let $\overline{A_P} = \{A_p | p \in P\}$, for $\overline{A} : S \to P$.This family of subsets of S has the following properties:

Proposition 1. *For a P–fuzzy set $\overline{A} : S \to P$,*

(1) *if $p, q \in P$ and $p \leq q$, then $A_q \subseteq A_p$;*

(2) *if for $P_1 \subseteq P$ there exists a supremum of P_1 $(\bigvee(p | p \in P_1))$, then $\bigcap(A_p | p \in P_1) = A_{\bigvee(p|p \in P_1)}$;*

(3) $\bigcup(A_p | p \in P) = S$;

(4) *for every $x \in S$, $\bigcap(A_p | x \in A_p) \in \overline{A_P}$.*

Proof.

(1) If $p \leq q$, then $\overline{A_q}(x) = 1$ implies $\overline{A_p}(x) = 1$, i.e. $A_q \subseteq A_p$;

(2) Suppose that for $P_1 \subseteq P$ the supremum $\bigvee(p | p \in P_1)$ exists in P. Then for $x \in S$,

$$\begin{aligned} x \in A_{\bigvee(p|p\in P_1)} \quad & \text{iff} \quad \overline{A}_{\bigvee(p|p\in P_1)}(x) = 1 \ \text{ iff } \ \overline{A}(x) \geq \bigvee(p|p \in P_1) \\ & \text{iff} \quad \overline{A}(x) \geq p \ \text{ for all } \ p \in P_1, \\ & \text{iff} \quad x \in \bigcap(A_p | p \in P_1)\,; \end{aligned}$$

(3) If $x \in S$, then $\overline{A}(x) = p \in P$ and $x \in A_p$. Thus, $x \in \bigcup(A_p | p \in P)$, i.e. $S \subseteq \bigcup(A_p | p \in P)$. Obviously, $\bigcup(A_p | p \in P) \subseteq S$, and the equality holds;

(4) Let $x \in S$. Then, $x \in A_p$ iff $\overline{A}(x) \geq p$, i.e. iff $\overline{A_p}(x) = 1$. By Theorem 1, $\overline{A}(x) = \bigvee(p | \overline{A_p}(x) = 1)$, and by (2)

$$A_{\bigvee(p|\overline{A_p}(x)=1)} = \bigcap(A_p | \overline{A_p}(x) = 1).$$

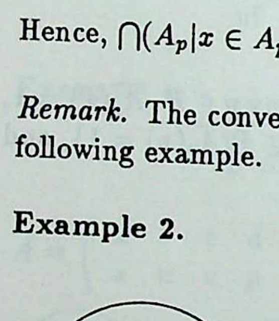
Hence, $\bigcap(A_p | x \in A_p) \in \overline{A_P}$. □

Remark. The converse of (2) in Proposition 1 is not true, as shown by the following example.

Example 2.

$$S = \{a, b\} \quad P = \{p, q, r, s\}$$

$$\overline{A} = \begin{pmatrix} a & b \\ r & s \end{pmatrix}$$



$(P, \leq)/\sim$

	a	b
	r	s
$\overline{A_p}$	0	0
$\overline{A_q}$	0	0
$\overline{A_r}$	1	0
$\overline{A_s}$	0	1

$A_p = A_q = \emptyset$
$A_r = \{a\}$
$A_s = \{b\}$

In this P-fuzzy set, $A_r \cap A_s \in \overline{A_P}$, but (2), (Proposition 1) is not true, since $r \vee s$ does not exist in P.

Theorem 2. *Let S be a nonempty set, and P a family of its subsets $(P \subseteq \mathcal{P}(S))$, such that:*

(1) $\bigcup P = S$;

(2) for every $x \in S, \quad \bigcap(p \in P | x \in p) \in P$.

Let $\overline{A} : S \to P$ be defined with

$$\overline{A}(x) = \bigcap(p \in P | x \in p).$$

Then, $\overline{A}$ is a P-fuzzy set, where $(P, \leq)$ is a partially ordered set under $p \leq q$ iff $q \subseteq p$ $(p, q \in P)$, and for every $p \in P$,

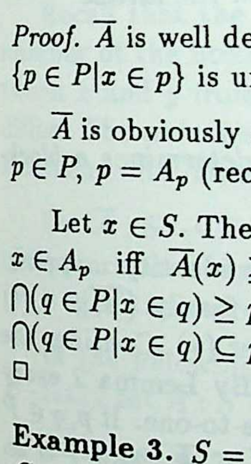
$$p = A_p.$$

Proof. $\overline{A}$ is well defined. Indeed, by (2), for every $x \in S$ the family $\{p \in P | x \in p\}$ is uniquely determined.

$\overline{A}$ is obviously a P-fuzzy set, and we have to prove that for every $p \in P$, $p = A_p$ (recall that $A_p = \{x \in S | \overline{A}(x) \geq p\}$).

Let $x \in S$. Then,
$x \in A_p$ iff $\overline{A}(x) \geq p$ iff (by the definition of $\overline{A}$)
$\bigcap(q \in P | x \in q) \geq p$ iff (by the definition of $\leq$)
$\bigcap(q \in P | x \in q) \subseteq p$ iff $x \in p$ (since by (1), the intersection is not empty).
□

Example 3. $S = \{a, b, c\} \quad P = \{\emptyset, \{a\}, \{b\}, \{b, c\}\}$
Conditions (1) and (2) are satisfied. Thus, we have the following P-fuzzy set:



$(P, \leq)$
$(p \leq q$ *iff* $q \subseteq p)$

$$\overline{A} = \begin{pmatrix} a & b & c \\ \{a\} & \{b\} & \{b,c\} \end{pmatrix}$$

	a	b	c	
$\overline{A_p}$	$\{a\}$	$\{b\}$	$\{b,c\}$	A_p
$\overline{A_\emptyset}$	0	0	0	$\emptyset$
$\overline{A_{\{a\}}}$	1	0	0	$\{a\}$
$\overline{A_{\{b\}}}$	0	1	0	$\{b\}$
$\overline{A_{\{b,c\}}}$	0	1	1	$\{b,c\}$

As shown in the table, for every $p \in P$, $A_p = p$.

2. Codes generated by P–fuzzy sets

Let $S = \{1, 2, \ldots n\}$ and let $(P, \leq)$ be a finite partially ordered set. Every P–fuzzy set on S *determines* a binary block-code V of length n, in the following way:

To every class $[p]_\sim$ $(p \in P)$, there corresponds a codeword

$v_{[p]} = x_1 x_2 \ldots x_n$, such that $x_i = j$ iff $\overline{A_p}(i) = j$, for $i \in S$, and $j \in \{0,1\}$.

We shall use the following componentwise defined order on the set of codewords belonging to a binary block-code V: for $x, y \in V$, $x = x_1 \ldots x_n$, $y = y_1 \ldots y_n$,

$$(*) \qquad x \leq y \quad \text{iff} \quad y_1 \leq x_1, \ldots, y_n \leq x_n,$$

where $\leq$ on the right is the ordinary ordering relation on the lattice $(\{0,1\}, \leq) : 0 < 1$.

Thus, for example, $101101 \leq 001001$.

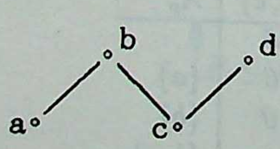
Theorem 3. *Every finite partially ordered set $(P, \leq)$ determines a block-code V, such that $(P, \leq)$ is isomorphic with $(V, \leq)$.*

Proof. Let $P = \{p_1, \ldots p_n\}$, and let $\overline{A} : P \to P$ be the identity mapping, as a P–fuzzy set on P. The decomposition of $\overline{A}$ gives a family $\{\overline{A_p} | p \in P\}$ which is the required code, under the above defined order $(*)$. Consider the mapping $f : P \to \{\overline{A_p} | p \in P\}$, such that $f(p) = \overline{A_p}$. By Lemma 2 every $(\sim)$-class contains exactly one element, and thus f is one-to-one. If $p, q \in P$ and $p \leq q$, then $A_q \subseteq A_p$, which by $(*)$ means that $\overline{A_p} \leq \overline{A_q}$, and f is an isomorphism. □

Example 4.

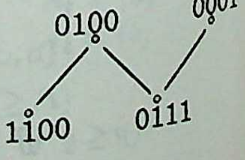
$$P = \{a, b, c, d\}$$

$$\overline{A} = \begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix} \qquad V = \{1100, 0100, 0111, 0001\}$$



$(P, \leq)$

$\overline{A_x}$	a	b	c	d
$\overline{A_a}$	1	1	0	0
$\overline{A_b}$	0	1	0	0
$\overline{A_c}$	0	1	1	1
$\overline{A_d}$	0	0	0	1



$(V, \leq)$

If V is a binary block-code, and $\overline{A}$ is a $P-$fuzzy set, then we say that $\overline{A}$ *corresponds* to V if the block-code determined by $\overline{A}$ (as defined at the beginning of the paragraph) is V.

Theorem 4. *Let* $V = \{v_1, ...v_k\} \subseteq \{0,1\}^n$ *be a binary block-code, such that for every* $i \in \{1, \dots n\}$ *at least one codeword has a nonzero* $i-$*th coordinate. Then there is a* $P-$*fuzzy set which corresponds to* V *if and only if for every* $i \in \{1, \dots n\}$

$$(a) \qquad \bigvee (v \in V | v(i) = 1) \in V.$$

(The supremum in (a) is induced by $\leq$ in $(*)$.)
Proof. If we think of the codewords of V as of the characteristic functions of the subsets of $\{1, ...n\}$, then the "if" part follows by Theorem 2, since the supremum in (a) is in fact the intersection of the corresponding subsets.

The converse is true by Proposition 1. □

Recall that the *Hamming weight* $||x||$ of a codeword $x \in \{0,1\}^n$ is the number of the nonzero coordinates in x, the *Hamming distance* $d(x,y)$ between x and y from $\{0,1\}^n$ is the number of coordinates in which x and y differ. The *code distance* of $V \subseteq \{0,1\}^n$ is the minimum Hamming distance between two different codewords in V, and is denoted by $d(V)$.

Let $\overline{A} : S \to P$ be a $P-$fuzzy set. We say that the number of elements of S which are mapped into the same element p of P is a *degree* of the class $[p]_\sim$, or of the corresponding codeword $v_{[p]}$, and we denote it by $s(v_{[p]})$.

In the following four propositions, let V be a code corresponding to a $P-$fuzzy set $\overline{A} : S \to P$.

As it was done in [3] for lattice valued fuzzy sets, we shall describe the above-mentioned parameters in terms of $P-$fuzzy sets.

Proposition 2.

$$d(V) \geq min_{p \in \overline{A}(S)} s(v_{[p]}).$$

(Recall that $\overline{A}(S) = \{p \in P | p = \overline{A}(x), \quad \text{for some } x \in S\}$.)
Proof. If two codewords from V differ in the coordinate mapped onto p, they differ in at least $s(v_{[p]})$ coordinates. □

Proposition 3. *If* $v_{[p]} \in V$, *then*

$$||v_{[p]}|| = \sum (s(v_{[q]}) | q \in \overline{A}(S), \textit{ and } v_{[p]} \leq v_{[q]}).$$

Proof. If $i \in S$, and $v_{[p]} \in V$, then $v_{[p]}(i) = 1$ if $q = \overline{A}(i) \geq r$, for every $r \in [p]_\sim$. Every $q \in \overline{A}(S)$ represents one class $[q]_\sim$ (by Lemma 2), and the number of these classes coincides with $||v_{[p]}||$. □

Proposition 4. *If* $v_{[p]} \leq v_{[q]}$, *then*

$$d(v_{[p]}, v_{[q]}) = \sum(s(v_{[r]}) | v_{[r]} \in K),$$

$$\textit{where} \quad K = \{v_{[r]} \in V | v_{[r]} \geq v_{[p]}, \textit{ and } \rceil(v_{[r]} \geq v_{[q]})\}.$$

Proof. If $v_{[r]} \geq v_{[p]}$, and $\rceil(v_{[r]} \geq v_{[q]})$, $r \in \overline{A}(S)$, then for every $i \in S$ such that $\overline{A}(i) = r$, $v_{[p]}(i) = 1$, and $v_{[q]}(i) = 0$. Moreover, every nonzero coordinate in $v_{[q]}$ is nonzero in $v_{[p]}$ as well. □

Theorem 5. *For any* $v_{[p]}, v_{[q]} \in V$,

$$(b) \qquad d(v_{[p]}, v_{[q]}) = \sum(s(v_{[r]}) | v_{[r]} \in K)$$

$$\textit{where} \quad \begin{aligned} K = \{v_{[r]} \in V | v_{[r]} \geq v_{[p]} \quad &\textit{and} \quad \rceil(v_{[r]} \geq v_{[q]}), \\ \textit{or } v_{[r]} \geq v_{[q]} \quad &\textit{and} \quad \rceil(v_{[r]} \geq v_{[p]})\}. \end{aligned}$$

Proof. If $v_{[p]} \leq v_{[q]}$, then the proof follows by Proposition 4. Now,let $v_{[p]}$ and $v_{[q]}$ be uncomparable. If $v_{[r]} \geq v_{[p]}$, and $\rceil(v_{[r]} \geq v_{[q]})$, then for every $i \in S$ such that $\overline{A}(i) = r$, $v_{[p]}(i) = 1$ and $v_{[q]}(i) = 0$. On the other hand, if $v_{[r]} \geq v_{[q]}$, $\rceil(v_{[r]} \geq v_{[p]})$ and $\overline{A}(i) = r$, then $v_{[q]}(i) = 1$, and $v_{[p]}(i) = 0$. Hence,

$$d(v_{[p]}, v_{[q]}) \geq \sum(s(v_{[r]}) | v_{[r]} \in K).$$

Now, if $v_{[p]}$ and $v_{[q]}$ differ in i−th coordinate, for example $v_{[q]}(i) = 1$ and $v_{[p]}(i) = 0$, then for $\overline{A}(i) = r$, $v_{[r]} \geq v_{[q]}$ and $\rceil(v_{[r]} \geq v_{[p]})$. Thus, $v_{[r]} \in K$, and the equality (b) holds. □

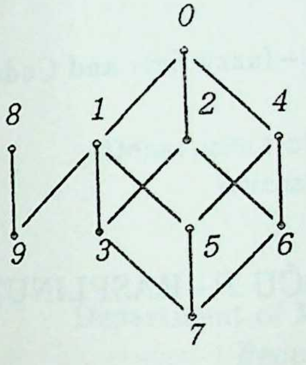
* * *

Some well known codes can be represented by $P-$fuzzy sets, i.e. they satisfy the conditions of Theorem 4.

Example 5. *a) For a BCD-code $V = \{0000, 0001, ...1001\}$, there is a corresponding $P-$fuzzy set, as shown in the sequel.*

$$\overline{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 8 & 4 & 2 & 1 \end{pmatrix},$$

$$S = \{1,2,3,4\},\ P = \{1,2,...9\}$$



$(P, \leq)$

	1	2	3	4
p	8	4	2	1
0	0	0	0	0
1	0	0	0	1
2	0	0	1	0
3	0	0	1	1
4	0	1	0	0
5	0	1	0	1
6	0	1	1	0
7	0	1	1	1
8	1	0	0	0
9	1	0	0	1

The construction is based on Theorem 4.

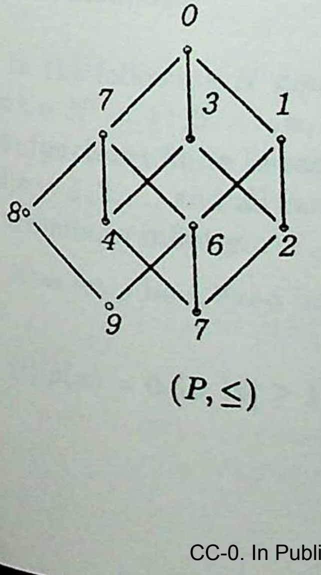
b) The following Gray's code can be represented by a $P-$fuzzy set in a similar way.

$V = \{0000, 0001, 0011, 0010, 0110, 0111, 0101, 0100, 1100, 1101\}$

$S = \{1,2,3,4\}$

$P = \{0,1,...9\}$

$$\overline{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 3 & 1 \end{pmatrix}$$



$(P, \leq)$

	1	2	3	4
p	8	7	3	1
0	0	0	0	0
1	0	0	0	1
2	0	0	1	1
3	0	0	1	0
4	0	1	1	0
5	0	1	1	1
6	0	1	0	1
7	0	1	0	0
8	1	1	0	0
9	1	1	0	1

References

[1] G. Birkhoff, T.C. Bartee, Modern Applied Algebra, McGrawHill, New York, 1970.

[2] B. Šešelja, G. Vojvodić, Fuzzy sets on S as closure operations on $P(S)$, Review of Research, Fac. of Sci. Univ. of Novi Sad, Vol. 14, 1 (1984), 117-127.

[3] B. Šešelja, A. Tepavčević, G. Vojvodić, $L-$fuzzy Sets and Codes Fuzzy Sets and Systems (to appear).

REZIME

O KONSTRUKCIJI KODOVA POMOĆU $P-$RASPLINUTIH SKUPOVA

Posmatraju se $P-$rasplinuti skupovi, kao funkcije iz proizvoljnog nepraznog skupa S u parcijalno uredjeni skup P. Daju se potrebni i dovoljni uslovi pod kojima familija podskupova skupa S predstavlja kolekciju na koju se razlaže dati rasplinuti skup na S. Time se dolazi do uslova pod kojima se binarni blok-kod može opisati pomoću $P-$rasplinutog skupa. Eksplicitno se opisuje norma, Hemingovo i uopšte kodno rastojanje, a daju se i primeri poznatih kodova (BCD, kodovi Greja) izraženih pomoću $P-$rasplinutih skupova.

Received by the editors May 25, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 81-89

Review of Research
Faculty of Science
Mathematics Series

THE PRODUCT OF THE DISTRIBUTIONS x_+^λ AND $x_-^{-\lambda-p}$ DEFINED BY A DISTRIBUTION VECTOR

Brian Fisher
Department of Mathematics, The University,
Leicester, LE1 7RH, England

Li Zhishen
Department of Mathematics, Lanzhou University,
People's Republic of China

Abstract

This paper proves the existence of the neutrix product $x_+^\lambda \circ x_-^{-\lambda-p}$ of the distributions x_+^λ and $x_-^{-\lambda-p}$ in terms of the distribution vector which was defined in [3].

AMS Mathematics Subject Classification (1980): 46F10.
Key words and phrases: Neutrix, neutrix limit, Dirac delta-function, distribution.

In the following N denotes the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions $n^\lambda \ln^{r-1} n$, $\ln^r x$ for $\lambda > 0$ and $r = 1, 2, \ldots$ and all functions which converge to zero in the normal sense as n tends to infinity.

Now let ρ be a fixed infinitely differentiable function having the properties

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

The function δ_n is defined by $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$. It is obvious that the sequence $\{\delta_n\}$ is regular and converges to the Dirac delta-function δ.

The following definition for the product of two distributions was given in [2].

Definition 1. *Let f and g be distributions and let $g_n = g * \delta_n$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to h on the interval (a,b) if*

$$\underset{n\to\infty}{\mathrm{N-lim}} \langle f g_n, \phi \rangle = \langle h, \phi \rangle$$

for all test functions ϕ with compact support contained in the interval (a,b).

In order to give more information about the behaviour of the product $f \circ g$ the notion of a distribution vector was introduced in [3], where the following definitions, theorems and lemma were given.

Definition 2. *Let h_r be a distribution for $r = 0, 1, 2, \ldots$. We say that*

$$\mathbf{h} = [h_0, h_1, \ldots, h_r, \ldots]$$

is a distribution vector.

If α is any real number we define $\alpha\mathbf{h}$ to be the distribution vector

$$[\alpha h_0, \alpha h_1, \ldots, \alpha h_r, \ldots]$$

and if $\mathbf{k} = [k_0, k_1, \ldots, k_r, \ldots]$ is a second distribution vector we define $\mathbf{h} + \mathbf{k}$ to be the distribution vector

$$[h_0 + k_0, h_1 + k_1, \ldots, h_r + k_r, \ldots].$$

If $h_{r+i} = 0$ for $i = 1, 2, \ldots$ we write

$$\mathbf{h} = [h_0, h_1, \ldots, h_r, 0, 0, \ldots] = [h_0, h_1, \ldots, h_r]$$

and if $h_i = 0$ for $i = 1, 2, \ldots$ we write

$$\mathbf{h} = [h_0] = h_0.$$

Definition 3. *Let* $\mathbf{h} = [h_0, h_1, \ldots, h_r, \ldots]$ *be a distribution vector and let* ϕ *be an arbitrary test function with compact support. We define* $\langle \mathbf{h}, \phi \rangle$ *by the sequence of real numbers*

$$\langle \mathbf{h}, \phi \rangle = (\langle h_0, \phi \rangle, \langle h_1, \phi \rangle, \ldots, \langle h_r, \phi \rangle, \ldots).$$

Definition 4. *Let* $\mathbf{h} = [h_0, h_1, \ldots, h_r, \ldots]$ *be a distribution vector. We define the derivative* $\mathbf{h}'$ *of* $\mathbf{h}$ *by*

$$\mathbf{h}' = [h_0', h_1', \ldots, h_r', \ldots].$$

Definition 5. *Let* f *and* g *be distributions and let* $g_n = g * \delta_n$*. We say that the neutrix product* $f \circ g$ *of* f *and* g *exists and is equal to the distribution vector* $\mathbf{h} = [h_0, h_1, \ldots, h_r, \ldots]$ *on the interval* (a, b) *if*

$$\mathop{\mathrm{N-lim}}_{n\to\infty} n^{-r} \langle f g_n, \phi \rangle = \langle h_r, \phi \rangle$$

for $r = 0, 1, 2, \ldots$ *and all test functions* ϕ *with compact support contained in the interval* (a, b).

Definition 6. *Let* f *and* g *be distributions and suppose that the neutrix product* $f \circ g$ *exists as the distribution vector* $\mathbf{h} = [h_0, h_1, \ldots, h_r, \ldots]$ *on the interval* (a, b)*. We say that* h_0 *is the finite part of* $f \circ g$ *on the interval* (a, b)*. If* $h_r \neq 0$ *for some* $r \geq 1$ *we write*

$$\text{p.f. } f \circ g = h_0$$

on the interval (a, b) *and if* $h_r = 0$ *for* $r = 1, 2, \ldots$ *we write*

$$f \circ g = h_0$$

on the interval (a, b).

Theorem 1. *Let* $\mathbf{h} = [h_0, h_1, \ldots, h_r, \ldots]$ *be a distribution vector and let* ϕ *be an arbitrary test function with compact support. Then*

$$\langle \mathbf{h}', \phi \rangle = -\langle \mathbf{h}, \phi' \rangle.$$

Theorem 2. *Let f and g be distributions and suppose that the neutrix products $f \circ g$ and $f' \circ g$ (or $f \circ g'$) exist as distribution vectors on the interval (a,b). Then the neutrix product $f \circ g'$ (or $f' \circ g$) exists as a distribution vector and*

$$(f \circ g)' = f' \circ g + f \circ g'$$

on the interval (a,b).

Lemma 1.

$$\int_0^{1/n} x^p \delta_n^{(q)}(x)\, dx = (-1)^{p+1} n^{q-p} p! \rho_{q-p-1}$$

for $p = 0,1,2,\ldots,q-1$ and $q = 1,2,\ldots$,

$$\int_0^{1/n} x^p \delta_n^{(p)}(x) dx = \frac{1}{2}(-1)^p p!$$

for $p = 0,1,2,\ldots$ and

$$\int_0^{1/n} \left| x^p \delta_n^{(q)}(x) \right| dx = o(n^{q-p})$$

for $q = 0,1,\ldots,p-1$ and $p = 1,2,\ldots$, where

$$\rho_{2r} = \rho^{(2r)}(0), \quad \rho_{2r+1} = 0$$

for $r = 0,1,2,\ldots$.

We now prove the following theorem.

Theorem 3. *The neutrix product $x_+^\lambda \circ x_-^{-\lambda-q}$ exists as a distribution vector and*

$$(1) \qquad x_+^\lambda \circ x_-^{-\lambda-q} = \mathbf{h}(\lambda,q) = [h_0(\lambda,q), h_1(\lambda,q), \ldots, h_{q-1}(\lambda,q)]$$

for $q = 1,2,\ldots$ and $\lambda \neq 0, \pm1, \pm2, \ldots$, where

$$h_i(\lambda,q) = \begin{cases} -\frac{\pi \operatorname{cosec}(\pi\lambda)}{2(q-1)!}\delta^{(q-1)}, & i = 0, \\ \frac{\Gamma(\lambda+q-i)\pi \operatorname{cosec}(\pi\lambda)}{\Gamma(\lambda+q)(q-i-1)!}\rho_{i-1}\delta^{(q-i-1)}, & 1 \leq i < q. \end{cases}$$

In particular

$$x_+^\lambda \circ x_-^{-\lambda-1} = -\frac{1}{2}\pi \operatorname{cosec}(\pi\lambda)\delta$$

for $\lambda \neq 0, \pm1, \pm2, \ldots$.

Proof. The distribution x_+ is defined as a locally summable function for $\lambda > -1$ by

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0, \\ 0, & x < 0 \end{cases}$$

and is defined for $\lambda < -1$ and $\lambda \neq -2, -3, \ldots$ inductively by

$$x_+^\lambda = (\lambda+1)^{-1}(x_+^{\lambda+1})'.$$

The distribution x_-^λ is defined by

$$x_-^\lambda = (-x)_+^\lambda$$

for $\lambda \neq -1, -2, \ldots$.

We will first of all suppose that $p - 1 < \lambda < p$, for some non-negative integer p. Then

$$x_-^{-\lambda-q} = \frac{\Gamma(\lambda-p+1)}{\Gamma(\lambda+q)} \frac{d^{p+q-1}}{dx^{p+q-1}} x_-^{-\lambda+p-1}$$

and so

$$\begin{aligned} (x_-^{-\lambda-q})_n &= x_-^{-\lambda-q} * \delta_n(x) \\ &= \frac{\Gamma(\lambda-p+1)}{\Gamma(\lambda+q)} \int_x^{1/n} (t-x)^{-\lambda+p-1} \delta_n^{(p+q-1)}(t)\,dt \end{aligned}$$

for $-1/n < x < 1/n$. Thus

$$\frac{\Gamma(\lambda+q)}{\Gamma(\lambda-p+1)} \int_{-\infty}^{\infty} x_+^\lambda (x_-^{-\lambda-q})_n x^j\,dx =$$

$$= \int_0^{1/n} x^{\lambda+j} \int_x^{1/n} (t-x)^{-\lambda+p-1} \delta_n^{(p+q-1)}(t)\,dt\,dx =$$

$$= \int_0^{1/n} \delta_n^{(p+q-1)}(t) \int_0^t x^{\lambda+j}(t-x)^{-\lambda+p-1}\,dx\,dt =$$

$$= \int_0^{1/n} t^{p+j} \delta_n^{(p+q-1)}(t) \int_0^1 v^{\lambda+j}(1-v)^{-\lambda+p-1}\,dv\,dt,$$

where the substitution $x = tv$ has been made. Since

$$\int_0^1 v^{\lambda+j}(1-v)^{-\lambda+p-1}\,dv = \frac{\Gamma(\lambda+j+1)\Gamma(-\lambda+p)}{(p+q)!},$$

it follows on using the lemma that

$$\int_{-\infty}^{\infty} x_+(x_-^{-\lambda-q})_n x^j \, dx = \tag{2}$$

$$= \begin{cases} \frac{(-1)^j\Gamma(\lambda+j+1)\pi\operatorname{cosec}(\pi\lambda)}{\Gamma(\lambda+q)}\rho_{q-j-2}n^{q-j-1}, & 0 \le j \le q-2 \\ \frac{1}{2}(-1)^q\pi\operatorname{cosec}(\pi\lambda), & j = q-1 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \left| x_+^\lambda(x_-^{-\lambda-q})_n x^q \right| dx = o(n^{-1}). \tag{3}$$

Now let ϕ be an arbitrary test function with compact support. Then

$$\phi(x) = \sum_{j=0}^{q-1} \frac{x^j}{j!}\phi^{(j)}(0) + \frac{x^q}{q!}\phi^{(q)}(\xi x),$$

where $0 < \xi < 1$. It follows that

$$\begin{aligned} \langle x_+^\lambda, (x_-^{-\lambda-q})_n\phi\rangle &= \sum_{j=0}^{q-1}\frac{\phi^{(j)}(0)}{j!}\int_{-\infty}^{\infty} x_+^\lambda(x_-^{-\lambda-q})_n x^j\,dx + \\ &\quad + \frac{1}{q!}\int_{-\infty}^{\infty} x_+^\lambda(x_-^{-\lambda-q})_n x^q\phi^{(q)}(\xi x)\,dx = \\ &= \sum_{j=0}^{q-2}\frac{\Gamma(\lambda+j+1)\pi\operatorname{cosec}(\pi\lambda)}{\Gamma(\lambda+q)j!}\rho_{q-j-2}n^{q-j-1}\langle\delta^{(j)},\phi\rangle + \\ &\quad + \frac{\pi\operatorname{cosec}(\pi\lambda)}{2(q-1)!}\langle\delta^{(q-1)},\phi\rangle + o(n^{-1}) \end{aligned}$$

on using equations (2) and (3). Thus

$$\mathrm{N}\!-\!\lim_{n\to\infty} n^{-i}\langle x_+^\lambda, (x_-^{-\lambda-q})_n\phi\rangle =$$

$$= \begin{cases} -\frac{\pi\operatorname{cosec}(\pi\lambda)}{2(q-1)!}\langle\delta^{(q-1)},\phi\rangle, & i = 0 \\ \frac{\Gamma(\lambda+q-i)\pi\operatorname{cosec}(\pi\lambda)}{\Gamma(\lambda+q)(q-i-1)!}\rho_{i-1}\delta^{(q-i-1)}, & 1 \le i \le q-1 \end{cases}$$

and equation (1) follows for $p-1 < \lambda < p$ and $p = 0, 1, 2, \dots$.

Now assume that equation (1) holds for $p-1 < \lambda < p$ and $q = 1, 2, \dots,$ where p is now some negative integer. Then the neutrix products $x_+^\lambda \circ x_-^{-\lambda-q}$

and $x_+^\lambda \circ x_-^{-\lambda-q-1}$ exist and so by Theorem 2 the neutrix product $x_+^{\lambda-1} \circ x_-^{-\lambda-q}$ exists and

$$\begin{aligned}\lambda x_+^{\lambda-1} \circ x_-^{-\lambda-q} &= (x_+^\lambda \circ x_-^{-\lambda-q})' - (\lambda+q)x_+ \circ x_-^{-\lambda-q-1} \\ &= \mathbf{h}'(\lambda,q) - (\lambda+q)\mathbf{h}(\lambda,q+1) \\ &= [k_0(\lambda,q), k_1(\lambda,q), \dots, h_q(\lambda,q)],\end{aligned}$$

where

$$\begin{aligned} k_0(\lambda,q) &= h_0'(\lambda,q) - (\lambda+q)h_0(\lambda,q+1) \\ &= \frac{\pi\lambda\mathrm{cosec}(\pi\lambda)}{2q!}\delta^{(q)} \\ &= \lambda h_0(\lambda-1,q+1), \\ k_i(\lambda,q) &= h_i'(\lambda,q) - (\lambda+q)h_i(\lambda,q+1) \\ &= -\frac{\lambda\Gamma(\lambda+q-i)\pi\mathrm{cosec}(\pi\lambda)}{\Gamma(\lambda+q)(q-i)!}\rho_{i-1}\delta^{(q-i)} \\ &= \lambda h_i(\lambda-1,q+1),\end{aligned}$$

for $i = 1,2,\dots,q-1$ and

$$\begin{aligned} k_q(\lambda,q) &= -\frac{\lambda\Gamma(\lambda)\pi\mathrm{cosec}(\pi\lambda)}{\Gamma(\lambda+q)}\rho_q\delta \\ &= \lambda h_q(\lambda-1,q+1).\end{aligned}$$

Equation (1) follows for $p-2 < \lambda < p-1$ and $q = 1,2,\dots$. Equation (1) now holds by induction for $p-1 < \lambda < p$, $q = 1,2,\dots$ and $p = -1,-2,\dots$. This completes the proof of the theorem. □

Corollary 1. *The neutrix product $x_-^\lambda \circ x_+^{-\lambda-q}$ exists as a distribution vector and*

$$x_-^\lambda \circ x_+^{-\lambda-q} = (-1)^{q-1}\mathbf{h}(\lambda,q)$$

for $q = 1,2,\dots$ and $\lambda \neq 0,\pm1,\pm2,\dots$.

In particular

$$x_-^\lambda \circ x_+^{-\lambda-1} = -\frac{1}{2}\pi\mathrm{cosec}(\pi\lambda)\delta$$

for $\lambda \neq 0,\pm1,\pm2,\dots$.

Proof. The results follow from the theorem on noting that $\rho_i = 0$ for odd i and

$$(-x)_+^\lambda = x_-^\lambda, \quad (-x)_-^{-\lambda-q} = x_+^{-\lambda-q},$$
$$\delta^{(i)}(-x) = (-1)^i \delta^{(i)}(x). \quad \square$$

Corollary 2.

$$\text{p.f. } x_+^\lambda \circ x_-^{-\lambda-q} = -\frac{\pi \operatorname{cosec}(\pi\lambda)}{2(q-1)!}\delta^{(q-1)},$$

$$\text{p.f. } x_-^\lambda \circ x_+^{-\lambda-q} = \frac{(-1)^q \pi \operatorname{cosec}(\pi\lambda)}{2(q-1)!}\delta^{(q-1)}$$

for $q = 1, 2, \ldots$ *and* $\lambda \neq 0, \pm 1, \pm 2, \ldots$.

The results of this corollary are immediate.

Corollary 3.

$$(\lambda+1)x_+^\lambda \circ x_-^{-\lambda-2} - (-1)^q(\lambda+q+1)x_+^{\lambda+q} \circ x_-^{-\lambda-q-2} = \tfrac{1}{2} q\pi \operatorname{cosec}(\pi\lambda)\delta'$$

for $q = \pm 1, \pm 2, \ldots$ *and* $\lambda \neq 0, \pm 1, \pm 2, \ldots$.

Proof. The result follows on noting that

$$x_+^\lambda \circ x_-^{-\lambda-2} = \frac{\pi \operatorname{cosec}(\pi\lambda)}{\lambda+1}[-\frac{1}{2}(\lambda+1)\delta', \rho_0\delta],$$

$$x_+^{\lambda+q} \circ x_-^{-\lambda-q-2} = \frac{(-1)^q \pi \operatorname{cosec}(\pi\lambda)}{\lambda+q+1}[-\frac{1}{2}(\lambda+q+1)\delta', \rho_0\delta]. \ \square$$

Corollary 4.

$$\pi x_+^{q-1} \circ \delta^{(q)} - (\lambda+q+1)(q-1)! \sin(\pi\lambda) x_+^{\lambda+q} \circ x_-^{-\lambda-q-2} = \frac{1}{2}(-1)^q(\lambda+1)(q-1)!\delta'$$

for $q = 1, 2, \ldots$ *and* $\lambda \neq 0, \pm 1, \pm 2, \ldots$.

Proof. It was proved in [3] that

$$x_+^{q-1} \circ \delta^{(q)} = (-1)^q (q-1)![-\frac{1}{2}q\delta', \rho_0\delta]$$

and the result follows on noting that

$$(\lambda+q+1)(q-1)! \sin(\pi\lambda) x_+^{\lambda+q} \circ x_-^{-\lambda-q-2} = (-1)^q \pi (q-1)![\frac{1}{2}(\lambda+q+1)\delta', \rho_0\delta].$$

References

[1] van der Corput, J.G.: Introduction to the neutrix calculus, J. Analyse Math., 7(1959-60), 291-398.

[2] Fisher, B.: A non-commutative neutrix product of distributions, Math. Nachr., 108 (1982), 117-127.

[3] Fisher, B.: Products of distributions defined by a vector, Proc. of International Conference on Complex Analysis and Applications, (Varna, 1985), 212-218.

REZIME

PROIZVOD DISTRIBUCIJA x_+^λ I $x_-^{-\lambda-p}$ DEFINISAN PREKO DISTRIBUCIONOG VEKTORA

Koristeći pojam distribucionog vektora iz [3], ispituje se proizvod distribucija x_+^λ i $x_-^{-\lambda-p}$, gde je λ realan broj koji nije ceo, a p prirodan broj.

Received by the editors November 21, 1988.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 91-94

Review of Research
Faculty of Science
Mathematics Series

ARBITRARINESS OF MULTIPLICITY FOR A LINEAR NON – ANTICIPATIVE TRANSFORMATION OF THE WIENER PROCESS

Zoran A. Ivković
Faculty of Mathematics, University of Belgrade
Studentski trg 16, 11000 Beograd, Yugoslavia

Abstract

In the paper the example is given of the process $\{X(t), t > 0\}$ defined by $X(t) = \int_0^t f(t;u)dW(u)(\{W(t)\}$ is Winer process), having an arbitrary finite multiplicity N and a maximal spectral type which is absolutely continuous.

AMS Mathemetics Subject Classification (1980): 60G12
Key words and phrases: Spectral type and multiplicity of second order process.

The notions of the spectral multiplicity theory in the time-domain analysis of continuous second-order processes used in this paper are form the classical papers [1] and [3]. Also, we shall refer to [6].

Let the process $\{X(t), t > 0\}$ be a linear non-anticipative transformation of a standard Wiener process (considered as a wide-sense martingale) $\{W(t), t > 0\}$ i.e.

(1) $$X(t) = \int_0^t f(t;u)dW(u), f(t;\cdot) \in \mathcal{L}_2(du).$$

It is stated in [5] that for the arbitrary chain

(2) $$F_0(t) \succeq F_1(t) \succeq \ldots \succeq F_{N-1}(t) \;(N \text{may be } \infty)$$

of distribution functions ordered by absolute continuity, there exists the process $\{X(t), t > 0\}$ for which (2) is its spectral type for $t \geq \varepsilon$, where $\varepsilon > 0$ is arbitrary and fixed. We cannot see a way to remove the restriction $t \geq \varepsilon$. Paper [2] (according to [7]) contains the example of the process $\{X(t), t > 0\}$ with the multiplicity $N > 1$, but the spectral type of $\{X(t)\}$ involves discontinuity points.

In this paper we shall give one construction of the process $\{X(t)\}$ with, arbitrary finite multiplicity N and all of whose functions $F_k(t), k = \overline{0, N-1}$ in (2) are absolutely continuous.

Let a partition of $(0, \infty)$ on disjoint sets $S_0, S_1, \ldots, S_{N-1}$ be defined by $S_k = \bigcup_{i=-\infty}^{\infty}(2^{k+Ni}, 2^{k+1+Ni}]$ and let $I_k(u), u > 0$, be the indicator function of S_k. Let $I(2^{-k}t; u)$, $u > 0$, be the indicator function of $(0, 2^{-k}t]$. We define the processes $\{Z_k(t),\ t > 0\}$, $k = \overline{0, N-1}$ by

$$Z_k(t) = \int_0^t I(2^{-k}t; u) I_k(u) dW(u). \tag{3}$$

Proposition 1. *The processe $\{Z_k(t)\}, k = \overline{0, N-1}$ are mutually orthogonal wide-sense martingales. Continuous functions $F_k(t) = \|Z_k(t)\|^2 = EZ_k^2(t), t > 0$, (linearly) increasie on S_0 only.*

Proof. Let $0 < s \leq t$, then $< Z_k(s), Z_k(t) >= EZ_k(s)Z_k(t) =$ $= \int_0^s I(2^{-k}s; u) I(2^{-k}(t); u) I_k^2(u) du = \int_0^s I(2^{-k}s; u) I_k(u) du$ is a function of s only. This means that $\{Z_k(t)\}$ is that wide-sense martingale. Also for $k \neq l$ and arbitrary $s, t > 0$ we have $\langle Z_k(s), Z_l(t)\rangle = \int_0^s I(2^{-k}s; u) I(2^{-k}t; u) I_k(u) I_l(u) du =$ 0 because $I_k(u) I_l(u) = 0$ for $k \neq l$. Finally, from the fact that $t \in S_0$ is equivalent to $2^{-k}t \in S_k, k = \overline{1, N-1}$, we conclude that if $t \in S_0$, then $I(2^{-k}t; u) I_k(u) = 1$ for $u \in (0, 2^{-k}t] \cap S_k$. Hence, $F_k(t)$ linearly increase for $t \in S_0$. If $t \notin S_0$, then $2^{-k}t \notin S_k$ and $I(2^{-k}t; u) I_k(u) = 0$ for $\underline{\text{all } u > 0}$ i.e. $F_k(u)$ is constant on each interval $\bigcap_{k=1}^{N-1}(2^{k+Ni}, 2^{k+1+Ni}], i = \overline{-\infty, \infty}$.

There are several constructions of the continuous process $\{X(t)\}$ having $\{Z_k\ (t)\}, k = \overline{0, N-1}$ as its innovation process:

$$X(t) = \sum_{k=0}^{N-1} \int_0^t g_k(t; u) dZ_k(u) \tag{4}$$

Let $\mathcal{H}(Y;t)$ be the mean-square linear closure of $\{Y(u), u \leq t\}$. In the Cramér-Hida representation (4) $\mathcal{H}(X;t) = \bigotimes\sum_{k=0}^{N-1}\mathcal{H}(Z_k;t)$. Since $\mathcal{H}(Z_k;t) \subset \mathcal{H}(W;t)$, we conclude that $\mathcal{H}(X;t) \subset \mathcal{H}(W;t)$ or $X(t) \in \mathcal{H}(W;t)$. So, $\{X(t)\}$ is the non-anticipative transformation (1) of $\{W(t)\}$.

One the simplest constructions of $\{X(t)\}$ is in [4]: Let $\varphi(t), t > 0$, be a continuous function but not absolutely continuous in any interval. Then, $\{X(t), t > 0\}$ defined by $X(t) = \sum_{k=0}^{N-1}\varphi^k(t)Z_K(t)$ is the N-ple Markov process of the multiplicity N. In our construction, $\{X(t)\}$ is the following non-anticipative transformation.

$$X(t) = \int_0^t \{\sum_{k=0}^{N-1}\varphi^k(t)I(2^{-k}t;u)I_k(u)\}dW(u).$$

Remark. For the sake of simplicity let $N = 2$:

$$(5)\quad X(t) = Z_0(t) + \varphi(t)Z_1(t) = \int_0^t \{I_0(u) + \varphi(t)I(2^{-1}t;u)I_1(u)\}dW(u).$$

Relation (5) defines a linear transformation A of $\mathcal{H}(W;t)$ onto $\mathcal{H}(X;t)$ by $X(t) = AW(t)$. But A is unbounded. Indeed, $\|A(W(t+h) - W(t))\|^2 = \|X(t+h) - X(t)\|^2 = \|Z_1(t+h) - Z_1(t)\|^2 + \varphi^2(t+h)\|Z_2(t+h) - Z_2(t)\|^2 + [\varphi(t+h) - \varphi(t)]^2\|Z_2(t)\|^2 = F_1(t+h) - F_1(t) + \varphi^2(t+h)[F_2(t+h) - F_2(t)]^2 + [\varphi(t+h) - \varphi(t)]^2 F_2(t)$, or, for $h \to 0$

$$\frac{\|A(W(t+h) - W(t))\|^2}{\|W(t+h) - W(t)\|^2} \longrightarrow F_1(t) + \varphi^2(t)F_2^1(t) + F_2(t)\lim_{h\to 0}\frac{[\varphi(t+h) - \varphi(t)]^2}{h}$$

The last limit is ∞ for some t. We would mention, in connection with the above remark, the hypothesis in [6], p.46. The simplified version of this hypothesis is the following: Let B be a regular linear transformation (i.e. B and B^{-1} are linear and bounded) defined by $Y(t) = BW(t)$. Then, the process $\{Y(t), t > 0\}$ has the spectral type equivalent to the spectral type of $\{W(t), t > 0\}$ i.e. the ordinary Lebesgue measure dt.

References

[1] Cramér, H.: Stochastic processes as curves in Hilbert Space, Teor. Veroya. i ee prim. 9(1964), 169-179.

[2] Ephremides, A. and Thomas, J.B.: On random processes linearly equivalent to white noise, Inform. Sci. 7(1974), 133-156.

[3] Hida, T.: Canonical representation of Gaussian processes and their applications, Mem. Coll. Sci., Univ. Kyoto, Ser. A, 33 (1960), 109-155.

[4] Pitt, L.D.: Hida-Cramer multiplicity theory for multiple Markov processes and Goursat representation, Nagoya Math.J. 57(1975), 199-228.

[5] Rozanov, Yu.A.: Innovation and non-anticipative processes, Multivariate Analysis-III Proceeding, Dayton, Ohio, (1972) New York, 1973.

[6] Rozanov, Yu.: Teoriya obnovlyayushchih procesov, Nauka, Moskva, 1974.

[7] Stoyanov, J.M.: Counterexamples in Probability, John Wiley and Sons, 1987.

REZIME

PROIZVOLJNOST MULTIPLICITETA ZA JEDNU LINEARNU NE - ANTICIPATIVNU TRANSFORMACIJU WIENEROVOG PROCESA

U radu je dat primer procesa $\{X(t), t > 0\}$ definisanog sa $X(t) = \int_0^t f(t;u)dW(u)$, ($\{W(t)\}$ je Vinerov proces) koji ima proizvoljni konačni multiplicitet N i apsolutno neprekidni maksimalni spektralni tip.

Received by the editors May 20, 1989.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 95-103

Review of Research
Faculty of Science
Mathematics Series

R^n-VALUED FUZZY RANDOM VARIABLE

Mila Stojaković[1]
Faculty of Technical Sciences, University of Novi Sad
Trg Dositeja Obradovića 6, 21000 Novi Sad, Yugoslavia

Abstract

A new metric and structure in the space of integrably bounded fuzzy random variables is introduced and the notion of conditional expectation for fuzzy random variable in R^n is defined.

AMS Mathematical Subject Classification (1980): 28A45, 60G45.
Key words and phrases: Conditional expectation, fuzzy random variable, integrably bounded.

1. Introduction

The concept of a fuzzy set was introduced by Zadeh (1965). Subsequent developments focused on applications of this concept to pattern recognition and system analysis among other areas. Puri and Ralescu [9] studied fuzzy random variables as a generalization of random sets. The purpose of this generalization was the introduction of statistical techniques. The notion of conditional expectation is one of steps in that direction.

After some preliminaries on random sets introduced in § 2, we introduce a new distance on the set of integrably bounded fuzzy random variables in § 3. The main purpose of this paper is to present a theory of conditional expectation for fuzzy random variable and it is done in § 4.

---

[1] This research was supported by Science Fund of Serbia, grant number 0401A, through Matematički institut

2. Preliminaries

In this paper we restrict our attention to the set of fuzzy random variables on the base space R^n, adapting in what follows definitions and results from Feron [3] and Puri, Ralescu [9]. A fuzzy set $u \in \mathcal{F}(R^n)$ is a function $u : R^n \to [0,1]$ for which

1. $u_0 = \overline{co}\{x \in R^n; u(x) > 0\}$ is compact,
2. the α-level set u_α of u, defined by

$$u_\alpha = \{x \in R^n : u(x) \geq \alpha\}$$

is nonempty, closed and convex subset of R^n for all $\alpha \in (0,1]$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space where P is a probability measure. A fuzzy random variable is a function $X : \Omega \to \mathcal{F}(R^n)$ such that

$$\{(\omega, x) : x \in (X(\omega))_\alpha\} \in \mathcal{A} \times \mathcal{B}, \text{ for every } \alpha \in [0,1],$$

where $\mathcal{B}$ denotes the Borel subsets of R^n.

It is obvious that the function $X_\alpha : \Omega \to 2^{R^n}$ defined by $X_\alpha(\omega) = (X(\omega))_\alpha$ is the R^n-valued random set. If H is Hausdorff metric defined on $\mathcal{P}(R^n)$ (the space of all compact and convex subsets of R^n)

$$H(A,B) = \max\{\sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\|\}, \ A, B \in \mathcal{P}(R^n),$$

then $(\mathcal{P}(R^n), H)$ is a complete metric space.

For any multufunction $F : \Omega \to \mathcal{P}(R^n)$ we can define the set

$$S_F = \{f \in L(\Omega, \mathcal{A}) : f(\omega) \in F(\omega) \quad P - a.e.\}$$

where $L(\Omega, \mathcal{A}) = L$ denotes the set of all functions $h : \Omega \to R^n$ which are integrable with respect to the probability measure P.

The set $S_F \subset L$ is closed with respect to a norm in L defined by

$$\|h\| = \int_\Omega \|h(\omega)\| dP, \quad h \in L.$$

Using S_F we can now define an integral for F (first introduced by Aumann [1])

$$\int_\Omega F dP = \{\int_\Omega f(\omega) dP(\omega) : f \in S_F\}.$$

The integrals $\int_\Omega f(\omega)dP(\omega)$ are defined in the sense of Bochner. $F : \Omega \to \mathcal{P}(R^n)$ is integrably bounded if there exists integrable real valued function $h : \Omega \to R$ such that $\sup_{x \in F(\omega)} \|x\| \leq h(\omega)$ $P - a.e.$ The fuzzy random variable $X : \Omega \to \mathcal{F}(R^n)$ is integrably bounded if X_α is integrably bounded for all $\alpha \in [0,1]$. Let $\mathcal{L} = \mathcal{L}(\Omega, \mathcal{A})$ denote the set of all integrably bounded multivalued functions $F : \Omega \to \mathcal{P}(R^n)$ and let $\Lambda = \Lambda(\Omega, \mathcal{A})$ be the set of all integrably bounded fuzzy random variables $X : \Omega \to \mathcal{F}(R^n)$.

We shall close this section, by recalling a lemma which we shall use in the sequel.

Lemma 1. *([7]) Let M be a set and let $\{M_\alpha : \alpha \in [0,1]\}$ be a family of subsets of M such that*

1. $M_0 = M$
2. $\alpha \leq \beta \Rightarrow M_\beta \subseteq M_\alpha$
3. $\alpha_1 \leq \alpha_2 \leq \ldots, \lim_{n\to\infty} \alpha_n = \alpha \Rightarrow M_\alpha = \bigcap_{n=1}^\infty M_{\alpha_n}$.

Then, the function $\phi : M \to [0,1]$ defined by $\phi(x) = sup\{\alpha \in [0,1] : x \in M_\alpha\}$ has the property that $\{x \in M : \phi(x) \geq \alpha\} = M_\alpha$ for every $\alpha \in [0,1]$.

3. Space of fuzzy random variables

For all $X, Y \in \Lambda$ we can define the function $\mathcal{D} : \Lambda \times \Lambda \to R$

$$\mathcal{D}(X,Y) = \sup_{\alpha \geq 0} \Delta(X_\alpha, Y_\alpha).$$

Two fuzzy variables $X, Y \in \Lambda$ are considered to be identical if $\mathcal{D}(X,Y) = 0$. It is obvious that $\mathcal{D}$ is a metric in Λ since Δ is metric in $\mathcal{L}$ (Th. 3.3 [4]).

Theorem 1. *$(\Lambda, \mathcal{D})$ is a complete metric space.*

Proof. Since Δ is metric in $\mathcal{L}$ we have that $\mathcal{D}$ is metric too. Let $\{X^n\}_{n\in N}$ be a Cauchy sequence in Λ, that is, for every $\varepsilon > 0$ there exists $n_0(\varepsilon)$

such that $\mathcal{D}(X^n, X^k) < \varepsilon$ for all $n, k > n_0(\varepsilon)$. For every $\alpha \in [0,1]$ the sequence $\{X^n_\alpha\}_{n\in N}$ is a Cauchy sequence in complete metric space $(\mathcal{L}, \Delta)$ and $\lim_{n\to\infty} X^n_\alpha = \tilde{X}_\alpha \in \mathcal{L}$. Let $S^n_\alpha = S_{X^n_\alpha}$ and $S_\alpha = S_{\tilde{X}_\alpha}$. We can define the random set X_α by

$$X_\alpha(\omega) = cl\{f(\omega) : f \in S_\alpha\}.$$

It is obvious that $\Delta(X_\alpha, \tilde{X}_\alpha) = 0$ which implies that $lim_{n\to\infty} X_\alpha \in \mathcal{L}$. Let $X(\omega)(x) = \sup_{\alpha>0} \{x \in X_\alpha(\omega)\}$. We shall show that for every $\omega \in \Omega$ all the conditions of Lemma 1 are satisfied which will mean that X is a fuzzy random variable.

1. By $X^n_\theta(\omega)$ we denote the set $\{x \in R^n : X^n(\omega)(x) \geq 0\}$. Since $X^n_\theta(\omega) = R^n$ for all $n \in N$ and all $\omega \in \Omega$ we have that $\lim_{n\to\infty} X^n_\theta(\omega) = R^n$ for all $\omega \in \Omega$.

2. For every $\omega \in \Omega$ and every $n \in N, X^n(\omega)$ is a fuzzy set. The inequality $\alpha < \beta$ implies $X^n_\beta(\omega) \subseteq X^n_\alpha(\omega)$, that is, $S^n_\beta \subseteq S^n_\alpha$. Since $S_\beta = \lim_{n\to\infty} S^n_\beta \subseteq \lim_{n\to\infty} S^n_\alpha = S_\alpha$ we have

$$X_\alpha(\omega) = cl\{f(\omega) : f \in S_\alpha\} \subseteq \{f(\omega) : f \in S_\beta\} = X_\beta(\omega),$$

for all $\omega \in \Omega$.

3. Let $\{\alpha_i\}_{i\in N} \subset (0,1]$ be a nondecreasing sequence and let $\lim_{i\to\infty} \alpha_i = \alpha$. In order to prove that for every $\omega \in \Omega$ $X_\alpha(\omega) = \bigcap_{i=1}^\infty X_{\alpha_i}(\omega)$ we shall prove that $S_\alpha = \bigcap_{i=1}^\infty S_{\alpha_i}$. As it was already proved, from $\alpha_i \leq \alpha$ we get $S_\alpha \subseteq S_{\alpha_i}, S_\alpha \subseteq \bigcap_{i=1}^\infty S_{\alpha_i}$. Using the Hausdorff semimetric r $(r(A,B) = \sup_{a\in A} \inf_{b\in B} \| a - b \|,\ A, B \subseteq L)$ we get

$$r(\bigcap_{i=1}^{\infty} S_{\alpha_i}, S_\alpha) \leq r(\bigcap_{i=1}^{\infty} S_{\alpha_i}, \bigcap_{i=1}^{\infty} S^j_{\alpha_i}) + r(\bigcap_{i=1}^{\infty} S^j_{\alpha_i}, S^j_\alpha) + r(S^j_\alpha, S_\alpha)$$

for a fixed j. But, $r(\bigcap_{i=1}^\infty S^j_{\alpha_i}, S^j_\alpha) = 0$, consequently, for every $\varepsilon > 0$, there exists j_0 such that

$$r(\bigcap_{i=1}^{\infty} S_{\alpha_i}, S) \leq \varepsilon + r(\bigcap_{i=1}^{\infty} S_{\alpha_i}, S^j_{\alpha_i})$$

for $j > j_0$, since $S^j_\alpha \to S_\alpha$ uniformly in $\alpha \in [0,1]$.
Now

$$r(\bigcap_{i=1}^{\infty} S_{\alpha_i}, \bigcap_{i=1}^{\infty} S^j_{\alpha_i}) \leq r(\bigcap_{i=1}^{\infty} S_{\alpha_i}, S_{\alpha_n}) + r(S_{\alpha_n}, S^j_{\alpha_n}) + + r(S^j_{\alpha_n}, \bigcap_{i=1}^{\infty} S^j_{\alpha_i})$$

for any $n \geq 1$. Since $\bigcap_{i=1}^{\infty} S_{\alpha_i} \subseteq S_{\alpha_n}$, we obtain

$$r(\bigcap_{i=1}^{\infty} S_{\alpha_i}, \bigcap_{i=1}^{\infty} S_{\alpha_i}^j) \leq r(S_{\alpha_n}, S_{\alpha_n}^j) + r(S_{\alpha_n}^j, \bigcap_{i=1}^{\infty} S_{\alpha_i}^j).$$

Now $r(S_{\alpha_n}, S_{\alpha_n}^j) < \varepsilon$ for $j > j_0$. Note that (since the convergence $S_\alpha^j \to S_\alpha$ is uniform in α) j_0 does not depend on n. Since $S_{\alpha_n}^j$ decreases to $\bigcap_{i=1}^{\infty} S_{\alpha_i}^j$ when $n \to \infty$, it follows that

$$r(S_{\alpha_{n_0}}^j, \bigcap_{i=1}^{\infty} S_{\alpha_n}^j) < \varepsilon \text{ for some } n_0 \text{ (depending on } j).$$

Thus

$$r(\bigcap_{i=1}^{\infty} S_{\alpha_i}, S_{\alpha_i}^j) < 2\varepsilon \text{ for } j > j_0, \text{ that is,}$$

$$r(\bigcap_{i=1}^{\infty} S_{\alpha_i}, S_\alpha) \leq 3\varepsilon \Rightarrow \bigcap_{i=1}^{\infty} S_{\alpha_i} \subseteq S_\alpha.$$

So we have proved that $S_\alpha = \bigcap_{i=1}^{\infty} S_{\alpha_i}$.

Now we have to prove that $X_\alpha(\omega) = \bigcap_{i=1}^{\infty} X_{\alpha_i}(\omega)$ for all $\omega \in \Omega$. We know that $X_\beta(\omega)$ is compact for all $\beta \in [0,1]$ and $\omega \in \Omega$. Let $Z(\omega) = \bigcap_{i=1}^{\infty} X_{\alpha_i}(\omega)$, $\omega \in \Omega$. Since the family of sets $X_{\alpha_i}(\omega)$ has the finite intersection property (every finite subsystem has a non-void intersection), $Z(\omega) \neq 0$ and $Z(\omega)$ is a compact set. Further, we have the next implication

$$\alpha \geq \alpha_i \Rightarrow X_\alpha(\omega) \subseteq X_{\alpha_i}(\omega) \Rightarrow X_\alpha(\omega) \subseteq \bigcap_{i=1}^{\infty} X_{\alpha_i}(\omega),$$

that is

(1) $$X_\alpha(\omega) \subseteq Z(\omega) \text{ for all } \omega \in \Omega.$$

Since $Z \in \mathcal{L}$ it follows (*Lemma* 1.1 [4]) that there exists the sequence $\{f_n\} \subset L$ such that $Z(\omega) = \text{cl } \{f_n(\omega)\}$ for all $\omega \in \Omega$,and we get

$$Z(\omega) \subseteq X_{\alpha_i}(\omega) \Rightarrow \{f_n\} \subseteq S_{\alpha_i} \Rightarrow \{f_n\} \subseteq \bigcap_{i=1}^{\infty} S_{\alpha_i} = S_\alpha \Rightarrow$$

$$\text{cl } \{f_n(\omega)\} \subseteq \text{cl } \{g(\omega) : g \in S_\alpha\} \text{ if } \omega \in \Omega \setminus A$$

and

$$\text{cl } \{f_n(\omega)\} = B \text{ if } \omega \in A,$$

which means that

$$Z(\omega) \subseteq X_\alpha(\omega). \tag{2}$$

Lemma 1 is applicable and there exists fuzzy random variable X with $[X(\omega)]_\alpha = X_\alpha(\omega)$ for every $\alpha \in [0,1]$. From the completness of $(\mathcal{L},\Delta)$ it follows that $X_\alpha \in \mathcal{L}$ for all $\alpha \in (0,1]$, which shows that $X \in \Lambda$.

It remains to show that $X^n \to X$ in $(\Lambda, \mathcal{D})$. For $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that $\mathcal{D}(X^n, X^k) < \varepsilon,\ n,k > n_0(\varepsilon)$. Let $n > n_0$ be fixed. Then

$$\Delta(X^n_\alpha, X_\alpha) = \lim_{k\to\infty} \Delta(X^n_\alpha, X^k_\alpha) \leq$$

$$\lim_{k\to\infty} \sup_{\alpha>0} \Delta(X^n_\alpha, X^k_\alpha) = \lim_{k\to\infty} \mathcal{D}(X^n, X^k) < \varepsilon.$$

Hence $\sup_{\alpha>0} \Delta(X^n_\alpha, X_\alpha) \leq \varepsilon$ for all $n > n_0(\varepsilon)$, implying that $X^n \to X$ in the metric $\mathcal{D}$. That completes the proof.

4. Conditional expectation

Motivated by definition of conditional expectation for random set we introduce the notion of fuzzy conditional expectation for fuzzy random variable.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{F}$ a sub-σ-algebra of $\mathcal{A}$ and $F \in \mathcal{L}$. The conditional expectation of F with respect to $\mathcal{F}$, which is in $\mathcal{L}(\Omega,\mathcal{F})$, is determined by setting

$$S_{E(F|\mathcal{F})} = cl\{g \in L(\Omega,\mathcal{F}) : g = E(f|\mathcal{F}),\quad f \in S_F\}.$$

Finally if X is a fuzzy random set we can define the conditional expectation of $X \in \Lambda$ in such a way that the following conditions are satisfied:

$$E(X|\mathcal{F}) \in \Lambda(\Omega,\mathcal{F}),$$

$$\{x \in R^n : E(X|\mathcal{F})(\omega)(x) \geq \alpha\} = E(X_\alpha(\omega)\,|\mathcal{F}).$$

The next theorem shows that there exists a unique fuzzy random variable satisfying these requirements. The proof is based on Lemma 1.

Theorem 2. *If $X \in \Lambda(\Omega,\mathcal{A})$, then there exists a unique fuzzy random variable $Y \in \Lambda(\Omega,\mathcal{F})$ such that*

$$Y_\alpha(\omega) = E(X_\alpha(\omega)\ |\mathcal{F}).$$

Proof.

1. Since $X_\theta(\omega) = R^n$ for every $\omega \in \Omega$ then $S_{X_\theta} = L(\Omega, \mathcal{A})$ and $S_{E(X_\theta|\mathcal{F})} = \tilde{S}_\theta = L(\Omega, \mathcal{F})$, which means that $\{f(\omega) : f \in L(\Omega, \mathcal{F})\} = R^n = Y(\omega)$ for all $\omega \in \Omega$.

2. If $\alpha \le \beta$, then, clearly, $S_{X_\alpha} = S_\alpha \supseteq S_\beta = S_{X_\beta}$ which implies that $\tilde{S}_\alpha = S_{E(X_\alpha|\mathcal{F})} \supseteq S_{E(X_\beta|\mathcal{F})} = \tilde{S}_\beta$, and $Y_\alpha(\omega) \supseteq Y_\beta(\omega)$ for all $\omega \in \Omega$.

3. Now, we suppose that $\alpha_1 \le \alpha_2 \ldots, \lim_{n\to\infty} \alpha_n = \alpha$. We denote $S_{X_{\alpha_i}}, S_{X_\alpha}, S_{E(X_{\alpha_i}|\mathcal{F})}$ and $S_{E(X_\alpha|\mathcal{F})}$ by $S_i, S, \tilde{S}_i$ and $\tilde{S}$ respectively. Our aim is to show that $\tilde{S} = \bigcap_{i=1}^\infty \tilde{S}_i$. Since $\tilde{S} \subseteq S_i$ it follows $\tilde{S} \subseteq \bigcap_{i=1}^\infty \tilde{S}_i$. But, from the nonexpansivity of conditional expectation for every $\varepsilon > 0$ there exists $i_0(\varepsilon)$ such that $h(\tilde{S}_i, \tilde{S}) \le h(S_i, S) < \varepsilon$ for all $i > i_0(\varepsilon)$.

In order to prove that $Y_\alpha(\omega) = \bigcap_{i=1}^\infty Y_{\alpha_i}(\omega)$ for all $\omega \in \Omega$ we proceed as follows. If $\omega \in \Omega$, then $Y_\beta(\omega) \subseteq Y_0(\omega) \in K_{cc}(\mathcal{X})$ wich implies that $Y_\beta(\omega) \neq \emptyset$ is compact for all $\beta \in [0,1]$. Let

$$Z(\omega) = \bigcap_{i=1}^{\infty} Y_{\alpha_i}(\omega) \ \text{ for all } \ \omega \in \Omega$$

(that is $H(Y_{\alpha_i}(\omega), Z(\omega)) \to 0$ for all $\omega \in \Omega$), and let $\{f_n\} \in S_Z(\mathcal{F})$ be the sequence (*Lemma* 1.1 [4]) such that

$$Z(\omega) = \text{ cl } \{f_n(\omega)\} \ \text{ for all } \ \omega \in \Omega.$$

Since $Z(\omega) \subseteq Y_{\alpha_i}(\omega)$ for all $\omega \in \Omega$, we have that

$$\{f_n\} \subseteq \tilde{S}_{\alpha_i} \ \text{ for all } \ i \in N,$$

that is

$$\{f_n\} \subseteq \bigcap_{i=1}^{\infty} \tilde{S}_i = \tilde{S}.$$

Then

$$Z(\omega) = \text{ cl } \{f_n(\omega)\} \subseteq \text{ cl } \{g(\omega) : g \in \tilde{S}\} = Y_\alpha(\omega).$$

On the other hand, from $\alpha \ge \alpha_i$ for all $i \in N$, we get

$$Y_\alpha(\omega) \subseteq Y_{\alpha_i}(\omega) \Rightarrow Y_\alpha(\omega) \subseteq \bigcap_{i=1}^{\infty} Y_{\alpha_i}(\omega) = Z(\omega).$$

From last two relations we get that

$$Y_\alpha(\omega) = \bigcap_{i=1}^{\infty} Y_{\alpha_i}(\omega) \text{ for all } \omega \in \Omega.$$

Since $Y_\alpha = \mathcal{E}[X_\alpha|\mathcal{F}]$ we have (Th. 5.6. [4]) that $Y_\alpha \in \mathcal{L}(\Omega, \mathcal{F},)$ which implies that Y is integrably bounded. The uniqueness of Y follows from the uniqueness of conditional expectation of random sets.

The fuzzy random variable $Y \in \Lambda(\Omega, \mathcal{F},)$ defined below is fuzzy conditional expectation of $X \in \Lambda(\Omega, \mathcal{A},)$. We shall use notation $Y = E(X|\mathcal{F})$.

Theorem 3. *The fuzzy conditional expectation has the following properties:*

1. $\mathcal{D}(E(X_1|\mathcal{F}), E(X_2|\mathcal{F})) \leq \mathcal{D}(X_1, X_2)$ *for all* $X_1, X_2 \in \Lambda$.

2. If $\mathcal{F}_1 \subset \mathcal{F} \subset \mathcal{A}$ *and* $X \in \Lambda$, *then* $E(X|\mathcal{F}_1)$ *taken on the base space* $(\Omega, \mathcal{A}, P)$ *is equal to* $E(X|\mathcal{F}_1)$ *taken on the base space* $(\Omega, \mathcal{F}, P)$.

3. If $\mathcal{F}_1 \subset \mathcal{F} \subset \mathcal{A}$ *and* $X \in \Lambda$, *then* $E(E(X|\mathcal{F})|\mathcal{F}_1) = E(X|\mathcal{F}_1)$.

4. If $X_n : \Omega \to \mathcal{F}(R^n)$ *are uniformly integrable bounded and* $X_n \xrightarrow{\mathcal{D}} X$, *then* $E(X_n|\mathcal{F}) \xrightarrow{\mathcal{D}} E(X|\mathcal{F})$.

This theorem is a fuzzy generalization of Th. 5.2. (1), Th. 5.3. [4] and Th. 6.2. [8]. The proof is quite similar to the case of random sets so it is omited.

References

[1] Aumann, R.J.: Integrals of set–valued functions, J. Math. Anal. Appl. 12 (1965) 1–12.

[2] Castaing C., Valadier M.: Convex analysis and measurable multifunctions, Lecture Notes in Math., No 560, Springer Verlag, Berlin.

[3] Feron R.: Ensambles aleatoires flus, C.R.Acad. Sci. Paris Ser. A 282 (1976), 903–906.

[4] Hiai F., Umegaki H.: Integrals, conditional expectations and martingales of multivalued functions, J. Multivar. Anal. 7 (1977) 149–182.

[5] Kwakernaak H.: Fuzzy random variables, Inform. Sci. 15 (1978), 1–29.

[6] Matheron G.: Random sets and integral geometry, Wiley, New York, 1975.

[7] Negoita C.V., Ralescu D.: Applications of fuzzy sets to systems analysis, Wiley, New York, 1975.

[8] Papageorgiou N.: On the theory of Banach space valued multifunctions, J. Multivar. Anal. 17 (1985) 185–227.

[9] Puri M., Ralescu D.: Fuzzy random variables, J. Math. Anal. Appl. 114 (1986) 409–422.

[10] Stojaković M., Gajić Lj.: Fuzzy Markov chains, Zbornik FTN (1990), 126–131.

REZIME

RASPLINUTA SLUČAJNA PROMENLJIVA SA VREDNOSTIMA U R^n

Nad skupom integrabilno ograničenih fazi slučajno promenljivih je definisana metrika i dokazano je da je prostor kompletan. Dalje se definiše i ispituju se osobine uslovnog matematičkog očekivanja fazi slučajne promenljive.

Received by the editors November 15, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 105-107

Review of Research
Faculty of Science
Mathematics Series

REMARK ON THE PAPER OF CHITRA AND SUBRAHMANYAM ABOUT THE SECTION THEOREM OF KY FAN

G. Isac
Département de Mathématiques, Collège Militaire Royal
St - Jean, Quebec, Canada JOJ 1RO

Abstract

In 1961 Ky Fan proved in [2] an interesting result with important applications to the variational inequality theory and the minimax theory.

This theorem is known now as "the Section Theorem" and it has been the subject of several papers [4], [5] etc.

Recently in paper [1] Chitra and Subrahmanyam proved a generalization of this theorem and proof is based essentially on Brouwe's fixed point theorem.

Now, we show in tis note that, this generalization is, in fact, a direct consequence of Section Theorem.

AMS Mathematics Subject Classification (1980): 47H10
Key words and phrases: Section Theorem, variational inequalities

We recall that in 1972, in paper [3, pg. 105], Ky Fan reformulated the Section Theorem as the following result.

Theorem 1. *[Ky Fan][3]. Let X be a nonempty compact convex set in a Hausdorff topological vector space. Let D be subset of $X \times X$ having the following properties:*

(a) *for every* $x \in X, (x,x) \in D$,

(b) *for each fixed* $y \in X$, *the set* $D_y = \{x \in X | (x,y) \notin D\}$ *is convex (possibly empty),*

(c) *for each fixed* $x \in X$, *the set* $D_x = \{y \in X | (x,y) \in D\}$ *is closed.*

Then there exists a point $y_0 \in X$ *such that* $X \times \{y_0\} \subset D$.

The generalization of Section Theorem proved in [1] by Chitra and Subramanyam is the following.

Theorem 2. *Let* K *be a nonempty compact convex subsets of Hausdorff topological vector space* E.

Let $A \subset K \times K$ *and* $g : K \to K$ *such that the following conditions are satisfied:*

i. *for every* $y \in K, \{x \in K | (g(x), y) \in A\}$ *is closed,*

ii. *for every* $x \in K, (g(x), x) \in A$,

iii. *for each* $x \in K, \{y \in K | (g(x), y) \notin A\}$ *is empty or convex.*

Then there exists $x_0 \in K$ *such that* $\{g(x_0)\} \times K \subset A$.

We prove now that Theorem 2 is an immediate consequence of Theorem 1.

Indeed, if we denote $X = K$ and $D = \{(x_1, x_2) \in K \times K | (g(x_2), x_1) \in A\}$ we constat that the all assumptions of Theorem 1 are satisfied.

Obviously, $D \subseteq X \times X$ and we have:

(a) for every $x \in X, (x,x) \in D$ by assumption (ii),

(b) for every $y \in X$ the set
$D_y = \{x \in X | (x,y) \notin D\}$ is empty or convex since,
$D_y = \{x \in X | (g(y), x) \notin A\}$ and the second part is an empty or convex set by assuption (iii),

(c) for each fixed $x \in X$ the set $D_x = \{y \in X | (x,y) \in D\}$ is closed since, $D_x = \{y \in X | (g(y), x) \in A\}$ and the second part is a closed set by assumption (i).

Hence from Theorem 1 there exists $x_0 \in X$ such that $(y, x_0) \in D$, for every $y \in X$, that is $(g(x_0), y) \in A$ for every $y \in K$ and Theorem 2 is proved. □

In consequence Theorem 2 is a corollary of Section Theorem.

References

[1] Chitra A. and Subrahmanyam P.V.: A generalization of a section theorem of Ky Fan and its applications to variational ineqyalities, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., 17,1(1987), 17-37.

[2] Fan K.: A generalization of Tychonoff's fixed point theorem, Mat. Ann., 142(1961), 305-310.

[3] Fan K.: A minimax inequality and applications, Inequalities III. Academic Press. New York (1972).

[4] Iohvidov I.S.: On a lemma of Ky Fan generalizing the fixed point principle of A.N. Tikhonoff. Soviet. Math. Dokl., 5(1964), 1523-1526.

[5] Itoh S., Takahashi W. and Ynagi K.: Variational inequalities and complementary problems, J. Math. Soc. Japan, 30(1978), 23-28.

REZIME

PRIMEDBE NA RAD CHITRE I SUBRAHMANYAMA O TEOREMI O SEČENJU KY FANA

1961. godine Ky Fan [2] je dokazao zanimljiv rezultat sa važnim primenama na varijacione nejednakosti i minimaks teoriju.

Ova je teorema poznata kao "Teorema o sečenju" i bila je predmet mnogih radova [4],[5].

Nedavno su u [1] Chitra i Subrahmanyam dokazali uopštenje ove teoreme i dokaz je u osnovi zasnovan na Brauerovoj teoremi o nepokretnoj tački.

U ovoj noti je pokazano da je ovo uopštenje zapravo direktna posledica teoreme o sečenju.

Received by the editors September 20, 1988.

Hence from Theorem 1 there exists $x_0 \in X$ such that $(y, x_0) \in D$, for every $y \in X$, that is $\phi(y, x_0) \le 0$ for every $y \in X$ and Theorem 2 is proved. □

In consequence Theorem 2 is a corollary of Berge's theorem.

References

[1] Chitra A. and Subrahmanyam P.V.: A generalization of a section theorem of Ky Fan and its applications to variational inequalities, Zb. Rada Novom Sadu Zb. Rad. Prirod. Mat. Fak. Ser. Mat., 17.1(1987), 17-32.

[2] Fan K.: A generalization of Tychonoff's fixed point theorem, Math. Ann., 142(1961), 305-310.

[3] Fan K.: A minimax inequality and applications, Inequalities III, Academic Press, New York (1972).

[4] Ishikov S.: On a lemma of Ky Fan generalizing the fixed point principle of A.N. Tikhonoff, Soviet Math. Dokl., 5(1964), 1622-1626.

[5] Yen S., Takahashi W., and Yanagi K.: Variational inequalities and complementary problems, J. Math. Soc. Japan, 30(1978), 23-28.

REZIME

PRIMEDBE NA RAD CHITRE I SUBRAHMANYAMA O TEOREMI O SEČENJU KY FANA

[illegible] Ky Fan [2] je dokazao zanimljiv rezultat sa važnim primenama u varijacione nejednakosti i minimaks teoriju.

Ova je teorema postala, kao što [illegible] (teorija) i [illegible] je predmet mnogih radova [4],[5].

Autori [1] Chitra i Subrahmanyam [illegible] dokazuju nove teoreme [illegible] Bergeove teoreme [illegible]

U ovoj noti je pokazano da je [illegible] direktna posledica Bergeove teoreme.

Received by the editors September 20, 1988

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 109-112

Review of Research
Faculty of Science
Mathematics Series

ON POWERS OF HEAVISIDE'S FUNCTION

Brian Fisher
Department of Mathematics, The University,
Leicester, LE1 7RH, England

Abstract

We show that the non-associativity of the distribution product must be taken into account when differentiating powers of Heaviside's function.

AMS Mathematics Subject Classification (1980): 46F10
Key words and phrases: Product of distributions, Heaviside's function.

Heaviside's function is the locally summable function Y defined by

$$Y(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

It follows that

(1)
$$Y^n = Y.$$

Formal differentiation of equation (1) appears to give

(2)
$$nY^{n-1}\delta = \delta = nY\delta$$

and so

$$Y\delta = \frac{1}{n}\delta$$

for $n = 2, 3, \dots$, which is clearly impossible.

However, it is well known that the product of distributions is in general non-associative and taking this into account, equation (2) is false, since $(Y^n)' \neq nY^{n-1}\delta$ for $n = 3, 4, \ldots$. It is desirable and indeed possible for equation (1) to hold for $n = 2, 3, \ldots$ and for formal differentiation of the product Y^n to give δ. Indeed, if ϕ is an arbitrary test function with compact support, then

$$\langle Y^n, \phi \rangle = \langle Y, \phi \rangle$$

for $n = 2, 3, \ldots$ and so equation (1) follows.

A definition for a commutative product of distributions was given in [1] and a definition for a non-commutative product of distributions was given in [2]. Both definitions lead to the result that

$$Y\delta = \delta Y = \frac{1}{2}\delta \tag{3}$$

and so we will accept equation (3).

It is easily seen that both of these definitions lead to a non-associative product since with either definition we have

$$Y(Y\delta) = \frac{1}{2}Y\delta = \frac{1}{4}\delta$$

whereas

$$(YY)\delta = Y\delta = \frac{1}{2}\delta.$$

The non-associativity of the product must therefore be taken into account when differentiating Y^n.

We will now consider a product of n distributions $f_1, \ldots, f_n$. We will further suppose that the product exists and is taken in the form

$$f_1(f_2 \ldots (f_{n-1} f_n) \ldots). \tag{4}$$

If the resulting products exist, formal differentiation of (4) gives

$$\begin{aligned}[f_1(f_2 \ldots (f_{n-1} f_n) \ldots)]' \;=\;& f_1'(f_2 \ldots (f_{n-1} f_n) \ldots) + f_1(f_2' \ldots \\ & \ldots (f_{n-1} f_n) \ldots) + \ldots + f_1(f_2 \ldots f_{n-1}' f_n) \ldots) + \\ & + f_1(f_2 \ldots (f_{n-1} f_n') \ldots).\end{aligned}$$

In particular, when

$$f_1 = \ldots = f_n = Y,$$

we have

$$\begin{aligned}[Y(Y\ldots(YY)\ldots)]' &= \delta(Y\ldots(YY)\ldots)+Y(\delta\ldots(YY)\ldots)+\ldots+\\ &\quad +Y(Y\ldots(\delta Y)\ldots)+Y(Y\ldots(Y\delta)\ldots)\\ &= \delta Y+Y(\delta Y)+Y(Y(\delta Y))+\ldots+Y(Y\ldots\\ &\quad \ldots(\delta Y)\ldots)+Y(Y\ldots(Y\delta)\ldots)\\ &= \frac{1}{2}\delta+(\frac{1}{2})^2\delta+(\frac{1}{2})^3\delta+\ldots+(\frac{1}{2})^{n-1}\delta+(\frac{1}{2})^{n-1}\delta\\ &= \delta = Y'.\end{aligned}$$

We therefore see that it is equation (2) that is false.

Notice that in the above, the product Y^n has been taken in the form

$$Y^n = Y(Y\ldots(YY)\ldots) \tag{5}$$

to obtain the result

$$(Y^n)' = \delta. \tag{6}$$

However, the product Y^n could have been formed in many different ways. For example, we could have

$$Y^n = Y^r Y^{n-r},$$

where $1 < r < n$ and Y^r and Y^{n-r} are of the form (5), but whichever way the product Y^n is formed, equation (6) will always hold.

To see this, assume that for some $n > 1$

$$(Y^r)' = \delta$$

for $r = 2, 3, \ldots, n$ for whichever way the product Y^r is formed. This is certainly true when $n = 2$. The product Y^{n+1} must be of one of the following three forms

$$Y(Y^n), \quad (Y^n)Y, \quad (Y^r)(YY^{n-r}),$$

where $1 < r < n$. By our assumption,

$$(Y^n)' = (Y^r)' = (Y^{n-r})' = \delta,$$

whichever way the products Y^n, Y^r, Y^{n-r} have been formed. Thus

$$(Y(Y^n))' = \delta(Y^n) + Y(Y^n)' = \delta Y + Y\delta = \delta,$$

$$((Y^n)Y)' = (Y^n)'Y + (Y^n)\delta = \delta Y + Y\delta = \delta$$

and

$$\begin{aligned}[(Y^r)(Y(Y^{n-r}))]' &= (Y^r)'(Y(Y^{n-r})) + (Y^r)(\delta(Y^{n-r})) + (Y^r)(Y(Y^{n-r})') \\ &= \delta Y + Y(\delta Y) + Y(Y\delta) \\ &= \frac{1}{2}\delta + (\frac{1}{2})^2\delta + (\frac{1}{2})^2\delta = \delta.\end{aligned}$$

It follows that however the product Y^{n+1} has been formed

$$(Y^{n+1})' = \delta$$

and our result follows by induction.

References

[1] Fisher, B.: The product of distributions, Quart. J. Math. (2), 22 (1971), 291-298.

[2] Fisher, B.: On defining the product of distributions, Math. Nachr., 99 (1980), 239-249.

REZIME

O STEPENIMA HEAVISIDEOVE FUNKCIJE

Autor pokazuje da se neasocijativnost proizvoda distribucija mora uzeti u obzir kod diferenciranja stepena Heavisideove funkcije.

Received by the editors August 13, 1988.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 113-120

Review of Research
Faculty of Science
Mathematics Series

REGULAR BOREL t-DECOMPOSABLE MEASURES

Endre Pap
Institute of Mathematich, University of Novi Sad,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

The aim of the present paper is to study the regularity of Borel t-conorm decomposable measures.

AMS *Mathematics Subject Classification* (1980): 28B10, 28C15
Key words and phrases: t-conorm, Borel sets, order continuous, regular Borel ⊥ -decomposable measure.

1 . Introduction

In this paper we shall investigate the regularity property of the ⊥-decomposable measure with respect to a t-conorm ⊥. We have considered such measures in previous papers [6]- [8] (see also [1] and [10]) on families of abstract sets. In this paper we shall restrict ourselves to the ⊥-decomposable measures defined on the class $\mathcal{B}$ of Borel sets of a locally compact set X. Among other results, we shall prove the singleton characterization of atoms of the regular Borel ⊥-decomposable measure - Theorem 1. We shall also give some conditions which ensure the regularity of the ⊥-decomposable measures with respect to continuous at zero t-conorms ⊥-Theorem 4. We shall prove the order continuity of ⊥-decomposable measures which are decomposable with respect to t-conorm ⊥, which have a series property - Theorem 5.

2 . Consequences of the regularity

Let X be locally compact Hausdorff topological space and let $\mathcal{K}$ be the lattice of all the compact subsets of X. Borel σ-ring $\mathcal{B}$ is the smallest σ-ring containing $\mathcal{K}$. We shall denote by $\mathcal{O}$ the class of all the open sets belonging to $\mathcal{B}$.

A function $\perp : [0,1] \times [0,1] \to [0,1]$ will be called t-conorm, if it is asociative, commutative, monotone and has 0 as a neutral element (see [1], [6] - [8], [10]). A set function $m : \mathcal{B} \to [0,1]$ with $m(\emptyset) = 0$ will be caled a (Borel) $\perp$-decomposable measure if

$$m(A \cup B) = m(A) \perp m(B)$$

holds for all $A, B \in \mathcal{B}$, such that $A \cap B = \emptyset$.

A $\perp$-decomposable measure m is **regular**, if for each set $A \in \mathcal{B}$ and each $\varepsilon > 0$ there exist $K \in \mathcal{K}$ and $V \in \mathcal{O}$, such that $K \subset A \subset V$ and

$$m(V \setminus K) < \varepsilon.$$

A set function $m : \mathcal{B} \to [0,1]$ is **order continuous** if

$$\lim_{n\to\infty} m(E_n) = 0$$

for any sequence (E_n), $E_n \in \mathcal{B}$ $(n \in \mathcal{N})$, such that $E_n \searrow \emptyset$.

A set function $m : \mathcal{B} \to [0,1]$ is **exhaustive**, if

$$\lim_{n\to\infty} m(E_n) = 0$$

for any sequence (E_n) of pairwise disjoint sets from $\mathcal{B}$.

A set $A \in \mathcal{B}$ is an **atom** of $\perp$-decomposable measure m iff $m(A) > 0$ and either $m(A \cap B) = 0$ or $m(A \setminus B) = 0$ for any $B \in \mathcal{B}$.

We have in the following theorem as an important property of atoms of regular Borel $\perp$-decomposable measures.

Theorem 1. *Let m: $\mathcal{B} \to [0,1]$ be a regular Borel $\perp$-decomposable measure with respect to an arbitrary but fixed t-conorm $\perp$. If $A \in \mathcal{B}$ is an atom of m, then there exists a point $a \in A$ such that*

$$m(A) = m(\{a\}).$$

Proof. Let $A \in \mathcal{B}$ be an arbitrary, but fixed atom of m. We denote with $\mathcal{K}'$ the family of the compact sets $K \subset A$, such that $m(A \setminus K) = 0$. Since for of any fixed $K \in \mathcal{K}'$,

$$K \cap B \subset A \cap B \text{ and } K \setminus B \subset A \setminus B$$

hold for any $B \in \mathcal{B}$, we have by the monotonicity of m

$$m(K \cap B) \cdot m(K \setminus B) = 0.$$

Then, since we have

$$m(K) = m(A \setminus K) \perp m(K) = m(A) > 0 ,$$

K is an atom of m.

Further, for each K_1 and K_2 from $\mathcal{K}'$ we have by the monotonicity and $\perp$-subaddivity of m

$$m(A \setminus (K_1 \cap K_2)) = m((A \setminus K_1) \cup (A \setminus K_2)) \leq$$

$$\leq \; m(A \setminus K_1) \perp m(A \setminus K_2).$$

Hence, by $m(A \setminus K_i) = 0 \;\; (i = 1, 2)$ we have

$$m(A \setminus (K_1 \cap K_2)) = 0, \text{ i.e. } K_1 \cap K_2 \in \mathcal{K}'.$$

Now, let

$$K_0 = \bigcap_{K \in \mathcal{K}'} K.$$

Then K_0 is a non-emty compact set. Indeed, if we would suppose the contrary, i.e. $K_0 = \emptyset$, then, since X is a Hausdorff topological space, there would exist some finite subcollection of $\{K\}_{K \in \mathcal{K}'}$ with an empty intersection. This is imposible, since this finite subcollection would belong to $\mathcal{K}'$, but it is an atom as an element of $\mathcal{K}'$, which is non-empty. We shall show that $K_0 \in \mathcal{K}'$. Namely, for an arbitrary but fixed $K \in \mathcal{K}'$ there has to be $m(K \setminus K_0) = 0$. If we suppose that this is not true, then, since for any $B \in \mathcal{B}$, we have

$$B \cap (K \setminus K_0) \subset A \cap B \text{ and}$$

$$(K \setminus K_0) \setminus B \subset A \setminus B, \quad .$$

we obtain

$$m(B \cap (K \setminus K_0)) \cdot m((K \setminus K_0) \setminus B) = 0,$$

i.e. $K \setminus K_0$ would be an atom of m. We have

$$m(A) = m(A \setminus K) \perp m(K \setminus K_0) \perp m(K_0). \tag{1}$$

Since we supposed that $m(K \setminus K_0) > 0$ and A , K are atoms of m (which implies $m(A) > 0$ and $m(K_0) = m(K \cap K_0) = 0$) , we obtain by (1)

$$m(A) = m(K \setminus K_0).$$

Hence,

$$m(A \setminus (K \setminus K_0)) = 0.$$

These facts imply that $K \setminus K_0$ has to contain an element of $\mathcal{K}'$. Since K_0 is non-empty, this is a contradiction. So, we have $m(K \setminus K_0) = 0$. Then, by (1), we obtain $m(A) = m(K_0)$, i.e. $m(A \setminus K_0) = 0$. That means $K_0 \in \mathcal{K}'$. We shall show that K_0 reduces on a set with one element. If we suppose the contrary that K_0 contains at least two distinct elements a_1 and a_2, then, since X is a locally compact Hausdorff topological space, there exists an open neighbourhood V of a_1 such that $\bar{V}$ does not contain a_2 . Then, we have

$$K_0 = (K_0 \setminus V) \cup (K_0 \cap \bar{V}).$$

Since one of the sets $K_0 \setminus V$, $K_0 \cap \bar{V}$ has to belong to $\mathcal{K}'$, but K_0 is the least element from $\mathcal{K}'$, we obtain a contradiction. So, we have for some $a \in A$ $m(A) = m(K_0) = m(\{a\})$.

In the special case $X = \mathbf{R}$, we have

Theorem 2. *Each continuous from above regular Borel $\perp$-decomposable measure m: $\mathcal{B}(\mathbf{R}) \to [0,1]$ with respect to an arbitrary but fixed t-conorm $\perp$ on the Borel σ-algebra $\mathcal{B}(\mathbf{R})$, which has the property*

$$m((a,b)) = g(b-a) \ \ ((a,b) \subset [0,1])$$

for some continuous at 0 function g with $g(0) = 0$, is non-atomic.

Proof. Suppose the contrary. Then, there exists an atom $A \in \mathcal{B}(\mathbf{R})$ of m. Then, by Theorem 1, there exists $a \in A$, such that $m(A) = m(\{a\})$. Then,

$$m(\{a\}) = m(\bigcap_{n=1}^{\infty}(a - \frac{1}{n}, a + \frac{1}{n})) =$$

$$= \lim_{n\to\infty} m((a - \frac{1}{n}, a + \frac{1}{n})) =$$

$$= \lim_{n\to\infty} g(\frac{2}{n}) = 0 \text{ i.e. } m(\{a\}) = 0.$$

Contradicton.

Theorem 3. *A regular Borel* $\perp$*-decomposable measure* m *with respect to an arbitrary but fixed t-conorm* $\perp$ *is exhaustive on the family of open sets* $\mathcal{O}$.

Proof. Let (O_n) be a sequence of open sets from $\mathcal{O}$ which are pairwise disjoint. Since $\cup_{n=1}^{\infty} O_n$ is an open set, for any $\varepsilon > 0$, there exists a compact set K such that $K \subset \cup_{n=1}^{\infty} O_n$ and

$$m(\bigcup_{n=1}^{\infty} O_n \setminus K) < \varepsilon.$$

Since (O_n) is an open cover of K, there exists a natural number n_0 such that $K \subset \cup_{n=1}^{n_0} O_n$. Then, we have by the monotonicity of m for $k \geq n_0 + 1$

$$m(O_k) \leq m(O_k \bigcup (\bigcup_{n=1}^{n_0} O_n \setminus K)) \leq m(\bigcup_{n=1}^{\infty} O_n \setminus K)$$

3 . Regularity and continuity

We have in the following theorem conditions which ensure the regularity of the Borel t-conorm decomposable measure.

Theorem 4. *Let* $m\colon \mathcal{B} \to [0,1]$ *be a* $\perp$*-decomposable measure with respect to a continuous at zero t-conorm* $\perp$*, which is exhaustive on the family of compact sets* $\mathcal{K}$ *and satisfies the condition*

$$m(A) = sup\{m(K) : K \in \mathcal{K}, K \subset A\} \ (A \in \mathcal{B}). \quad (2)$$

Then m *is a regular Borel* $\perp$*-decomposable measure.*

Proof. Suppose that m is not regular. Then, there exist a set A from $\mathcal{B}$ and a number $\varepsilon > 0$ such that

$$m(V \setminus K) > \varepsilon \quad (3)$$

for each $K \in \mathcal{K}, V \in \mathcal{O}, K \subset A \subset V$. Let us fix such sets K_0 and V_0. Then, by (2), there exist $C_1, D_1 \in \mathcal{K}$, $C \subset V_0 \setminus A$, $D_1 \subset A \setminus K_0$, such that

$$m(C_1) \geq \frac{1}{2} m(V_0 \setminus A) \text{ and } m(D_1) \geq \frac{1}{2} m(A \setminus K_0).$$

If we denote by $V_1 = V_0 \setminus C_1$ and $K_1 = K_0 \cup D_1$, then we have $V_1 \in \mathcal{O}$ and $K_1 \in \mathcal{K}$ and

$$K_1 \subset A \subset V_1.$$

Hence, by (3), we have

$$m(V_1 \setminus A_1) > \varepsilon.$$

Now, we are going to repeat the preceding procedure. After n-steps we have

$$C_n, D_n \in \mathcal{K},\ C_n \subset V_{n-1} \setminus A,\ D_n \subset A \setminus K_{n-1},$$

such that

$$(4) \qquad m(C_n) \geq \frac{1}{2} m(V_{n-1} \setminus A); \quad m(D_n) \geq \frac{1}{2} m(A \setminus K_{n-1}).$$

So, we have two sequences (C_n) and (D_n) of pairwise disjoint sets from $\mathcal{K}$, whice are subsets of V_0 . Since m is exhaustive on $\mathcal{K}$,we obtain by (4)

$$\lim_{n\to\infty} m(V_n \setminus A) = 0 \text{ and } \lim_{n\to\infty} m(A \setminus K_{n-1}) = 0.$$

Hence, by continuity at zero of t-conorm $\perp$, we have

$$\lim_{n\to\infty} m(V_n \setminus K_n) = \lim_{n\to\infty} ((V_n \setminus A) \cup (A \setminus K_n)) = 0.$$

Contradiction with (3).

Now, we are going to introduce an important condition for the t-conorm $\perp$.

A t-conorm $\perp$ has the **series property** if for some $k_0 \in N$

$$\perp(\frac{x}{2}, \frac{x}{2^{k_0}}) \leq x \quad (x \in [0,1])$$

holds. It is obvious that t-conorm with the series property is also continuous at zero.

Many important known t-conorms have the series property. For example: $\perp_m(x,y) = min(x+y,1)$, $U_\lambda(x,y) = min(x+y+\lambda xy, 1)$ for $\lambda > -1$, $S_p(x,y) = (x^p + y^p - x^p y^p)^{\frac{1}{p}}$, for fixed $p > 0$, etc.

Theorem 5. *Each regular Borel $\perp$-decomposable measure m with respect to a t-conorm $\perp$ which has the series property is order continuous and exhaustive.*

Proof. Let (A_n) be a sequence from $\mathcal{B}$ such that $A_n \searrow \emptyset$ and let $\varepsilon > 0$. Since m is regular, we can choose $K_k \in \mathcal{K}$ $(k \in N)$ such that

$$m(A_k \setminus K_k) < \frac{\varepsilon}{2^{(k-1)+k_0}} \ (k \in N).$$

Hence, by the series property of $\perp$ and $\perp$-subadditivity of m, we obtain

$$m(\bigcup_{k=1}^{n}(A_k \setminus K_k)) \leq$$

$$\leq (\ldots((m(A_n \setminus K_n) \perp m(A_{n-1} \setminus K_{n-1})) \perp m(A_{n-2} \setminus K_{n-2})) \perp \ldots \perp m(A_1 \setminus K_1)) \leq$$

$$\leq (\ldots((\frac{\varepsilon}{2^{(n-1)+k_0}} \perp \frac{\varepsilon}{2^{(n-2)+k_0}}) \perp \frac{\varepsilon}{2^{(n-3)+k_0}}) \perp \ldots \perp \frac{\varepsilon}{2^{k_0}}) \leq$$

$$\leq (\ldots(\frac{\varepsilon}{2^{(n-2)}} \perp \frac{\varepsilon}{2^{(n-3)+k_0}}) \perp \ldots \perp \frac{\varepsilon}{2^{k_0}}) \leq \varepsilon.$$

We have $\bigcap_{i=1}^{n} K_i \in \mathcal{K}$ and $\bigcap_{i=1}^{n} K_i \subset A_n$. Since $A_n \searrow \emptyset$, we obtain $\bigcap_{i=1}^{n} K_i \searrow \emptyset$. Then, there exists a natural number n_0 such that $\bigcap_{i=1}^{n} K_i = \emptyset$ for $n \geq n_0$. Hence, for $n \geq n_0$

$$A_n = A_n \setminus \bigcap_{i=1}^{n} K_i = \bigcup_{i=1}^{n}(A_n \setminus K_i) \subset \bigcup_{i=1}^{n}(A_i \setminus K_i).$$

Then, by the monotonicity of m, we have

$$m(A_n) \leq m(\bigcup_{i=1}^{n}(A_i \setminus K_i)) < \varepsilon$$

for $n \geq n_0$, i.e. m is order continuous. If (E_n) is a sequence of pairwise disjoint sets from $\mathcal{B}$, then the inequality

$$m(E_n) \leq m(\bigcup_{k=n}^{\infty} E_k) \text{ holds}$$

Since $\bigcup_{k=n}^{\infty} E_k \searrow \emptyset$, the preceding inequality implies the exhaustivity of m.

Remark 1. *Theorem 5 implies that the set function* $m(A) = sup_{x \in A} f(x)$, *from Example 2.4 of [8], is not regular, although it is* σ*-sup-decomposable and the sup has the series property.*

Corollary 1. *Each regular Borel* $\perp$*-decomposable measure* m *with respect to an Archimedean t-conorm* $\perp$*, which has the series property, is* $\sigma - \perp$*-additive.*

Remark 2. *For* $\perp$*-decomposable measure with respect to continuous (specially Archimedean) t-conorm, we can apply the results of H. Weber [9] and P. Morales [5] on uniform semigroup valued measures.*

References

[1] D'Apuzzo, L., Squillante, M., Ventre, A.G.S.: A survey on decomposable measures, Control and Cybernetics (to appear)

[2] Dinculeanu, N., Kluvanek, I.: On vector measures, Proc. London Math.Soc., III. Ser. 17 (1967), 505–512.

[3] Dobrakov, I.: On submeasures I, Dissertationes Math. 112, Warszawa, 1974.

[4] Khurana, S.S.: Extension and regularity of group-valued Baire measures, Bull. Acad. Polon. Sci. Ser Sci. Math. Astron. Phys. 22 (1974), 891–895.

[5] Morales, P.: Regularity and extension of semigroup-valued Baire measures, Lecture Notes in Math. 794, Proc. Measure Theory, Oberwolfach 1979, Springer-Verlag, 1980, 317–323.

[6] Pap, E.: Lebesgue and Saks decompositions of ⊥-decomposable measures, Fuzzy Sets and Systems, 38 (1990), 345–353.

[7] Pap, E.: On non-additive set functions, Atti Sem. Mat. Fis. Univ. Modena, 39 (1991), 345–360.

[8] Pap, E.: Extension of the continuous t-conorm decomposamble measure, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. (in print).

[9] Weber, S.: Fortsetzung von Massen mit Werten in uniformen Halbgruppen, Arch. der Math. 27 (1976), 412–423.

[10] Weber, S.: ⊥-decomposable measures and integrals for Archimedean t-conorms, J. Math. Anal. Appl. 101 (1984), 114–138.

REZIME

REGULARNE BORELOVE t-KONORMA DEKOMPOZABILNE MERE

U radu se ispituje regularnost ⊥-dekompozabilnih mera u odnosu na t-konormu ⊥ a koje su definisane na Borelovim skupovima lokalno kompaktnog Hausdorffovog topološkog prostora.

Received by the editors March 5, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 121-130

Review of Research
Faculty of Science
Mathematics Series

EXTENSION OF THE CONTINUOUS t-CONORM DECOMPOSABLE MEASURE

Endre Pap
Institute of Mathematics, University of Novi Sad ,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

The aim of the present paper is to study the extensions of the $\perp$-decomposable measure m, with respect to a continuous t-conorm $\perp$, from a ring $\mathcal{R}$ to the σ-ring Σ generated by $\mathcal{R}$ and to the class $\mathcal{R}_\sigma$ of sets which are limits of increasing sequences from $\mathcal{R}$.

AMS Mathematics Subject Classification (1980): 28A10, 28B10
Key words and phrases: t-conorm, $\perp$-decomposable measure, order continuous, exhaustive, $\perp$-subadditive set function

1. Introduction

This paper is a continuation of investigations on $\perp$-decomposable measures with respect to a t-conorm $\perp$, which were considered in the papers [5] - [8]. The interest for these non-additive set function (non-additive with respect to the usual addition of the real numbers) is growing. We have proved in previous papers [5] and [6] the analogies of classical measure theory theorems: Lebesgue decomposition, Saks decomposition, Darboux property, compactness of the range. In this paper we shall investigate the extension of the order continuous $\perp$-decomposable measure m, with respect to a continuous t-conorm $\perp$, from a ring $\mathcal{R}$ to the σ-ring Σ generated by $\mathcal{R}$ - Theorem 3.2.

The unique extension is monotone order continuous $\perp$-subadditive set function $\overline{m}$. It is interesting that without the supposition of the order continuity of m we lose the uniqueness of the extension - Example 3.3. We have proved in Theorem 3.1. that there exists a unique σ $\perp$-decomposable extension of the σ $\perp$-decomposable measure from $\mathcal{R}$ to the class $\mathcal{R}_\sigma$ of sets which are limits of increasing sequences from $\mathcal{R}$.

2. $\perp$-subadditive set functions

Definition 2.1. *A function* $\perp : [0,1] \times [0,1] \to [0,1]$ *will be called t-conorm if it satisfies:*

(A) $\perp(x,0) = \perp(0,x) = x \quad (x \in [0,1])$;

(B) if $x_1 \le x_3$ and $x_2 \le x_4$ for $x_1, x_2, x_3, x_4 \in [0,1]$
then $\perp(x_1,x_2) \le \perp(x_3,x_4)$;

(C) $\perp(x,y) = \perp(y,x) \quad (x,y \in [0,1])$;

(D) $\perp(\perp(x,y),z) = \perp(x,\perp(y,z)) \quad (x,y,z \in [0,1])$.

A t-conorm $\perp$ will be called continuous at zero if it satisfies the condition

(E) for all sequences (x_n) and (y_n) such that $x_n, y_n \in [0,1]$

and $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = 0$, $\lim\limits_{n\to\infty} \perp(x_n,y_n) = 0$ holds.

There are many important t-conorms: $\perp_m(x,y) = \min(x+y,1)$, $U_\lambda(x,y) = min(x+y+\lambda xy,\, 1)$ for $\lambda > -1$, $S_p(x,y) = (x^p + y^p - x^p y^p)^{1/p}$ for fixed $p > 0$, etc. (see [5], [8]).

There exist t-conorms which are continuous at zero but they are not continuous. For example,

$$\perp(x,y) = \begin{cases} max\{x,y\} & \text{for} \quad y \in [0,1/2) \\ max\{x,y\} & \text{for} \quad x \in [0,1/2) \\ 1 & \text{otherwise} \end{cases}$$

In the whole paper, $\mathcal{R}$ always denotes a ring of subsets of the given nonempty set X.

Definition 2.2. *A set function* $m : \mathcal{R} \to [0,1]$ *with* $m(\emptyset) = 0$ *will be called* $\perp$*-subadditive, if*

$$m(A \cup B) \leq m(A) \perp m(B)$$

holds for all $A, B \in \mathcal{R}$, *such that* $A \cap B = \emptyset$.
If in the preceding inequality equality holds, then m *will be called* $\perp$*-decomposable measure.*

A set function $\eta : \mathcal{R} \to [0, \infty]$ is called a submeasure if it is monotone nondecreasing, subadditive and $\eta(\emptyset) = 0$.

Theorem 2.3. *If* m *is a monotone* $\perp$*-subadditive set function on a ring* $\mathcal{R}$ *of sets, then the following hold*

(i) $m(A \cup B) \leq m(A) \perp m(B)$ *for arbitrary* $A, B \in \mathcal{R}$,

(ii) if $\perp$ *is continuous at zero, then* $m(A_n) + m(B_n) \to 0$ as $n \to \infty$, for $A_n, B_n \in \mathcal{R}$ $(n \in \mathbb{N})$ *implies*

$$m(A_n \cup B_n) \to 0$$

Proof.

(i) For $A \cap B = \emptyset$, the inequality is true by Definition 2.2. Let $A, B \in \mathcal{R}$ and $A \cap B \neq \emptyset$. Then, we have

$$m(A \cup B) = m(A \cup ((A \cup B) \backslash A)) \leq m(A) \perp m((A \cup B) \backslash A) \leq m(A) \perp m(B).$$

(ii) Follows by property **(i)**.

A set function m : $\mathcal{R} \to [0,1]$ is order continuous (continuous from above at $\emptyset$), if $\lim_{n\to\infty} m(E_n) = 0$ for any sequence (E_n), $E_n \in \mathcal{R}\, (n \in \mathbb{N})$, such that $E_n \searrow \emptyset$

A set function m : $\mathcal{R} \to [0,1]$ is exhaustive, if $\lim_{n\to\infty}(E_n) = 0$ for any sequence (E_n) of pairwise disjoint sets from $\mathcal{R}$.

There exists $\sigma-$ $\perp$-decomposable measure with respect to a continuous t-conorm $\perp$, which is not order continuous.

Example 2.4. Let Σ be the Borel σ-algebra of subsets of the set of real numbers and let f be a continuous function on $\mathbb{R}$ such that $f(0) \neq 0$ and $0 \leq f(x) \leq 1$ $(n \in \mathbb{R})$. Then, the function

$$m(A) = \sup_{x \in A} f(x) \quad (A \in \Sigma)$$

is a $\sigma-$ $\perp$-decomposable measure with respect to the continuous t-conorm $\perp = sup$, but it is not order continuous. Namely, if we take the sequence of open intervals $(0, \frac{1}{n})$, $n \in \mathbb{N}$, then

$$\bigcap_n (0, \frac{1}{n}) = \emptyset, \quad \text{but}$$

$$\lim_{n \to \infty} m((0, \frac{1}{n})) = \lim_{n \to \infty} \sup_{x \in (0, \frac{1}{n})} f(x) = f(0) \neq 0.$$

We have the following generalization of Theorem 3.2. from [5].

Theorem 2.5. *Let $m : \mathcal{R} \to [0,1]$ be a monotone $\perp$-subadditive set function with respect to a continuous at zero t-conorm $\perp$. Then, there exists a submeasure η on $\mathcal{R}$ such that*

$$\lim_{n \to \infty} m(E_n) = 0 \quad \text{iff} \lim_{n \to \infty} \eta(E_n) = 0.$$

The proof is quite analogous to the proof of Theorem 3.2. from [5], using Theorem 2.3. instead of Theorem 3.1. from [5].

Theorem 2.6. *A $\perp$-decomposable measure m on $\mathcal{R}$ is exhaustive iff for every monotone sequence (A_n) for $\mathcal{R}$ holds*

$$m(A_n \bigtriangleup A_m) \to 0 \text{ as } n, m \to \infty,$$

where $A \bigtriangleup B = (A \backslash B) \cup (B \backslash A)$.

Proof. „ $\Rightarrow$" Suppose that the theorem is not true, i.e. that for nondecreasing sequence (A_n) there exist $\epsilon > 0$ and an increasing sequence (i_n) of natural numbers, such that

$$m(A_{i_{n+1}} \bigtriangleup A_{i_n}) > \epsilon \quad (n \in \mathbb{N}).$$

Since $(A_{i_{n+1}} \bigtriangleup A_{i_n})$ is a sequence of pairwise disjoint sets from $\mathcal{R}$, we have a contradiction with the exhaustivity of m.

„ $\Leftarrow$" Suppose now that m is not exhaustive. Then, there exists a sequence (E_n) of pairwise disjoint sets from $\mathcal{R}$ and $\epsilon > 0$ such that

$$m(E_n) > \epsilon \quad (n \in \mathbb{N}).$$

Let $A_n = \bigcup_{k=1}^{n} E_k$. Then, the sequence (A_n) is nondecreasing and since m is monotone, we have for $m > n$

$$m(A_n \,\Delta A_m) = m(\bigcup_{k=n}^{m} E_k) \geq m(E_n) > \epsilon.$$

Contradiction.

3. Extensions

Let $\mathcal{R}_\sigma$ be the class of sets which are limits of increasing sequences from $\mathcal{R}$.

Theorem 3.1. *Let $m : \mathcal{R} \to [0,1]$ be a $\sigma-\perp$-decomposable measure with respect to a continuous t-conorm $\perp$. Then, the function $m^+ : \mathcal{R}_\sigma \to [0,1]$ defined by*

$$(3.1) \qquad m^+(A) := \lim_{n\to\infty} m(A_n) \qquad (A \in \mathcal{R}_\sigma),$$

where (A_n) is any sequence in $\mathcal{R}$ such that $A_n \nearrow A$, is a unique $\sigma\ \perp$-decomposable extension of m on $\mathcal{R}_\sigma$.

Proof. First we shall prove that m^+ is independent from the choice of the sequence (A_n) in (3.1.). Let (A_n') and (A_n'') be two increasing sequences from $\mathcal{R}$ such that $A_n' \nearrow A$ and $A'' \nearrow A$. Since m is continuous from below (Theorem 3.2 (iii) from [8]), we have

$$(3.2) \qquad \lim_{n\to\infty}(\lim_{k\to\infty} m(A_n' \cap A_k'')) = \lim_{n\to\infty} m(A_n')$$

and

$$(3.3) \qquad \lim_{k\to\infty}(\lim_{n\to\infty} m(A_n' \cap A_k'')) = \lim_{k\to\infty} m(A_k'').$$

On the other hand, by the monotonicity of m we have

$$(3.4) \qquad \lim_{n\to\infty}(\lim_{k\to\infty} m(A_n' \cap A_k'')) \leq \lim_{k\to\infty} m(A_k'')$$

and

$$(3.5) \qquad \lim_{k\to\infty}(\lim_{n\to\infty} m(A_n' \cap A_k'')) \leq \lim_{n\to\infty} m(A_n').$$

Then by (3.2) and (3.4) we have

$$(3.6) \qquad \lim_{n\to\infty} m(A_n') \leq \lim_{k\to\infty} m(A_k''),$$

and by (3.3) and (3.5) we obtain

$$(3.7) \qquad \lim_{k\to\infty} m(A''_k) \le \lim_{n\to\infty} m(A'_n).$$

(3.6) and (3.7) imply

$$\lim_{n\to\infty} m(A'_n) = \lim_{n\to\infty} m(A''_n).$$

It is obvius that m^+ is an extension of m to $\mathcal{R}_\sigma$. We shall prove that m^+ is $\perp$-decomposable. Let $A, B \in \mathcal{R}_\sigma$, such that $A \cap B = \emptyset$. Let (A_n) and (B_n) be two sequences from such that $A_n \nearrow A$ and $B_n \nearrow B$. Hence, $A_n \cap B_n = \emptyset$ $(n \in \mathbb{N})$. Since the t-conorm $\perp$ is continuous, we have

$$m^+(A \cup B) =$$

$$= \lim_{n\to\infty} m(A_n \cup B_n) = \lim_{n\to\infty} (m(A_n) \perp m(B_n)) =$$

$$= \lim_{n\to\infty} m(A_n) \perp \lim_{n\to\infty} m(B_n) = m^+(A) \perp m^+(B).$$

m^+ is continuous from below, i.e. if $A, A_n \in \mathcal{R}_\sigma$ $(n \in \mathbb{N})$ and $A_n \nearrow A$, then $m^+(A) = \lim_{n\to\infty} m^+(A_n)$. Hence, by Theorem 3.2 (iii) from [8] follows that m^+ is $\sigma-\perp$-decomposable.

Theorem 3.2. *Let $m : \mathcal{R} \to [0,1]$ be an order continuous $\perp$-decomposable measure with respect to a continuous t-conorm and let Σ be the σ-ring generated by $\mathcal{R}$. Then, m can be extended to a unique monotone order continuous $\perp$-subadditive set function $\overline{m} : \Sigma \to [0,1]$, iff the following conditions hold:*

(a) *If (A_n) is a sequence from $\mathcal{R}$ such that $m(A_n \triangle A_m) \to 0$ as $n, m \to \infty$, then there exists the limit of the sequence $(m(A_m))$,*

(b) *m is exhaustive.*

Proof First we shall prove that conditions (a) and (b) are sufficient. By Theorem 2.5. there exists an order continuous submeasure η or $\mathcal{R}$, such that $m(A_n) \to 0 \Leftrightarrow \eta(A_n) \to 0$. By 7.2. from [3] there exists an order continuous extension $\overline{\eta}$ of η to Σ.

We define the required extension $\overline{m}$ of m to Σ in the following way: for $A \in \Sigma$ we choose a sequence (A_n) from $\mathcal{R}$ such that $\overline{\eta}(A \triangle A_n) \to 0$ as $n \to \infty$ and we take

$$(3.8) \qquad \overline{m}(A) = \lim_{n\to\infty} m(A_n).$$

The function $\overline{m}$ is independent of the choice of the sequence (A_n). It is obvious that $\overline{m}$ extends m. We shall prove that $\overline{m}$ is $\perp$-subadditive. Let $A, B \in \Sigma$, such that $A \cap B = \emptyset$. Let (A_n) and (B_n) be two sequences from $\mathcal{R}$, such that $\overline{\eta}(A \bigtriangleup A_n) \to 0$ and $\overline{\eta}(B \bigtriangleup B_n) \to 0$. Then we have $\overline{m}(A) = \lim_{n\to\infty} m(A_n)$ and $\overline{m}(B) = \lim_{n\to\infty} m(B_n)$.
Using the inclusion

$$(A \cup B)\triangle(A_n \cup B_n) \subset (A\triangle A_n) \cup (B\triangle B_n),$$

we have

$$\overline{\eta}((A \cup B)\triangle(A_n \cup B_n)) \leq \overline{\eta}((A\triangle A_n) \cup (B\triangle B_n)) \leq$$

$$\leq \overline{\eta}((A\triangle A_n)\backslash(B\triangle B_n)) + \overline{\eta}((B\triangle B_n) \leq$$

$$\leq \overline{\eta}((A\triangle A_n) + \overline{\eta}((B\triangle B_n),$$

where we have used that $\overline{\eta}$ is monotone and subadditive. The preceding inequality implies

$$\overline{\eta}((A \cup B)\triangle(A_n \cup B_n)) \to 0 \quad \text{as} \quad n \to \infty.$$

By the definition of the set function $\overline{m}$ (3.8), we have

$$\overline{m}(A \cup B) = \lim_{n\to\infty} m(A_n \cup B_n).$$

Hence, using the continuity of the t-conorm $\perp$

$$\overline{m}(A \cup B) = \lim_{n\to\infty} m(A_n \cup B_n) \leq \lim_{n\to\infty} (m(A_n) \perp m(B_n)) =$$

$$= \lim_{n\to\infty} m(A_n) \perp \lim_{n\to\infty} m(B_n) = \overline{m}(A) \perp \overline{m}(B).$$

The set function $\overline{m}$ is monotone. This follows by the fact that m^+ (for the definition see (3.1)) is monotone on $\mathcal{R}_\sigma$ and that for every $A \in \Sigma$ there exists a sequence (A_n) from $\mathcal{R}_\sigma$, such that $A \subset A_n$ $(n \in \mathbb{N})$, $A_n \searrow$ and

$$\overline{\eta}(A\triangle A_n) = \overline{\eta}(A_n\backslash A) \to 0.$$

Since $\overline{m}$ is monotone and $\perp$-subadditive, by Theorem 2.5., there exists an order continuous submeasure γ on Σ such that $\gamma(A_n) \to 0| \Leftrightarrow m(A_n) \to 0.$

Hence, the restriction of γ to $\mathcal{R}$, $\gamma|I\!R$, satisfies $(\gamma|I\!R)(A_n) \to 0 \Leftrightarrow m(A_n) \to 0$. This implies by 7.3 from [3]

$$\gamma(A_n) \to 0 \Leftrightarrow \overline{\eta}(A_n) \to 0.$$

So, we have $\gamma \sim \overline{\eta}$. Since $\mathcal{R}$ is dense in the complete space $(\Sigma, \overline{\eta})$ (see [3], 3.1, 5.2 and 7.1), the set function $\overline{m}$ is by the condition (a) a continuous (unique) $\perp$-subadditive extension of m from $(\mathcal{R}, \eta)$ to $(\Sigma, \overline{\eta})$.

Since $\overline{m}$ is $\overline{\eta}$-continuous, $\overline{m}$ is order continuous.

Using the preceding facts about submeasures we can prove that condition (a) is also necessary.

If we suppose that $\overline{m}$ is the extension of m to Σ, then by Theorem 2.5 there exists an order continuous submeasure $\overline{\eta}$ on Σ such that

$$\overline{m}(E_n) \to 0 \Leftrightarrow \overline{\eta}(E_n) \to 0.$$

Since each order continuous submeasure on σ-ring is exhaustive, there follows by the preceding equivalence the exhaustivity of $\overline{m}$.

The assumption of the order continuity of the set function m in the preceding theorem is important. Namely, if we drop this assumption, then it can happen that the extension is not unique, as the following example shows.

Example 3.3. We are taking the sup-decomposable measure on the algebra $\mathcal{R}$ generated by closed intervals $[a, b]$, where a and b are rational numbers. We have for special functions f_1 and f_2, which are defined in the following way

$$f_1(x) = \begin{cases} \frac{1}{2} & \text{for} \quad x \in [0, i] \\ \frac{1}{3} & \text{for} \quad x \in (i, 1] \\ 0 & \text{otherwise} \end{cases}, \quad f_2(x) = \begin{cases} \frac{1}{2} & \text{for} \quad x \in [0, i) \\ \frac{1}{3} & \text{for} \quad x \in [i, 1] \\ 0 & \text{otherwise} \end{cases}$$

where i is some irational number from $(0, 1)$, the corresponding measures

$$m_1(A) = \sup_{x \in A} f_1(x) \quad \text{and} \quad m_2(A) = \sup_{x \in A} f_2(x) \quad (A \in \mathcal{R}).$$

Measures m_1 and m_2 are equal on $\mathcal{R}$. The algebra $\mathcal{R}$ generates the minimal σ-algebra Σ, Borel σ-algebra. m_1 is extended by $\overline{m_1}$ defined by

$$\overline{m_1}(A) = \sup_{x \in A} f_1(x) \quad (A \in \Sigma)$$

and m_2 is extended by $\overline{m_2}$ defined by

$$\overline{m_2}(A) = \sup_{x \in A} f_2(x) \quad (A \in \Sigma).$$

Although the sup-decomposable measures m_1 and m_2 are equal on the algebra $\mathcal{R}$, they have different extensions to the σ-algebra Σ. Namely, we have $m_1(\{i\}) = \frac{1}{2}$ and $m_2(\{i\}) = \frac{1}{3}$. As in Example 2.4, we have that $\overline{m_1}$ and $\overline{m_2}$ are σ-sup-decomposable, but they are not order continuous.

References

[1] Dobrakov, I.: On submeasures I, Dissertationes Math. 112, Waszawa, 1974.

[2] Dobrakov, I.: On extension of submeasures, Math. Slovaca 34, 1984, 265-271.

[3] Drewnowski, L.: Topological rings of sets, continuous set function, integration II, Bull. Acad. Polon. Sci. 20 (1972), 277-286.

[4] Drewnowski, L.: On the continuity of certain non-additive set functoins, Colloquium Math. 38 (1978), 243 - 253.

[5] Pap, E.: Lebesgue and Saks decompositions of $\perp$-decomposable measures, Fuzzy Sets and Systems. 38(1990), 345-353.

[6] Pap, E.: On non-additive set functions, Atti Sem. Mat. Fis. Universitá Modena 39(1991), 35-360.

[7] Squillante, M., Ventre, A., Weber, S.: Closure and convergence properties for classes of decomposable measures, Riv. di Math. Universitá Parma (to appear).

[8] Weber, S.: $\perp$-decomposable measures and integrals for Archimedean t-conorms, J. Math. Anal. Appl. 101, (1984), 114-138.

REZIME

PROŠIRENJE NEPREKIDNO t-CONORMA DEKOMPOZABILNE MERE

U radu se ispituju proširenja $\perp$-dekompozabilne mere (u odnosu na neprekidnu t-konormu $\perp$) sa prstena skupova $\mathcal{R}$ na σ-prsten Σ generisan sa $\mathcal{R}$ i na klasu $\mathcal{R}_\sigma$ koja se sastoji od skupova koji su granice rastućih nizova skupova iz $\mathcal{R}$.

Received by the editors February 12, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 131-144

Review of Research
Faculty of Science
Mathematics Series

ON SUMMABLE BOUNDEDNESS IN CONVERGENCE LINEAR SPACES

Endre Pap [1]
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Charles Swartz
New Mexico State University, Department of Mathematical Sciences,
Las Cruses, New Mexico 88003, USA

Abstract

Some properties of K-bounded sets and N-bounded sets in sequential convergence linear spaces are proved and some criterions for N-boundedness are obtained. Examples of K-konvergent, (N-konvergent) and K-bounded (N-bounded) sets are presented.

AMS Mathematics Subject Classification (1980): 54A20, 46A15
Key words and phrases: Convergent linear space, K-convergence, N-convergence, K-boundedness, N-boundedness.

1. Introduction

A kind of summable property of convergences in topological vector spaces was orginally introduced by S.Mazur and W.Orlicz in 1953 [12]. Recently

---

[1] *Partially supported by NSF and SMCI for Research of SAP Vojvodina through funds made available to the US - Yugoslav Joint Board on Scientific and Technological Cooperation (Grant JF 838).*

this property was rediscovered by members of the Katowice Branch of the Mathematical Institute of Polish Academy of Scince. Many authors developed the theory of K-convergent spaces (K-Katowice) for example in [1,2,3,6,9,13-17,20]. The notion of K-convergence has proven to be quite useful and interesting in studying various topics in functional analysis. This is particularly true in the case of the uniform boundedness principle, where the notion of K-boundedness has led to versions of the uniform boundedness principle which are valid in the absence of any completeness assumptions ([1,2,3,20]).
Recently it has been applied in the theory of Adjoint Theorem ([15],[16],[17]), which has an application in the theory of Closed Graph Theorems.

Another kind of summable property of convergences, the so-called N-property (N = Novosibirsk) has been introduced in [19]. In this paper we shall introduce the notion of N-boundedness. We shall also prove some properties of K-bounded sets in convergences linear spaces. We shall obtain some criteria for N-boundedness. We shall present an example which illustrates the relationship between the notion of K-convergent (N-convergent) sequences and K-boundedness (N-boundeness).

2. Convergences, K-sequences and N-sequences

The convergence of sequences of elements in topological spaces is well known. It can be seen as a function which to some sequences assigns elements called limits. It can be also seen as a set of pairs $((x_n), x)$ such that x is a limit of sequence (x_n). In analysis we also deal with cases when convergence of sequences is introduced directly without using topology. Examples for this are pointwise convergence, convergence almost everywhere and others.

In this paper by a convergence in a set X we mean a set G of pairs $((x_n), x)$, where x_n for $n \in N$ and x are elements of X. The meaning of expression $((x_n), x) \in G$ is that the sequence (x_n) converges to x, or equivalently, x is a limit of (x_n). Instead of $((x_n), x) \in G$ we shall equivalently write $x_n \xrightarrow{G} x$, or simply, $x_n \to x$ ([4],[14]). To be able to prove properties of a convergence, or theorems concerning notions defined by the convergence we have to postulate some properties as axioms of the convergence. In this paper we are interested in relations between properties of a convergence and properties of K-convergent, N-convergent sequences and boundedness.

In the sequel we shall be concerned with convergences in Abelian groups and linear spases. We start with recalling the notion of K-sequences and N-sequences. Assume that X is an Abelian group endowed with a convergence G and (x_n) is a sequence of elements x_n in X. It will be convinient to say that a sequence (x_n) is series convergent to x and write

$$\sum_{n=1}^{\infty} x_n = x \quad \text{if} \quad \sum_{k=1}^{n} x_k \to x.$$

It may happen that there is no series convergent sequence when no property is required from G. However, we have the following evident proposition.

Proposition 1.

(a) *If G satisfies the condition*
(S') $x_n \to 0$ *whenever* $x_n = 0$ *for* $n \in N$, *then zero sequences are series convergent to zero.*

(b) *If G satisfies the conditions*
(F) $x_n \to x$ *implies* $x_{m_n} \to x$

and
(L_1) $x_n \to x$ *and* $y_n \to y$ *implies* $x_n - y_n \to x - y$,

then series convergent sequences converge to zero.

It is well known that the completness of metric spaces is of great importance. The procedure of completion makes sense in the case of metric spaces but this is not the case when we deal with convergence spaces. It apears that in many cases the completeness property may be successfully replaced by a summable property of sequences ([1], ,[2]). This was first noticed by Orlicz and Mazur in their paper [12] and independently it was rediscovered and studied at the seminar at Katowice ([1],[2],[5],[9]) conducted by J.Mikusinski and P.Antosik. To be more exact we adopt the following definition.

Definition 1. *A sequence (x_n) of elements in an Abelian group X endowed with a convergence is said to be a K-convergent sequence or shorter K-sequence if each subsequence of (x_n) has a series convergent subsequence.*

If (S') holds, then zero sequences are K-sequences. Sequences converging to zero may not be a K-sequence. For instance, if X is the group of rational numbers with the usual convergence, then the sequence (10^{-10^n}) is not a K-sequence.

Proposition 2. *If a convergence in X satisfies (F), (L_1) and the condition*
(U) *$x_n \not\to 0$ implies that there exists a subsequence (x_{m_n}) of (x_n) whose no subsequence converges to zero,*
then K-sequences converge to zero,

If $x_n \to x$ implies $(x_n - x)$ is a K-sequence we say that X is a K-convergence group. In [9] is given an example of a K-normed linear space, but not complete. In [6] it is proved that K metric groups are of the second category. There are examples of K-topological groups but not of the second category (see [8]).

The more restrictive summable property of sequences has been introduced and studied by S.L.Sobolev in his monograph [19].

Definition 2. *A sequence (x_n) of elements in an Abelian group X endowed with a convergence is said to be an N-sequence, if each subsequence of (x_n) has a subsequence (y_n) which is subseries convergent, i. e. for each subsequence (z_n) of (y_n) the series*

$$\sum_{n=1}^{\infty} z_n$$

converges to an element in X.

Obviously, N-sequences are K-sequences. The converse is not true (see [9]). If $x_n \to x$ in X implies $(x_n - x)$ is an N-sequence, we say that X is an N-convergence group. Recently *Burzyk* [5] has constructed an N-normed space (a dense subspace of c_o) wich is not complete.

3. Boundedness, K-boundedness and N-boundedness

In Funcional Analysis we deal with theorems on uniform bounded ness of families of continuous and linear mappings on bounded subset of the domain of mappings. The theorems are valid under the assumption of completness or Baire category or barelledness type. From the results of paper [2] (see, Theorem 1, p.4), it folows that pointwise boundedfamilies of linear and continuous mappings from a convergence linear space to a topological vector space are uniformly bounded on so-colled K-bounded sets. Bearing in mind to above mentioned result, it is interesting to now the conditions under which bounded sets are K-bounded.

Throughout this section we assume that X is a linear space endowed with a convergence G. We assume that the field of scalars is equipped with the usual convergence. So, the field of scalars is an N-convergence field.

Definition 3. *A subset $B$of X is bounded, (K-bounded, N-bounded), if for each sequence (x_n) of elements x_n in B and for each sequence (α_n) of scalars α_n tending to zero the sequence $(\alpha_n x_n)$ is convergent to zero (K-sequence, N-sequence).*

Obviously, N-bounded subsets of X are K-bounded. The converse is not true even for normed spaces ([9]). In general, N-bounded sets may not be bounded. Take, for instance, the space of measurable function on [0,1] with the almost everywhere convergence.

This can be true even in the case of N-convergence space.

Example 1. Let R be the field of real numbers endowed with convergence G such that $x_n \xrightarrow{G} x$ if $\sum_{n=1}^{\infty} |x_n - x| < \infty$. Then (R, G) is an N-convergence space in which the sequence $(\frac{1}{n})$ is N-bounded but not bounded. The reason for that lies in the fact that $\frac{1}{n} \xrightarrow{G} 0$, but condidion (U) does not hold.

A K-bounded sequence can be not N-bounded even in a K-normed space.

Example 2. We shall take the spaces from the Corollary of Theorem 3 from [11]. Let e_n $(n \in N)$ be unit vectors from l^2. Let $Y = lin\{e_n, n \in N\}$. Then there are two K-subspaces X_1 and X_2 of l^2, such that $X_1 \cap X_2 = Y$ and for each linearly independent sequence (x_n) tending to zero in l^2 there are two subsequences (x_{1n}) and (x_{2n}) of (x_n), such that

$$\sum_{n=1}^{\infty} x_{1n} \in X_1 \text{ but } \sum_{n=1}^{\infty} x_{1n} \notin X_2 \text{ and}$$

$$\sum_{n=1}^{\infty} x_{2n} \notin X_1 \text{ but } \sum_{n=1}^{\infty} x_{2n} \in X_2$$

Then (e_n) is a K-bounded sequence in X_1, but it is not N-bounded. Namely, if we take a sequence (α_n) of scalars α_n $(n \in N)$tending zero, then the sequence $(\alpha_n\, e_n)$ is a K-sequence in X_1 and it is linearly independent and $\alpha_n\, e_n \to 0$. Then , each subsequence of $(\alpha_n\, e_n)$ has a subsequence whose sum belongs to X_2 and does not belong to X_1, i.e. (e_n) is not N-bounded.

If X is a FLUSH linear convergence space (S - stacionarity and H - Hausdorff property,see [4],[14]), then each K-sequence is convergent to zero. Hence, each K-bounded set is bounded but the converse is not always true. But, there are a lot of important spaces for which the converse is also true.

Example 3. A topological vector space (X, τ) is said to be an A-space, if every τ-bounded subset of X is τ-K-bounded ([10],[17]). There are many interesting A-spaces (see [10]). For example, if X is a B-space, then $(X, \sigma(X, X'))$ is an A-space which is not barrelled and not even infrabarrelled (Theorem 5. from [10]). There exist A-spaces which are not K-spaces. For example $(l^p, weak)$, $1 < p < \infty$.

Proposition 3.

(a) *If the convergence in X satisfies (S'), then the set $\{0\}$ is bounded, (K-bounded, N-bounded).*

(b) *If convergence in X satisfies (F), (L_1) and (U), then K-bounded sets are bounded.*

(c) *Subset of X are bounded (K-bounded, N-bounded) if their countable subsets are bounded, (K-bounded, N-bounded).*

(d) *Finite unions of K-bounded (N-bounded) sets are K-bounded (N-bounded).*

(e) *Products by scalaras of bounded (K-bounded, N-bounded) sets are (K-bounded, N-bounded).*

Under additional conditions imposed on the convergence in X we get further properties similar to properties of bounded subsets od topological vectors.

Proposition 4.

(a) *If G satisfies the condition (M_α) $\alpha_n \to 0$ implies $\alpha_n x \to 0$ for each $x \in X$, then finite subsets of X are K-bounded (N-bounded). If, additionally, (U) holds, then finite subsets of X are bounded.*

(b) *Finite algebraic sums of N-bounded subsets of X are N-bounded, whenever G satisfies the condition*
(L_2) $x_n \to x$ and $y_n \to y$ implies $x_n + y_n \to x + y$.

We note that (S) and (L_1) imply L_2, but (S) and (L_2) do not imply (L_1).

Proposition 4(b) is not valid for K bounded sets. This is due to the fact that Cartesian products of K-convergence groups may not be K-convergence groups. This can be shown in the following way .

Example 4. Let e_i for $i \in N$ be unit vectors in l^2 and let $E = span\{e_i : i \in N\}$. E is a dense linear subspace of l^2. Following the proof of Theorem 2 in [9], we find K-subspaces E^1 and E^2 of l^2 such that $E^1 \cap E^2 = E$. We note that $E^1 \times E^2$ is not a K-space. In other words, the Cartesian product of K-spaces may not be a K space. In fact, consider $x_n = \frac{1}{n} e_n$ for $n \in N$. Then, $x_n \in E$ for $n \in N$, $x_n \to 0$ and $(x_n, x_n) \to 0$. Suppose that there exists a summable subsequence (x_{m_n}, x_{m_n}), and $\sum_{n=1}^{\infty} (x_{n_m}, x_{n_m}) = (\sum_{n=1}^{\infty} x_{n_m}, \sum_{n=1}^{\infty} x_{n_m}) = (x, y)$. This implies that $x = y$, $x \in E^1$, $x \in E^2$. Consequently, $x \in E$, since $E^1 \cap E^2 = E$. On the other hand, $x = (\frac{1}{m_1}, \frac{1}{m_2}, \ldots) \notin E$. This contradiction shows that $E^1 \times E^2$ is not a K-space.

We note that the sequence $(x_n, 0)$, $(0, x_n)$ $(n \in N)$ are K-bounded in E^1 and E^2, respectively. But $(x_n, 0) + (0, x_n) = (x_n, x_n)$ and so (x_n, x_n) $(n \in N)$ is not K-bounded. This shows that the algebraic sums of K-bounded sets may not be K-bounded.

Proposition 5.

(a) *If X is a K-convergence group and Y is an N-convergence group, then the Cartesian product $X \times Y$ is a K-convergence group.*

(b) *If A is a K-bounded subsets of X and B is an N-bounded subsets of X, then $A + B$ is a K-bounded subsets of X.*

Example 5. It is interesting to note that the weak convergence in l^2 is not a K-convergence. The sequence (e_n) converges weakly to zero, but it is not K-convergent. At the same time, the sequence is weakly bounded and K-bounded. Moreover, from the results in [3] (see Proposition 1), it follows that weakly bounded subsets of l^2 are K-bounded.
In what follows we shall study the conditions under which bounded subsets of linear space X equipped with a convergence G are K-bounded. To do this we adopt the following definitions.

A sequence (x_n) of elements in X is said to be Cauchy, if for each increasing sequence (p_n) of positive integers $x_{p_{n+1}} - x_{p_n} \rightarrow 0$. Under conditions (L_1), convergent sequences are Cauchy. A subset A of X is complete, if each Cauchy sequence of elements in A converges to an element in A. Closed subsets of a complete topological group are complete.

A sequence (x_n) of elements in X is said to be convex convergent if for any sequence of finite systems of positive integers $p_{n1}, \dots, p_{nk_n}$, such that $min(p_{n1}, \dots, p_{nk_n}) \rightarrow \infty$ and for any sequences of finite systems of nonnegative scalars $\alpha_{p_{n1}}, \dots, \alpha_{p_{nk_n}}$ such that $\alpha_{p_{n1}} + \dots + \alpha_{p_{nk_n}} \leq 1$, the sequence of sums

$$\alpha_{p_{n1}} x_{p_{n1}} + \dots + \alpha_{p_{nk_n}} x_{p_{nk_n}}$$

converges to zero.

Proposition 6. *Subsequence of convex cconvergent sequences are convex convergent.*

Convex convergent sequences converge to zero. Sequences converging to zero in a convex topological vector space are convex convergent.In general, sequences converging to zero in F-spaces are not convex convergent (see [2], p.10). However, sequences converging to zero in F spaces have convex convergent subsequences.This observation leads us to the following definition. A convergence G in a linear spaces X is said to be subsequentially convex convergent if each sequence converging to zero in X has a convex convergent subsequence ([2], p.10). F-spaces are subsequentially convex convergent. We have the following theorem.
Theorem *If G is a complete subsequentially convex convergence in a linear space X, then bounded subsets of X are N-bounded.*

Proof. Assume that A is a bounded subset of linear space X endowed with a complete subsequentially convex convergence. Suppose that $x_n \in A$

for $n \in N$ and (α_n) is a sequence of scalars tending to zero. We should show that $(\alpha_n x_n)$ is an N-sequence. To this end we take a subsequence $(\beta_n y_n)$ of $(\alpha_n x_n)$ and, next, we take a subsequence $\gamma_n t_n$ of $(\beta_n y_n)$ such that

$$\sum_{n=1}^{\infty} \tau_n \leq 1$$

with

$$\tau_n = \sqrt{|\gamma_n|}$$

for $n \in N$. We note that $(\tau_n^{-1} \gamma_n t_n)$ is a subsequentially convex convergent sequence. We may assume that the sequence is convex convergent. Otherwise, we would take a convex convergent subsequence of the sequence. We assert that the series

$$\sum_{n=1}^{\infty} \gamma_n t_n \tag{1}$$

is convergent. In fact, assume that

$$u_n = \tau_n^{-1} \gamma_n t_n$$

for $n \in N$. We have $\tau_n u_n = \gamma_n t_n$ for $n \in N$. We put

$$S_n = \sum_{k=1}^{n} \tau_k u_k$$

. Let (p_n) be an increasing sequence of positive integers. We have

$$S_{p_{n+1}} - S_{p_n} = \sum_{k=p_n}^{p_{n+1}} \tau_k u_k$$

for $n \in N$. Since (u_k) is a convex convergent sequence and

$$\sum_{k=p_n}^{p_{n+1}} \tau_k \leq 1,$$

we have

$$S_{p_{n+1}} - S_{p_n} \to 0.$$

This shows that (1) is a Cauchy series. Since X is complete, then sum of (1) is in X. Hence, by Proposition 6, each subseries of (1) is convergent, which

means that $(\alpha_n x_n)$ is an N-convergent sequence.
Consequently A is an N-bounded set.

Under the same assumptions as in the Theorem, the conclusion of Proposition 2 in [2] is that bounded subsets of X are K-bounded. So, our Theorem is a stronger result than Proposition 2 in [2].

It is well known that a continuous linear map between topological vector spaces is bounded, i.e. preserves bounded sets. It is easy to prove

Proposition 7. *A continuous linear map between convergence linear spaces preserves K-bounded (N-bounded) sets.*

Taking the identity map in the preceding proposition, we obtain
Corollary. If G_1 and G_2 are two convergences on a linear space X such that $G_2 \subset G_1$, then each subset of X which is bounded, K-bounded or N-bounded in G_2 is also bounded, K-bounded or N-bounded, respectively in G_1.

4. Convergence and Boundedness

We note that in case of the topological vector spaces, convergent sequences are bounded, more exactly, sets of members of convergent sequences are bounded. We assume that X and G are as in the section 3.

If G satisfies the condition

$$(M) \quad \alpha_n \to 0 \text{ and } x_n \to x \text{ implies } \alpha_n x_n \to 0,$$

then convergent sequences in X are bounded.

It is interesting to note that if G is induced by a group topology on X and the multiplication by scalar αx is separately continuous, then (M) holds (see |1|). In light of the definitions of K-boundedness and N-boundedness natural questions which suggest themselves are:

(Q_1)Are K – convergent sequences K – bounded?

(Q_2)Are N – convergent sequences N – bounded?

Of course, in a metrizable sequentially complete linear topologocal space, the answer to Q_1 and Q_2 is obviously: yes. We shall show that the answer are generally: no.

First, we present an example of K-convergent sequence which is not K-bounded (see [3]).

Example 6. Let l^∞ be the space of all the bounded real sequences. Let m_0 be the subspace of l^∞ which consist of those sequences (t_j) with a finite range. Pick an element $= (x_j)$ in l^1 such that $x_j \neq 0$ for each j (for example, $x_j = \frac{1}{j^2}$). Define a norm on m_0 induced by x, by means of the formula,

$$\|(t_j)\| = \|(t_j)\|_x = \sum_{j=1}^{\infty} |t_j x_j|.$$

Let e_j be the element of m_0 which has a 1 in the j^{th} coordinate and 0 elsewhere.

We first claim that the sequence (e_j) in $(m_0, \| \quad \|_x)$ is N-convergent. This is immediate, since for any subsequence (e_{k_j}) if $y = (y_i) \in m_0$ is defined by $y_i = 1$ if $i = k_j$ for some j and $y_i = 0$ otherwise, we have

$$\|y - \sum_{j=1}^{n} e_{k_j}\| = \sum_{j=n+1}^{\infty} |x_{k_j}| \to 0.$$

The sequence (e_j) is not K-bounded in $(m_0, \| \|_x)$, since if (t_j) is any sequence of scalars which converges to 0 with $t_j \neq t_i$ for $i \neq j$, then no subseries of $\sum t_{k_j} e_{k_j}$ will converge to an element of m_0. (Any subseries $\sum t_{k_j} e_{k_j}$ will converge coordinatewise to an element of $l^\infty \backslash m_0$).

The method used in the construction of the example above can be generalized to yield a large number of examples of K-convergent sequences which are not K-bounded (see [3], p.15).

The above example 6 shows that even N-convergent sequences are not K-bounded. This gives negative answers to both questions.

K-bounded sets are also ipmortant in the construction of the locally convex topology which is used in the theory of the uniform Boundedness Principle ([2],[3],[20]) and in the theory of the Adjoint Theorem ([15],[16],[17]).

Example 7. Let X be a locally convex space and X', its dual. If we denote by $K(X, X')$, the topology of uniform convergence on $\sigma(X', X)$ - K-bounded subsets of X', then $K(X, X')$ is stronger than the Mackey topology $\tau(X, X')$ (Proposition 1 from [20]), but still have same bounded sets.

We have to express our gratitude to prof. *P. Antosik* for his valuable help in preparing this paper.

References

[1] Antosik, P.: Sur les suites d'applications, C.R. Acad. Sci., Paris, 287 A (1978), 75-77.

[2] Antosik, P.: On uniform boundedness of families of mappings, Proc. of the Conf. on Conv., Szczyrk (1979), 1-16.

[3] Antosik, P., Swartz, C.: Matrix Method in Analysis, Lecture Notes in Mathematics, Vol. 1113, Springer-Verlag,1985.

[4] Bednarek, A.R., Mikusinski, J.: Convergence and topology, Bull. Acad. Polon. Ser. Sci. Math. Astronom. Phys. 17 (1969), 437-442.

[5] Burzyk, J.: An example of non-complete normed N - space, Bull. Polish Acad. Sci. Math. 35(1987), 449-455.

[6] Burzyk, J., Kliś, Cz., Lipecki, Z.: On K-property of metric spaces, Colloq. Math. 49 (1984), 33-39.

[7] Drewnowski, L.: Equivalence of Brooks-Jelwett, Vitali-Hahn-Saks and Nikodym Theorems, Bull. Acad. Polon. Ser. Sci. Math. Astronom. Phys. 20 (1972). 725-731.

[8] Foged, L.: The Baire category for Frechet groups in which every null sequence has a summable subsequence, Proceedings of the Conference on Topologies in Huston, 1983.

[9] Kliś, Cz.: An example of non-complete normed (K)-space, Bull. Acad. Polon. Ser. Sci. Math. Astronom. Phys. 26 (1978),415-420.

[10] Li, R., Swartz, C.: Spaces for which the uniform boundedness principle holds (to appear).

[11] Lipecki, Z.: On some dense subspaces of topological linear spaces, Coll. Math.

[12] Mazur, S., Orlicz, W.: Sur les espaces metriques lineares II, Studia Math. 13 (1953), 137-179.

[13] Pap, E.: Contributions to functional analysis on convergence spaces, Proc. of the Fifth Prague Top. Symp. 1981,Heldermann-Verlag, (1983), 538-543.

[14] Pap E.: Funkcionalna analiza, Institute of Mathematics, Novi Sad,1982.

[15] Pap, E., Swartz, C.: A locally convex version of adjoint theorem (to appear).

[16] Pap, E., Swartz, C.: On the Closed Graph Theorem, Proc. of the Conference on Generalised Functions and Convergence, Katowice 1988, World Sci. Publ.,1990, 355-359.

[17] Pap, E., Swartz, C.: The Adjoint Theorem on A-spaces, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. (to appear).

[18] Rolewicz, S.: Metric Linear Spaces, Polish Scientific Publishers, Warsaw, 1972.

[19] Sobolev, L.S.: Vvedenije v teoriju kubaturnyuh formul, Nauka, Moskva, 1974.

[20] Swartz, C.: A generalization of Mackey's theorem and the uniform boundedness principle, Bull. Australian Math. Soc. Vol. 40 (1989), 123-128.

REZIME

O SUMABILNOJ OGRANIČENOSTI U KONVERGENTNOM VEKTORSKOM PROSTORU

U radu su dokazane neke osobine K-ograničenih i N-ograničenih skupova u nizovno konvergentnom vektorskom prostoru. Dobijeni su i neki kriterijumi za N-ograničenost. Dat je primer K-konvergentnog (N-konvergentnog) niza koji nije K-ograničen (N-ograničen).

Received by the editors February 19, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 145-151

Review of Research
Faculty of Science
Mathematics Series

ON THE BEHAVIOUR OF THE STIELTJES TRANSFORMATION AT THE ORIGIN

Danica Nikolić-Despotović and **Stevan Pilipović**
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

The behaviour of the Stieltjes transformation at the origin is investigated for a function f from L^1_{loc}, $supp\ f \subset [0,\infty)$ with the asymptotic expansion of the form

$$\sum_{i=p}^{\infty} a_i t^i,\ t \to 0.$$

This new result follows from the known one for the asymptotic expansion at ∞. Since the notion of the quasiasymptotic expansion of distributions is used, the result can be formulated for the distributional Stieltjes transformation of distributions.

AMS Mathematics Subject Classification (1980): 46F12, 44A15.
Key words and phrases: Stieltjes transformation of distributions, quasi-asymptotic expansion of distributions.

1. Notations and known results

By $\mathcal{S}$ and $\mathcal{S}'$ we denote the space of rapidly decreasing functions and tempered distributions, respectively. The space $J'(r)$, $r \in \mathbf{R} \setminus (-\mathbf{N})$ is defined in [2] as a subspace of $\mathcal{S}'_+ = \{f \in \mathcal{S}';\ supp\ f \subset [0,\infty)\}$ consisting of all f of the form

(1) $$f = F^{(m)} \text{ for some } m \in \mathbf{N}_0 = \mathbf{N} \cup \{0\},$$

$$F \in L^1_{loc},\ supp\ F \subset [0,\infty),$$

such that

$$\int_0^\infty \left|\frac{F(t)}{(t+x)^{r+m+1}}\right| dt < \infty,\ x > 0. \tag{2}$$

$I'(r)$ is the subspace of $J'(r)$ consisting of all $f \in J'(r)$ for which (1) holds and instead of (2) there holds

(3) $|F(t)| < C(1+t)^{r+m-\varepsilon}$, $t > 0$, for some $C = C(F)$, $\varepsilon = \varepsilon(F) > 0$.

The distributional Stieltjes tansformation S_r of index r, $r \in \mathbf{R} \setminus (-\mathbf{N})$,

$$(S_r f)(z) = (r+1)_m \int_0^\infty \frac{F(t)}{(t+z)^{r+m+1}} dt,\ z \in \mathbf{C} \setminus (-\infty, 0]. \tag{4}$$

where $(a)_m = a(a+1)\dots(a+m-1)$, $m \in \mathbf{N}$, $(a)_0 = 1$.
It is easy to see that $S_r f$ is a holomorphic function of the complex variable z on the domain $\mathbf{C} \setminus (-\infty, 0]$.
By L we denote a slowly varying function at ∞ (0^+). For the properties of such a function we refer to [5] .
In our investigations of the distributional Stieltjes transformation, the notion of quasiasymptotic behaviour at ∞ (0^+) plays a fundamental role ([1]).

Recall, $f \in \mathcal{S}'_+$ has the quasiasymptotic behaviour at ∞ (0^+) with respect to $k^\alpha L(k)$ $((1/k)^\alpha L(1/k))$ with the limit $g \in \mathcal{S}'_+$ if

$$\lim_{k\to\infty} < \frac{f(kt)}{k^\alpha L(k)}, \varphi(t) > = < g(t), \varphi(t) >,\ \varphi \in \mathcal{S}$$

(5)

$$(\lim_{k\to\infty} < \frac{f(t/k)}{(1/k)^\alpha L(1/k)}, \varphi(t) > = < g(t), \varphi(t) >,\ \varphi \in \mathcal{S}).$$

It is well known that g in (5) must be of the form $g = C f_{\alpha+1}$, where

$$f_{\alpha+1}(t) = \begin{cases} \frac{H(t)t^\alpha}{\Gamma(\alpha+1)}, & \alpha > -1 \\ f^{(n)}_{\alpha+n+1}(t), & \alpha \leq -1 \text{ for some } n \in \mathbf{N} \text{ with } n+\alpha > -1 \end{cases} \quad (t \in \mathbf{R})$$

where H is Heaviside' s function.
We shall need the following relation from [4].

Let $f \in J'(r)$ have the quasiasymptotic behaviour at 0^+ with respect to $(1/k)^\alpha L(1/k)$. Then,

$$(z/k)^{r+m+1}(S_r f)(z/k) = (r+1)_m (S_{r+m}\Phi)(k/z), \tag{6}$$

where $\Phi(t) = t^{r+m-1}F(1/t)$ for $t > 0$ and $\Phi(t) = 0$ for $t \leq 0$. Obviously,

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^{r-\alpha-1}L_1(t)} = \frac{C}{\Gamma(\alpha+m+1)}$$

($L_1(k) = L(1/k)$ is slowly varying at ∞) and Φ is locally integrable on $\mathbf{R}$.

We also need the following proposition from [3]

Proposition 1. *Let $f \in L^1_{loc}(0,\infty)$ and let*

$$f(t) \sim \sum_{i=1}^{\infty} \frac{a_i}{t^i},\ t \to \infty.$$

Then for $r \in \mathbf{N}_0$ and $n \geq 2$ we have the asymptotic expansion at infinity

$$\begin{aligned}(S_r f)(k) \sim{}& \frac{1}{k^{r+1}}(a_1 B_1 + T_1 + a_n m_0) + \sum_{m=1}^{n-1} \frac{(-1)^m a_m (r+1)_{m-1} \ln k}{(m-1)!k^{r+m}} + \\ &+ \sum_{l=2}^{n-2} \frac{(-1)^l}{k^{r+l}} \left[\Gamma(r+l) \sum_{i=1}^{l-1} \frac{a_i (r+1)_i}{(l-i)\Gamma(l+1-i)\Gamma(r+i)(i-1)!} - \right. \\ &\left. - \sum_{j=0}^{l-1} \frac{(r+1)_j T_{j+1}}{\Gamma(l-j)\Gamma(r+l+j)} + a_l \frac{(r+1)_l B_l^*}{(l-1)!} - m_{l-1}(r+1)_{l-1} \right] + \\ &+ \frac{(-1)^{n-1}\Gamma(r+n-1)}{k^{r+n-1}} \left[\sum_{i=1}^{n-2} \frac{a_i (r+1)_i}{(n-1-i)\Gamma(n-i)\Gamma(r+i)(i+1)!} - \right. \\ &\left. - \sum_{j=0}^{n-3} \frac{(r+1)_j T_{j+1}}{\Gamma(n-1-j)\Gamma(r+j)} - \frac{a_{n-1}(r+1)_{n-1} B_{n-1}^*}{(n-1)!} - m_{n-1}(r+1)_{n-2} \right] + \\ &+ \frac{(-1)^{n-1}}{(n-1)!}(r+1)_{n-1} \ln k / k^{r+n} + o(\ln k/k^{r+n}),\end{aligned}$$

where

$$\left\{ \begin{array}{ll} B_s = & \frac{(-1)^{r+s-1}}{\Gamma(r+s)} (1/z \ln z/(1+z))^{(r+s-1)}|_{z=1} + \\ & + \int_1^\infty \ln u/(1+u)^{r+s-1} du \\ B_s^* = & 1/(r+s) B_s, \end{array} \right. \qquad s \in \mathbf{N} \tag{7}$$

$$T_1 = a_2 + \frac{a_3}{2} + \cdots + \frac{a_{n-1}}{(n-2)},$$

$$T_2 = \frac{a_3}{2!} + \frac{a_4}{3 \cdot 2} + \cdots + \frac{a_{n-1}}{(n-2)(n-3)},$$

$$T_3 = \frac{a_4}{3!} + \frac{a_5}{4 \cdot 3 \cdot 2} + \cdots + \frac{a_{n-1}(n-5)!}{(n-2)!},$$

$$\vdots$$

$$T_{n-2} = \frac{a_{n-1}}{(n-2)!}$$

and

$$m_i = \int_0^\infty t^i \frac{f(t) - \sum_{j=1}^{n-1} a_j H(t-1)/t^j}{i!} dt, \quad i = 1, \ldots, n-2.$$

2. Application

We shall investigate the behaviour of the Stieltjes transformation at the orgin for a function f, *supp* $f \subset [0,\infty)$ and $f \in L^1_{loc}(A,\infty)$, $A > 0$, with the ordinary asymptotic expanion of the form

$$\sum_{i=p}^{\infty} a_i t^i, \quad t \to 0$$

using (6) and Proposition 1.

Let $m = 0$, $z = 1$, then from (6) we obtain

$$(1/k)^{r+1}(S_r f)(1/k) = (S_r \Phi)(k), \tag{8}$$

where

$$\Phi(t) = t^{r-1} f(1/t).$$

Let $r \in \mathbf{N}_0$, *supp* $f \subset [0,\infty)$, $f \in L^1_{loc}(A,\infty)$ for some $A > 0$ and

$$f(t) \sim \sum_{i=p}^{\infty} a_i t^i, \; t \to 0,$$

then

$$\Phi(t) = t^{r-1} f(1/t) \sim \sum_{i=p}^{\infty} \frac{a_i}{t^{i+1-p}} = \sum_{j=p+1-r}^{\infty} \frac{a_{j+r-1}}{t^j}, \; t \to \infty$$

and $\Phi \in L^1_{loc}(0, 1/A)$. If $p \geq r$ then the integral of Φ over $(1/A, \infty)$ is finite. From Proposition 1 we have:

Proposition 2. *Let supp* $f \subset [0, \infty)$, $f \in L^1_{loc}(A, \infty)$ *for some* $A > 0$ *and*

$$f(t) \sim \sum_{i=p}^{\infty} a_i t^i, \; t \to 0.$$

Then for $n \geq p + 2 - r$,

$$
\begin{aligned}
(9) \qquad & (S_r f)(s) \sim \tilde{T}_1 + a_{n+r-1}\tilde{m}_0 - \\
& - \sum_{m=p-r+1}^{n-1} \frac{(-1)^m a_{m+r-1}(r+1)_{m-1}}{(m-1)!} s^{m-1} \ln s + \\
& + \sum_{l=2}^{n-2} (-1)^l s^{l-1} \left[\Gamma(r+l) \sum_{i=p-r+1}^{l-1} \frac{a_{i+r-1}(r+1)_i}{(l-i)!\Gamma(l+1-i)\Gamma(r+i)(i-1)!} - \right. \\
& \left. - \sum_{j=0}^{l-1} \frac{(r+1)_j \tilde{T}_{j+1}}{\Gamma(l-j)\Gamma(r+1+j)} + a_{l+r-1}(r+1)_l \frac{B^*_l}{(l-1)!} - \tilde{m}_{l-1}(r+1)_{l-1} \right] + \\
& + (-1)^{n-1}\Gamma(r+n-1)s^{n+l-1} \left[\sum_{i=p-r+1}^{n-2} \frac{a_{i+r-1}(r+1)_i}{(n-1-i)\Gamma(n-i)\Gamma(r+1)(i+1)!} - \right. \\
& \left. - \sum_{j=0}^{n-3} \frac{(r+1)_j \tilde{T}_{j+1}}{\Gamma(n-1-j)\Gamma(r+j)} - \frac{a_{n+r-2}(r+1)_{n-1}}{(n-1)!} - \tilde{m}_{n-1}(r+1)_{n-2} \right] - \\
& - \frac{(-1)^{n-1}}{(n-1)!}(r+1)_{n-1} s^{n-1} \ln s + o(s^{n-1} \ln s), \; s \to 0
\end{aligned}
$$

where

$$\tilde{T}_1 = \frac{a_p}{p-r} + \cdots + \frac{a_{n+r-2}}{(n-2)},$$

$$\tilde{T}_2 = \frac{a_{p+1}}{(p-r+1)(p-r)} + \cdots + \frac{a_{n+r-2}}{(n-2)(n-3)},$$

$$(10) \qquad \tilde{T}_3 = \frac{a_{p+2}}{(p+r-2)(p-r+1)(p-r)} + \cdots + \frac{a_{n+r-2}(n-5)!}{(n-2)!},$$

$$\vdots$$

$$\tilde{T}_{n-2} = \frac{a_{n+r-2}}{(n-2)!}$$

$$(11) \qquad \tilde{m}_i = \int_0^\infty \frac{t^i}{i!}\left[t^{r-1}f(1/t) - \sum_{j=p-r+1}^{n-1} a_{j+r-1}H(t-1)/t^j\right]dt,$$

$i = 0, 1, \ldots, n-2$ and $B_s^*, s \in \mathbf{N}$, is introduced by (7).

Proof. From (8) it follows that

$$(1/k)^{r+1}(S_r f)(1/k) = (S_r\Phi)(k) \sim$$

$$\sim (1/k)^{r+1}(\tilde{T}_1 + a_{n+r-1}\tilde{m}_0) + \sum_{m=p-r+1}^{n-1} \frac{(-1)^m a_{m+r-1}(r+1)_{m-1}\ln k}{(m-1)!k^{r+m}} +$$

$$+ \sum_{l=2}^{n-2} \frac{(-1)^l}{k^{r+l}}\left[\Gamma(r+l)\sum_{i=p-r+1}^{l-1} \frac{a_{i+r-1}(r+1)_i}{(l-i)!\Gamma(l+1-i)\Gamma(r+i)(i-1)!} -\right.$$

$$\left. - \sum_{j=0}^{l-1} \frac{(r+1)_j \tilde{T}_{j+1}}{\Gamma(l-j)\Gamma(r+1+j)} + a_{l+r-1}\frac{(r+1)_l B_l^*}{(l-1)!} - \tilde{m}_{l-1}(r+1)_{l-1}\right] +$$

$$+ \frac{(-1)^{n-1}\Gamma(r+n-1)}{k^{r+n+l}}\left[\sum_{i=p-r+1}^{n-2} \frac{a_{i+r-1}(r+1)_i}{(n-1-i)\Gamma(n-i)\Gamma(r+1)(i+1)!} -\right.$$

$$\left. - \sum_{j=0}^{n-3} \frac{(r+1)_j \tilde{T}_{j+1}}{\Gamma(n-1-j)\Gamma(r+j)} - \frac{a_{n+r-2}(r+1)_{n-1}}{(n-1)!} - \tilde{m}_{n-1}(r+1)_{n-2}\right] +$$

$$+ \frac{(-1)^{n-1}}{(n-1)!}(r+1)_{n-1}\frac{\ln k}{k^{r+n}} + o(\frac{\ln k}{k^{r+n}}),$$

where $\tilde{T}_i$, $i = 0, 1, \ldots, n-1$, is given by (10) and $\tilde{m}_i$ is given by (11). If we put $1/k = s$, then we obtain the Proposition 2.

References

[1] Vladimirov, V. S., Drozzinov, Ju. N., Zavi'alov, B. I.: Multidimensional Tauberian Theorems for Generalized Functions. Nauka, Moscow, 1986, (In Russian).

[2] Lavoine, J., Misra, D. P.: Abelian theorem for the distributional Stieltjes transformation. Proc. Camb. Phil. Soc. 86(1979), 287–293.

[3] Nikolić-Despotović, D., Pilipović, S.: Quasiasymptotic expansion and the Stietjes transformation (to appear).

[4] Pilipović, S.: On the Behaviour of the Distributional Stieltjes Transformation at the Origin. Zeitschrift für Analysis und ihre Anwendungen Bd. 8(2)1989, 169–176.

[5] Seneta, E.: Regularly Varying Function (Lect. Notes Math. 508) Berlin-Heidelberg-New York, Springer-Verlag 1976.

REZIME

PONAŠANJE STIELTJESOVE TRANSFORMACIJE U NULI

Dokazano je tvrdjenje o ponašanju Stieltjesove transformacije u nuli, funkcije f sa osobinom $supp\ f \subset [0, \infty)$, $f \in L^1_{loc}(A, \infty)$ za neko $A > 0$ ako f ima asimptotski razvoj oblika

$$\sum_{i=p}^{\infty} a_i t^i, \ t \to 0.$$

Received by the editors February 8, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 153-159

Review of Research
Faculty of Science
Mathematics Series

A COMMON FIXED POINT THEOREM FOR TWO SEQUENCES OF MAPPINGS IN CONVEX METRIC SPACES

Olga Hadžić
Institute of Mathematics, University of Novi Sad,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In this paper we shall give a generalization of Theorem A from [3] in convex metric spaces.

AMS Mathematics Subject Classification (1980): 47H10
Key words and phrases: Common fixed point theorems, convex metric spaces.

1. Introduction

In [1] Assad and Kirk have proved the following theorem: Let (X, d) be a complete and convex metric space, C a nonempty closed subset of X, T a contraction mapping of C into $CB(X)$. If $T(\partial C) \subseteq C$ then there exists $u \in C$ such that $u \in Tu$.

In this theorem the convexity of X means that for each $x, y \in X$ with $x \neq y$, there exists $z \in X, \quad x \neq z, \quad y \neq z$, such that $d(x,z) + d(z,y) = d(x,y)$.

There are many fixed point theorems and common fixed point theorems in convex metric spaces for singlevalued and multivalued mappings and family of mappings.

T.Taniguchi generalized in [3] Theorem 1 from [2]. In this paper we shall prove a generalization of Theorem A from [3] in the case of a convex metrics space

2. The common fixed point theorem

Theorem 1. *Let (X,d) be a complete, convex metric space, K a nonempty, closed subset of X, $B_i : X \to X$ $(i \in \mathrm{N})$ and $A_i : K \to X$ continuous mappings so that $\partial K \subseteq B_i(K) \subseteq K$, $A_i K \cap K \subseteq B_{i+1} K$ and*

$$B_i x \in \partial K \Rightarrow A_i x \in K \quad \textit{for every} \quad i \in \mathrm{N}.$$

Suppose that the following conditions are satisfied for all $m, n \in \mathrm{N}$ and all $x, y \in K$:

a) there exists a constant $k < 1$, such that

$$d(A_{2n-1}x, A_{2n}y) \leq kd(B_{2n-1}x, B_{2n}y)$$

$$d(A_{2n}x, A_{2m+1}y) \leq kd(B_{2n}x, B_{2m+1}y), \quad \textit{for all} \quad m \geq n \geq 1.$$

b)

$$A_{2n}B_{2m}x = B_{2m}A_{2n}x \quad \textit{and} \quad A_{2n-1}B_{2m-1}x = B_{2m-1}A_{2n-1}x$$

c)

$$B_{2n}B_{2m}x = B_{2m}B_{2n}x \quad \textit{and} \quad B_{2m-1}B_{2n-1}x = B_{2n-1}B_{2m-1}x,.$$

Then there exists a unique common fixed point for two sequences $\{A_n\}_{n\in\mathrm{N}}$ and $\{B_n\}_{n\in\mathrm{N}}$.

Proof. Let $p \in \partial K$ and $p_0 \in K$ such that $p = B_1 p_0$. The existence of such an element follows from the condition $\partial K \subseteq B_1 K$. Since $B_1 p_0 \in \partial K$ it follows that $A_1 p_0 \in K$ and so from the condition $A_1 p_0 \in A_1 K \cap K \subseteq B_2 K$ it follows the existence of an element $p_1 \in K$ such that $B_2 p_1 = A_1 p_0$. Let $p_1' = A_1 p_0$ and $p_2' = A_2 p_1$. If $p_2' \in K$ it follows that $A_2 p_1 \in A_2 K \cap K \subseteq B_3 K$ and so the exists $p_2 \in K$ such that $B_3 p_2 = A_2 p_1$. If $p_2' \notin K$ then there exists $p_2 \in K$ such that $B_3 p_2 \in \partial K$ and

$$d(B_2 p_1, B_3 p_2) + d(B_3 p_2, A_2 p_1) = d(B_2 p_1, A_2 p_1).$$

If we proceed in this way we obtain two sequences $\{p_n\}_{n\in\mathbf{N}}$ and $\{p'_n\}_{n\in\mathbf{N}}$ such that for every $n \in \mathbf{N}$, $p_n \in K$, $p'_{n+1} = A_{n+1}p_n$ and the following implications hold for every $n \in \mathbf{N}$:

(i) $p'_{2n} \in K \Rightarrow p'_{2n} = B_{2n+1}p_{2n}$

$$p'_{2n} \notin K \Rightarrow B_{2n+1}p_{2n} \in \partial K \quad \text{and}$$

$$d(B_{2n}p_{2n-1}, B_{2n+1}p_{2n}) + d(B_{2n+1}p_{2n}, A_{2n}p_{2n-1}) =$$

$$= d(B_{2n}p_{2n-1}, A_{2n}p_{2n-1}).$$

(ii) $p'_{2n+1} \in K \Rightarrow p'_{2n+1} = B_{2n+2}p_{2n+1}$

$$p'_{2n+1} \notin K \Rightarrow B_{2n+2}p_{2n+1} \in \partial K \quad \text{and}$$

$$d(B_{2n+1}p_{2n}, B_{2n+2}p_{2n+1}) + d(B_{2n+2}p_{2n+1}, A_{2n+1}p_{2n}) =$$

$$= d(B_{2n+1}p_{2n}, A_{2n+1}p_{2n}).$$

We shall prove that there exists $z \in K$ such that $z = \lim_{n\to\infty} B_n p_{n-1}$. Let

$$P_0 = \{p_{2n};\ p'_{2n} = B_{2n+1}p_{2n}\},$$

$$P_1 = \{p_{2n};\ p'_{2n} \neq B_{2n+1}p_{2n}\},$$

$$Q_0 = \{p_{2n+1};\ p'_{2n+1} = B_{2n+2}p_{2n+1}\},$$

$$Q_1 = \{p_{2n+1};\ p'_{2n+1} \neq B_{2n+2}p_{2n+1}\}.$$

If $p_{2n} \in P_1$ then $B_{2n+1}p_{2n} \in \partial K$ which implies that $A_{2n+1}p_{2n} = p'_{2n+1} \in K$. From this it follows that $p'_{2n+1} = B_{2n+2}p_{2n+1}$ and so $p_{2n+1} \in Q_0$. It is easy to see that we have the following posibilities:

$$(p_{2n}, p_{2n+1}) \in P_0 \times Q_0; \ (p_{2n}, p_{2n+1}) \in P_0 \times Q_1;$$

$$(p_{2n}, p_{2n+1}) \in P_1 \times Q_0;$$

$$a) \qquad (p_{2n}, p_{2n+1}) \in P_0 \times Q_0$$

$$d(B_{2n+1}p_{2n}, B_{2n+2}p_{2n+1}) = d(A_{2n}p_{2n-1}, A_{2n+1}p_{2n}) \leq$$

$$\leq q d(B_{2n}p_{2n-1}, B_{2n+1}p_{2n}).$$

$$b) \qquad (p_{2n}, p_{2n+1}) \in P_0 \times Q_1$$

$$d(B_{2n+1}p_{2n}, B_{2n+2}p_{2n+1}) = d(B_{2n+1}p_{2n}, A_{2n+1}p_{2n}) -$$

$$d(B_{2n+2}p_{2n+1}, A_{2n+1}p_{2n}) \leq d(B_{2n+1}p_{2n}, A_{2n+1}p_{2n})$$
$$= d(A_{2n}p_{2n-1}, A_{2n+1}p_{2n}) \leq qd(B_{2n}p_{2n-1}, B_{2n+1}p_{2n})$$

$$c) \qquad (p_{2n}, p_{2n+1}) \in P_1 \times Q_0$$

$$d(B_{2n+1}p_{2n}, B_{2n+2}p_{2n+1}) \leq d(B_{2n+1}p_{2n}, A_{2n}p_{2n-1}) +$$
$$+ d(A_{2n}p_{2n-1}, B_{2n+2}p_{2n+1}) =$$
$$= d(B_{2n+1}p_{2n}, A_{2n}p_{2n-1}) + d(A_{2n}p_{2n-1}, A_{2n+1}p_{2n}) \leq$$
$$\leq d(B_{2n+1}p_{2n}, A_{2n}p_{2n-1}) + qd(B_{2n}p_{2n-1}, B_{2n+1}p_{2n}) \leq$$
$$\leq d(B_{2n}p_{2n-1}, B_{2n+1}p_{2n}) + d(B_{2n+1}p_{2n}, A_{2n}p_{2n-1}) +$$
$$= d(B_{2n}p_{2n-1}, A_{2n}p_{2n-1}).$$

Since $p_{2n} \in P_1$ implies that $p_{2n-1} \in Q_0$ it follows that $B_{2n}p_{2n-1} = A_{2n-1}p_{2n-2}$ and so

$$d(B_{2n+1}p_{2n}, B_{2n+2}p_{2n+1}) \leq d(B_{2n}p_{2n-1}, A_{2n}p_{2n-1}) =$$
$$= d(A_{2n-1}p_{2n-2}, A_{2n}p_{2n-1}) \leq qd(B_{2n-1}p_{2n-2}, B_{2n}p_{2n-1}).$$

Similarly we can prove the following implications:

$$(p_{2n-1}, p_{2n}) \in Q_0 \times P_0 \Rightarrow d(B_{2n}p_{2n-1}, B_{2n+1}p_{2n}) \leq$$
$$\leq qd(B_{2n-1}p_{2n-2}, B_{2n}p_{2n-1});$$
$$(p_{2n-1}, p_{2n}) \in Q_1 \times P_0 \Rightarrow d(B_{2n}p_{2n-1}, B_{2n+1}p_{2n}) \leq$$
$$\leq qd(B_{2n-1}p_{2n-2}, B_{2n-2}p_{2n-3});$$
$$(p_{2n-1}, p_{2n}) \in Q_0 \times P_1 \Rightarrow d(B_{2n}p_{2n-1}, B_{2n+1}p_{2n}) \leq$$
$$\leq qd(B_{2n-1}p_{2n-2}, B_{2n}p_{2n-1}).$$

It is easy to prove that

$$d(B_{2n+1}p_{2n}, B_{2n+2}p_{2n+1}) \leq q^{n-1} \cdot r$$

$$d(B_{2n+2}p_{2n+1}, B_{2n+3}p_{2n+2}) \leq q^{n} \cdot r$$

where $r = \max\{d(B_3p_2, B_4p_3),\ d(B_3p_2, B_2p_1)\}$, which imply the existence of $z \in K$ such that

$$z = \lim_{n \to \infty} B_np_{n-1}.$$

There exists at least one sequence $\{B_{2n_k+1}p_{2n_k}\}_{k\in\mathbf{N}}$ or $\{B_{2m_k+2}p_{2m_k+1}\}_{k\in\mathbf{N}}$ such that $p_{2n_k} \in P_0 (k \in \mathbf{N})$ or $p_{2m_k+1} \in Q_0 (k \in \mathbf{N})$.

Suppose that there exists $\{n_k\}$ such that

$$B_{2n_k+1}p_{2n_k} = A_{2n_k}p_{2n_k-1}, \quad \text{for every} \quad k \in \mathbf{N}.$$

We shall prove that

$$\lim_{k\to\infty} A_{2n_k+1}p_{2n_k} = z. \tag{1}$$

Relation (1) follows from

$$d(A_{2n_k+1}p_{2n_k}, B_{2n_k+1}p_{2n_k}) = d(A_{2n_k+1}p_{2n_k}, A_{2n_k}p_{2n_k-1})$$
$$\leq qd(B_{2n_k+1}p_{2n_k}, B_{2n_k}p_{2n_k-1})$$

since

$$\lim_{k\to\infty} d(A_{2n_k+1}p_{2n_k}, B_{2n_k+1}p_{2n_k})$$
$$\leq q \cdot \lim_{k\to\infty} d(B_{2n_k+1}p_{2n_k}, B_{2n_k}p_{2n_k-1}) = 0$$

and

$$\lim_{k\to\infty} B_n p_{n-1} = z.$$

Further, for every $m \in \mathbf{N}$

$$B_m z = B_{m+1} z. \tag{2}$$

In order to prove (2) we shall prove that $B_{2m}z = B_{2m+1}z$, and $B_{2m}z = B_{2m-1}z$ for every $m \in \mathbf{N}$.

Since

$$z = \lim_{k\to\infty} A_{2n_k}p_{2n_k-1} = \lim_{k\to\infty} A_{2n_k+1}p_{2n_k}$$

it follows from the continuity of B_n that

$$d(B_{2m}z, B_{2m+1}z) = d(B_{2m}(\lim_{k\to\infty} A_{2n_k}p_{2n_k-1}), B_{2m+1}(\lim_{k\to\infty} A_{2n_k+1}p_{2n_k})) =$$
$$= \lim_{k\to\infty} d(B_{2m}A_{2n_k}p_{2n_k-1} B_{2m+1}A_{2n_k+1}p_{2n_k})$$
$$= \lim_{k\to\infty} d(A_{2n_k}B_{2m}p_{2n_k-1}, A_{2n_k+1}B_{2m+1}p_{2n_k}) \leq$$
$$\leq q \lim_{k\to\infty} d(B_{2n_k}B_{2m}p_{2n_k-1}, B_{2n_k+1}B_{2m+1}p_{2n_k})$$
$$= q \lim_{k\to\infty} d(B_{2m}B_{2n_k}p_{2n_k-1}, B_{2m+1}B_{2n_k+1}p_{2n_k}) =$$

$$= qd(B_{2m}z, B_{2m+1}z).$$

Since $q < 1$ it follows that $B_{2m}z = B_{2m+1}z$, for every $m \in \mathbf{N}$. We shall prove that $B_{2m}z = B_{2m-1}z$, for every $m \in \mathbf{N}$. Since $z = \lim_{k\to\infty} A_{2n_k}p_{2n_k-1} = \lim_{k\to\infty} A_{2n_k+1}p_{2n_k}$ it follows that

$$\begin{aligned}
d(B_{2m}z, B_{2m-1}z) &= d(B_{2m}(\lim_{k\to\infty} A_{2n_k}p_{2n_k-1}), B_{2m-1}(\lim_{k\to\infty} A_{2n_k+1}p_{2n_k}))\\
&= \lim_{k\to\infty} d(B_{2m}A_{2n_k}p_{2n_k-1}, B_{2m-1}A_{2n_k+1}p_{2n_k}) =\\
&= \lim_{k\to\infty} d(A_{2n_k}B_{2m}p_{2n_k-1}, A_{2n_k+1}B_{2m-1}p_{2n_k})\\
&\leq q \lim_{k\to\infty} d(B_{2n_k}B_{2m}p_{2n_k-1}, B_{2n_k+1}B_{2m-1}p_{2n_k})\\
&= q \lim_{k\to\infty} d(B_{2m}B_{2n_k}p_{2n_k-1}, B_{2m-1}B_{2n_k+1}p_{2n_k})\\
&= qd(B_{2m}z, B_{2m-1}z)
\end{aligned}$$

and so $B_{2m}z = B_{2m-1}z$. Further, we have that $A_{2n}z = B_{2n+1}z$, $n \in \mathbf{N}$. Indeed for $n_k \neq n$ we have that

$$d(B_{2n+1}A_{2n_k+1}p_{2n_k}, A_{2n}z) = d(A_{2n_k+1}B_{2n+1}p_{2n_k}, A_{2n}z)$$

$$\leq qd(B_{2n_k+1}B_{2n+1}p_{2n_k}, B_{2n}z) = qd(B_{2n+1}B_{2n_k+1}p_{2n_k}, B_{2n}z)$$

and so

$$\lim_{k\to\infty} d(B_{2n+1}A_{2n_k+1}p_{2n_k}, A_{2n}z) \leq$$

$$\leq q \lim_{k\to\infty} d(B_{2n+1}B_{2n_k+1}p_{2n_k}, B_{2n}z).$$

Hence

$$d(B_{2n+1}z, A_{2n}z) \leq qd(B_{2n+1}z, B_{2n}z)$$

and since we can prove easily that $B_{2n+1}z = B_{2n}z$ we have that $B_{2n+1}z = A_{2n}z$, $n \in \mathbf{N}$. From the inequality

$$d(A_{2n-1}z, A_{2n}z) \leq qd(B_{2n-1}z, B_{2n}z)$$

we conclude that $A_{2n-1}z = A_{2n}z$, for every $n \in \mathbf{N}$.

Further, since $B_j : K \to K$ it follows that $A_jz \in K$, $j \in \mathbf{N}$ and

$$d(A_{2n-1}z, A_{2n}A_{2n}z) \leq qd(B_{2n-1}z, B_{2n}(A_{2n}z))$$

$$= qd(A_{2n-1}z, A_{2n}(B_{2n}z))$$

$$= qd(A_{2n-1}z, A_{2n}(A_{2n}z)).$$

This implies that $A_{2n-1}z = A_{2n}z = A_{2n}(A_{2n}z) = A_{2n}(B_{2n}z) = B_{2n}(A_{2n}z)$ and similarly

$$d(A_{2n}z, A_{2n+1}A_{2n+1}z) \leq qd(B_{2n}z, B_{2n+1}(A_{2n+1}z))$$

$$= qd(A_{2n}z, A_{2n+1}(B_{2n+1}z))$$

$$= qd(A_{2n}z, A_{2n+1}(A_{2n+1}z))$$

$A_{2n}z = A_{2n+1}(A_{2n+1}z) = A_{2n+1}(A_{2n}z) = A_{2n+1}(B_{2n+1}z) = B_{2n+1}(A_{2n+1}z) = B_{2n+1}A_{2n}z$.

Hence $u = A_{2n}z$ is a common fixed point for the families $\{A_n\}_{n\in\mathbf{N}}$ and $\{B_n\}_{n\in\mathbf{N}}$. The uniqueness of the common fixed point of the families $\{A_n\}_{n\in\mathbf{N}}$ and $\{B_n\}_{n\in\mathbf{N}}$ follows immediately since from $u = A_nu = B_nu$ and $w = A_nw = B_nw$ for every $n \in \mathbf{N}$ we have:

$$d(u, w) = d(A_{2n-1}u, A_{2n}w) \leq kd(B_{2n-1}u, B_{2n}w) \leq$$

$$\leq kd(u, w)$$

which implies $u = w$.

References

[1] Assad, N.A., Kirk, W.A.: Fixed point theorems of set-valued mappings of contractive type, Pacific J. Math. Vol. 43, No. 3 (1972), 553-562.

[2] Hadžić, O.: Common fixed point theorems for family of mappings in complete meric spaces, Math. Japan, 29 (1984), 127-134.

[3] Taniguchi, T.: A common fixed point theorem for two sequences of self - mappings, Internat. J. Math. Math. Sci. Vol. No. 3 (1991), 417-420.

REZIME

TEOREMA O ZAJEDNIČKOJ NEPOKRETNOJ TAČKI ZA DVA NIZA PRESLIKAVANJA U KONVEKSNIM METRIČKIM PROSTORIMA

Dokazano je uopštenje teoreme iz [3] u konveksnim metričkim prostorima.

Received by the editors June 12, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 161-171

Review of Research
Faculty of Science
Mathematics Series

ON MULTIVALUED CONTRACTIONS IN PROBABILISTIC METRIC SPACES

Olga Hadžić[1]
Institute of Mathematics, University of Novi Sad,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

Some properties of multivalued contractions in probabilistic metric spaces are investigated.

AMS Mathematics Subject Classification (1980): 47H10
Key words and phrases: Multivalued contractions, probabilistic metric spaces.

1. Introduction

For singlevalued mappings there are two types of contractions in probabilistic metric spaces. The first one is introduced by V.H.Sehgal and A.T.Bharucha-Reid in [9] and the second one by T.L.Hicks in [5].

Definition 1. *A mapping $f : S \to S$, where $(S, \mathcal{F})$ is a probabilistic semi-metric space, is a $B-$contraction if there is a $k \in (0,1)$ such that for every $p, q \in S$ and all $x > 0$*

$$F_{fp,fq}(kx) \geq F_{p,q}(x).$$

---

[1] This research was supported by Science Fund of Serbia, grant number 0401A, through Matematički institut

Definition 2. *A mapping $f : S \to S$, where $(S, \mathcal{F})$ is a probabilistic semimetric space, is an $H-$contraction if there is a $k \in (0,1)$ such that for every $p, q \in S$ and every $x > 0$*

$$F_{p,q}(x) > 1 - x \Rightarrow F_{fp,fq}(kx) > 1 - kx.$$

Let $(S, \mathcal{F})$ be a probabilistic semimetric space and for every $p, q \in S$

$$\beta(p,q) = \inf\{h; F_{p,q}(h^+) > 1 - h\}.$$

If $(S, \mathcal{F}, t)$ is a probabilistic metric space with $t \geq t_m$ $(t_m(x,y) = \max\{x + y - 1, 0\}, (x,y) \in [0,1]^2)$ then β is a metric on S [6].

In [6] the following Theorem is proved .

Theorem A. *The mapping $f : S \to S$ is an $H-$contraction on probabilistic metric space $(S, \mathcal{F}, t)$ with $t \geq t_m$ if and only if f is a metric contraction on the metric space (S, β).*

A $B-$contraction need not be an $H-$contraction and an $H-$contraction need not be a $B-$contraction [8].

In [8] some sufficient conditions for $\mathcal{F}$ are given such that every B contraction on $(S, \mathcal{F})$ is an $H-$contraction.

Theorem B. *Let $(S, \mathcal{F})$ be a probabilistic semimetric space such that $Ran(\mathcal{F})$ is finite and $Ran(\mathcal{F}) \setminus \{\epsilon_0\}$ is strictly increasing on $[0,1]$, where $\epsilon_0(x) = 0$ for $x \leq 0$ and $\epsilon_0(x) = 1$ for $x > 0$. Then every $B-$contraction on $(S, \mathcal{F})$ is an $H-$contraction.*

In general, it is not true that every $B-$contraction is an $H-$contraction and some examples are given in [8].

For multivalued mappings up to now there are three definitions of multivalued probabilistic contractions which are in some sense generalization of Definitions 1 and 2. The aim of this paper is to give some properties of a multivalued probabilistic contraction with respect to Definitions 3, 4 and 5 given below.

2. Preliminaries

Let $(S, \mathcal{F})$ be a propabilistic semimetric space and A a nonempty subset of S. The function $D_A(\cdot)$, defined by

$$D_A(u) = \sup_{s<u} \inf_{p,q \in A} F_{p,q}(s), \quad u \in \mathbf{R}^+$$

is called the probabilistic diameter of the set A and the set A is probabilistic bounded if and only if

$$\sup_{u \in \mathbf{R}^+} D_A(u) = 1.$$

For every two probabilistic bounded subsets A and B from S

$$\tilde{F}_{A,B}(u) = \sup_{s<u} \inf_{x \in A} \sup_{y \in B} F_{x,y}(s).$$

Definition 3. *Let $(S, \mathcal{F})$ be a probabilistic semimetric space, $f : S \to nB(S)$ (nonempty, bounded subsets of S) and there exists $k \in (0,1)$ such that*

$$\tilde{F}_{fp,fq}(ku) \geq F_{p,q}(u), \quad \textit{for every } p, q \in S \ \textit{ and every } \ u > 0.$$

Then f is a B – contraction type mapping .

Definition 4. *Let $(S, \mathcal{F})$ be a probabilistic semimetric space, $f : S \to n(S)$ and there exists $k \in (0,1)$ such that the following condition is satisfied:*

For every $p, q \in S$ and for every $v \in fp$
there exists $w \in fq$ such that for every $u > 0$

$$F_{v,w}(ku) \geq F_{p,q}(u).$$

Then f is a C – contraction type mapping.

Definition 5. *Let $(S, \mathcal{F})$ be a probabilistic semimetric space, $f : S \to n(S)$ and there exists $k \in (0,1)$ such that the following implication holds for every $p, q \in S$ and $u > 0$:*

$$F_{p,q}(u) > 1 - u \Rightarrow \tilde{F}_{fp,fq}(ku) > 1 - ku.$$

Then f is an H_1 – contraction type mapping.

Definition 6. *Let $(S,\mathcal{F})$ be a probabilistic semimetric space, $f : S \to n(S)$ and there exists $k \in (0,1)$ such that the following implication holds for every $p, q \in S$ and $x > 0$:*

$$F_{p,q}(x) > 1 - x \Rightarrow \textit{for every } u \in fp \textit{ there exists} \\ v(u) \in fq \textit{ such that } F_{u,v(u)}(kx) > 1 - kx.$$

Then f is an H_2 - contraction.

The Hausdorff function of noncompactness $\beta_A(\cdot)$ (A is a probabilistic bounded subset of S) is defined in the following way [10]:

$$\beta_A(u) = \sup\{\epsilon > 0; \quad \text{there exists a finite} \\ \text{subset } A_f \quad \text{of } S \quad \text{such that} \quad \tilde{F}_{A,A_f}(u) \geq \epsilon\}.$$

The function β has the following properties:

(i) $\beta_A \in \Delta$, where Δ is the set of distribution functions.

(ii) $\beta_A(u) \geq D_A(u)$, for every $u \in \mathbf{R}$.

(iii) $\emptyset \neq A \subset B \subset S \Rightarrow \beta_A(u) \geq \beta_B(u)$, for every $u \in \mathbf{R}$.

(iv) $\beta_{A \cup B}(u) = \min\{\beta_A(u), \beta_B(u)\}$, for every $u \in \mathbf{R}$.

(v) $\beta_A(u) = \beta_{\bar{A}}(u) (u \in \mathbf{R})$, where $\bar{A}$ is the closure of A.

(vi) $\beta_A = \epsilon_0 \Rightarrow A$ is precompact.

If $(S,\mathcal{F})$ is a probabilistic semimetric space , K a probabilistic bounded subset of S and T a mapping from K into $n(S)$ (nonempty subsets of S) we say that T is densifying on K with respect to β if $T(K)$ is probabilistic bounded and for every $B \subset K$:

$$\beta_{T(B)}(u) \leq \beta_B(u) \quad \text{foe every } u > 0 \Rightarrow B \quad \text{is precompact.}$$

A mapping $f : K \to n(S)$ is a k - set probabilistic contraction if $T(A) \in B(S)$, for every $A \subseteq K$ and

$$\beta_{T(A)}(ks) \geq \beta_A(s), \quad \text{for every} \quad s > 0.$$

The first result is a generalization of Theorem 3.1 from [8] .

Theorem 1. *Let $(S,\mathcal{F})$ be a probabilistic semimetric space and $f : S \to Com(S)$ (nonempty compact subsets of S) be a $B-$ contraction type mapping. If $Ran(\mathcal{F})$ is finite and each element of $Ran(\tilde{\mathcal{F}}) \setminus \{\epsilon_0\}$ is strictly increasing on $[0,1]$ then there exists $\gamma \in (0,1)$ such that*

$$D_\beta(fp, fq) \le \gamma\beta(p,q), \quad \textit{for every } \ p,q \in S$$

where D_β is defined by:

$$D_\beta(A,B) = \max\{\sup_{p\in A}\inf_{q\in B}\beta(p,q), \sup_{p\in B}\inf_{q\in A}\beta(p,q)\}.$$

First, we shall prove the following Lemma.

Lemma 1. *Let $(S,\tilde{\mathcal{F}})$ be a probabilistic semimetric space and $f : S \to Com(S)$ be a B – contraction type mapping such that each element of $Ran(\tilde{\mathcal{F}}) \setminus \{\epsilon_0\}$ is strictly increasing on $[0,1]$. Then*

$$D_\beta(fp, fq) \le \beta(p,q), \quad \textit{for every } \ p,q \in S.$$

If $(S,\mathcal{F},t)$ is a Menger space such that $t \ge t_m$ then $D_\beta(fp,fq) < \beta(p,q)$ for every $p \ne q$.

Proof. Let $r > 0$ be such that $0 < r < \frac{1-k}{k}\beta(p,q)$ where k is the contraction constant and $\beta(p,q) > 0$. Then $\beta(p,q) > k[\beta(p,q)+r]$ and since $\tilde{F}_{fp,fq}$ is strictly increasing we have that

$$\tilde{F}_{fp,fq}(\beta(p,q)) > \tilde{F}_{fp,fq}(k(\beta(p,q)+r)) \ge F_{p,q}(\beta(p,q)+r) > 1-\beta(p,q).$$

From the Definition of $\tilde{F}_{fp,fq}$ it follows that

$$\sup_{s<\beta(p,q)}\ \inf_{u\in fp}\ \sup_{v\in fq} F_{u,v}(s) > 1-\beta(p,q)$$

which implies

$$\inf_{u\in fp}\ \sup_{v\in fq} F_{u,v}(\beta(p,q)) > 1-\beta(p,q) \tag{1}$$

and similarly

$$\inf_{v\in fq}\ \sup_{u\in fp} F_{u,v}(\beta(p,q)) > 1-\beta(p,q). \tag{2}$$

From (1) it follows that for every $u \in fp$

$$\sup_{v \in fq} F_{u,v}(\beta(p,q)) > 1 - \beta(p,q)$$

and from (2) that for every $v \in fq$

$$\sup_{u \in fp} F_{u,v}(\beta(p,q)) > 1 - \beta(p,q).$$

This means that for every $u \in fp$ there exists $v(u) \in fq$ such that

$$F_{u,v(u)}(\beta(p,q)) > 1 - \beta(p,q) \tag{3}$$

and similarly for every $v \in fq$ there exists $u(v) \in fp$ such that

$$F_{u(v),v}(\beta(p,q)) > 1 - \beta(p,q). \tag{4}$$

Relation (3) implies that $\beta(u, v(u)) < \beta(p,q)$ and relation (4) implies that $\beta(u(v), v) < \beta(p,q)$. Hence

$$\inf_{v \in fq} \beta(u,v) < \beta(p,q), \inf_{u \in fp} \beta(u,v) < \beta(p,q)$$

which implies that

$$\sup_{u \in fp} \inf_{v \in fq} \beta(u,v) \le \beta(p,q) \tag{5}$$

$$\sup_{v \in fq} \inf_{u \in fp} \beta(u,v) \le \beta(p,q). \tag{6}$$

From (5) and (6) we have that

$$D_\beta(fp, fq) \le \beta(p,q), \quad \text{for every } p, q \in S.$$

If $(S, \mathcal{F}, t)$ is a Menger space such that $t \ge t_m$ then $\beta(\cdot,\cdot)$ is a metric on S and $\psi(u) = \inf_{v \in fq} \beta(u,v)$ and $\varphi(v) = \inf_{u \in fp} \beta(u,v)$ are continuous function $\psi : fp \to \mathbf{R}^+$, $\varphi : fq \to \mathbf{R}^+$. Since fp and fq are compact there exist $u_0 \in fp$ and $v_0 \in fq$ such that

$$\psi(u_0) = \sup_{u \in fp} \psi(u), \quad \varphi(v_0) = \sup_{v \in fq} \varphi(v).$$

Hence

$$\sup_{u \in fp} \inf_{v \in fq} \beta(u,v) = \inf_{v \in fq} \beta(u_0, v) < \beta(p,q)$$

and
$$\sup_{v\in fq}\inf_{u\in fp}\beta(u,v)=\inf_{u\in fp}\beta(u,v_0)<\beta(p,q).$$

This means that $D_\beta(fp,fq)<\beta(p,q)$ for $p\neq q$.

Proof of Theorem 1 . As in [8] for every $(p,q)\in S\times S$, $p\neq q$ there exists $\gamma_{p,q}\in(0,1)$ such that

$$D(fp,fq)<\gamma_{p,q}\beta(p,q).$$

Since the set $Ran(\mathcal{F})$ is finite there exists γ such that $\mathrm{Max}\{\gamma_{p,q};\ p,q\in S\}<\gamma<1$. Hence, for every $p,q\in S$

$$D_\beta(fp,fq)\le\gamma\beta(p,q). \tag{7}$$

Remark 1. If $(S,\mathcal{F},t)$ is a Menger space such that $t\ge t_m$ and that (S,β) is a compact metric space using the Lemma we can obtain a fixed point result for $f:S\to Cl(S)$ (the family of nonempty closed subsets of S) which is a continuous $B-$contraction and $\tilde{\mathcal{F}}$ is such that every element of $Ran(\tilde{\mathcal{F}})\setminus\{\epsilon_0\}$ is strictly increasing.

Namely, the following result is well known [2]: *Let (S,d) be a compact metric space and $f:S\to Cl(S)$ a continuous mapping such that*

$$D(fp,fq)<d(p,q),\ \ p\neq q.$$

Then f has a fixed point.
Hence, we have the following result:

Proposition 1. *Let $(S,\mathcal{F},t)$ be a Menger space such that $t\ge t_m$ and (S,β) is a compact metric space, $f:S\to Cl(S)$ a continuous $B-$ contraction and every element of $Ran(\tilde{\mathcal{F}})\setminus\{\epsilon_0\}$ is strictly increasing. Then there exists $x\in S$ such that $x\in fx$.*

Remark 2. Using the well known Nadler's fixed point theorem we have the following result.

Proposition 2. *Suppose that all the conditions of Theorem 1 are satisfied and that (S,β) is a complete metric space. Then there exists $x\in S$ such that $x\in fx$.*

Remark 3. Suppose that (7) holds for every $p, q \in S$ and that $F_{p,q}(x) > 1-x$. This implies that $\beta(p,q) < x$ and (7) implies that $D_\beta(fp, fq) < \gamma x$. Using the definition of D_β we have that $\sup_{u\in fp} \inf_{v\in fq} \beta(u,v) < \gamma x$ and $\sup_{v\in fq} \inf_{u\in fp} \beta(u,v) < \gamma x$. Hence $\inf_{v\in fq} \beta(u,v) < \gamma x$, for every $u \in fp$ and $\inf_{u\in fp} \beta(u,v) < \gamma x$, for every $v \in fq$. Since fp and fq are compact for every $u \in fp$ there exists $v(u) \in fq$ such that

$$F_{u,v(u)}(\gamma x) > 1 - \gamma x$$

and similarly for every $v \in fq$ there exists $u(v) \in fp$ such that

$$F_{u(v),v}(\gamma x) > 1 - \gamma x.$$

From this we have that f is an H_2 – contraction.

Theorem 2. *Let $(S, \mathcal{F})$ be a probabilistic semimetric space and $f : S \to nB(S)$ an H_1 – contraction. Then f is an H_2– contraction and*

$$D_\beta(fp, fq) \le k \cdot \beta(p,q) \quad \textit{for every } p, q \in S. \tag{8}$$

Proof . Let $\beta(p,q) < s$. We shall prove that

$$D_\beta(fp, fq) \le ks \tag{9}$$

which implies (8).

From $\beta(p,q) < s$ it follows that $F_{p,q}(s) > 1 - s$. Since f is an H_1-contraction we have that

$$\tilde{F}_{fp,fq}(ks) > 1 - ks$$

and so

$$\sup_{u<ks} \inf_{u\in fp} \sup_{v\in fq} F_{u,v}(u) > 1 - ks \tag{10}$$

$$\sup_{u<ks} \inf_{v\in fq} \sup_{v\in fp} F_{u,v}(u) > 1 - ks. \tag{11}$$

Relations (10) and (11) implies that

$$\inf_{u\in fp} \sup_{v\in fq} F_{u,v}(ks) > 1 - ks \tag{12}$$

$$\inf_{v\in fq}\sup_{u\in fp} F_{u,v}(ks) > 1-ks \tag{13}$$

and so for every $u \in fp$ and $v \in fq$ we have that

$$\sup_{v\in fq} F_{u,v}(ks) > 1-ks \tag{14}$$

$$\sup_{u\in fp} F_{u,v}(ks) > 1-ks. \tag{15}$$

Hence, for every $u \in fp$ there exists $v(u) \in fq$ such that $F_{u,v(u)}(ks) > 1-ks$ and for every $v \in fq$ there exists $u(v) \in fp$ such that $F_{v,u(v)}(ks) > 1-ks$. Thus, $\beta(u, v(u)) < ks$ and $\beta(v, u(v)) < ks$ which implies that

$$D_\beta(fp, fq) \leq ks.$$

Corollary 1. *If $(S, \mathcal{F}, t)$ is a complete Menger space such that $t \geq t_m$ and $f : S \to CB(S)$ (closed and bounded subsets of S) an H_1- contraction then there exists $x \in S$ such that $x \in fx$.*

Theorem 3. *Let $(S, \mathcal{F}, t)$ be a probabilistic bounded Menger space with continuous T–norm t and $f : S \to Com(S)$ a C - contraction. Then f is a k–set probabilistic contraction.*

Proof. The proof of this theorem is in fact given in a part of the proof of Theorem 1 from [4] but we shall give it here for the completeness.

In order to prove the inequality

$$\beta_{f(A)}(ks) \geq \beta_A(s), \quad \text{for every } s > 0 \tag{16}$$

and every $A \subset S$ we shall prove that for every $\epsilon \in (0, s)$

$$\beta_A(s-\epsilon) \leq \beta_{f(A)}(ks) \tag{17}$$

which implies (16), since $\beta_A(\cdot)$ is a left continuous function. If $\beta_A(s) = 0$ then (16) follows and suppose that $\beta_A(s) > 0$. In order to prove (17) we shall prove the implication:

$$\tau < \beta_A(s-\epsilon) \Rightarrow \tau < \beta_{f(A)}(ks) \tag{18}$$

since (18) implies (17). From $r < \beta_A(s - \epsilon)$ it follows that there exists a finite set $A_f = \{x_1, x_2, ..., x_n\} \subset S$ such that

$$\inf_{z \in A} \max_{w \in A_f} F_{z,w}(s - \epsilon) > r$$

and so for every $z \in A$ there exists $w(z) \in A_f$ so that $F_{z,w(z)}(s - \epsilon) > r$. If $y \in fz$ $(z \in A)$ then there exists $x \in f(w(z))$ so that

$$F_{y,x}(k(s - \epsilon)) \geq F_{z,w(z)}(s - \epsilon).$$

Let $\delta \in (0, r)$ and $\lambda(\delta) \in (0, 1)$ be such that $(t(r, 1) = r)$

$$1 \geq u > 1 - \lambda(\delta) \Rightarrow t(r, u) > r - \delta$$

and for every $i \in \{1, 2, ..., n\}$ let

$$fx_i \subset \cup_{j=1}^{n(i)} U_{z_j^i}(\frac{k\epsilon}{2}, \lambda(\delta))$$

where

$$U_v(\epsilon, \lambda) = \{p;\ p \in S, F_{p,v}(\epsilon) > 1 - \lambda\}.$$

It can be proved that

(19) $$\tilde{F}_{f(A), \cup_{i=1}^{n} \cup_{j=1}^{n(i)} \{z_j^i\}}(ks) > r\delta$$

and (19) implies $\beta_{f(A)}(ks) \geq r - \delta$. Hence $\beta_{f(A)}(ks) \geq r$ since δ is an arbitrary element from $(0, r)$.

Remark 4. It is obvious that f is densifying on S with respect to β if S is complete .

References

[1] Bocsan, Gh., Constantin, Gh.: The Kuratowski function and some applications to the probabilistic metric spaces, Sem. Teor. Funct. Si. Mat. Apl., Timisoara RS Romania, No. 1, 1973.

[2] Czerwik, S.: Fixed point theorems and special solutions of functional equations, Universytet Slaski, Katowice, 1980.

[3] Hadžić, O.: Some theorems on the fixed points in probabilistic metric and random normed spaces, Bull. Unione Mat. Ital. (6), 1-B (1982), 381-391.

[4] Hadžić, O.: Fixed point theorems for multivalued mappings in some classes of fuzzy metric spaces, Fuzzy Sets and Systems 29 (1989), 115-125.

[5] Hicks, T.L.: Fixed point theory in probabilistic metric spaces, Univ. u Novom Sadu Zb. Rad. Prirod. - Mat. Fak. Ser. Mat. 13 (1983), 63-72.

[6] Radu, V.: Some fixed point theorems in probabilistic metric spaces, Lectures Notes in Math. 1233, 125-133.

[7] Schweizer B., Sklar, A.: Probabilistic Metric Spaces, Elsevier North - Holland, 1983.

[8] Schweizer, B., Sherwood, H., Tardif, R.M.: Contractions on probabilistic metric spaces, Examples and counter examples, Stochastica XII-1 (1988), 5-17.

[9] Sehgal, V.M., Bharucha-Reid, A.T.: Fixed points of contraction mappings in probabilistic metric spaces, Math. Systems Theory 6(1972), 97-102.

[10] Tan, D.H.: On probabilistic condensing mappings, Rev. Roum. Math. Pures Appl. 26,10(1981), 1305-1317.

REZIME

O VIŠEZNAČNIM KONTRAKCIJAMA U VEROVATNOSNIM METRIČKIM PROSTORIMA

Neke osobine višeznačnih kontrakcija u verovatnosnim metričkim prostorima su ispitane.

Received by the editors June 12, 1990.

UDK 51
YU ISSN 0352–0900

УНИВЕРЗИТЕТ У НОВОМ САДУ — UNIVERSITY OF NOVI SAD

ЗБОРНИК РАДОВА
ПРИРОДНО-МАТЕМАТИЧКОГ ФАКУЛТЕТА
СЕРИЈА ЗА МАТЕМАТИКУ

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

21

No. 1

НОВИ САД — NOVI SAD

1991.

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

INSTITUTE OF MATHEMATICS
TRG DOSITEJA OBRADOVIĆA 4
21000 NOVI SAD
YUGOSLAVIA

EDITORIAL BOARD

Editor in Chief: **O. Hadžić**, Novi Sad
Associate Editor in Chief: **D. Herceg**, Novi Sad
Editorial Secretary: **R. Vulanović**, Novi Sad

Editors:
E. Allgower, Fort Collins, CO
K.Gilezan, Novi Sad
A. Ivić, Belgrade
Z. Ivković, Belgrade
A. Kamiński, Katowice
V. Marić, Novi Sad
G. Milovanović, Niš
E. Pap, Novi Sad
S. Pilipović, Novi Sad
M. Prvanović, Novi Sad
B. Stanković, Novi Sad
Z. Stojaković, Novi Sad
B. Šešelja, Novi Sad
R. Tošić, Novi Sad
J. Ušan, Novi Sad
V. Vladimirov, Moscow

The journal publishes original research papers from all fields of pure and applied mathematics. When preparing the manuscript please consider "Instructions to Authors" at the end of each number. Manuscripts and inquiries should be directed to the associate editor in chief. Twenty offprints of each paper are supplied free of charge. Additional copies may be ordered at cost price at time when proofs are returned.

Annual subscription rate for foreign countries: US $ 60 (one volume of two numbers).

Orders, exchange journals, manuscripts and all corespondence concering offprints should be sent to the associate editor in chief:

D. Herceg
Institute of Mathematics
University of Novi Sad
Trg Dositeja Obradovića 4
21000 Novi Sad, Yugoslavia

Published by the Institute of Mathematics, Novi Sad, Yugoslavia
Printed by FTN, Štamparija za grafičku delatnost, Novi Sad
Circulation: 500

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 1-13

Review of Research
Faculty of Science
Mathematics Series

ON A MODIFICATION OF NONLINEAR OVERRELAXATION METHODS

Dragoslav Herceg
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

A numerical composite VAOR-Newton iteration, with a modified nonlinear accelerated overrelaxation (VAOR) as the primary iteration and the Newton method as a secondary iteration, is considered. Some sufficient conditions for the local convergence of this method are given. In the linear case these conditions describe the area of convergence of the VAOR method, and also, as a subcase, the area of convergence of the AOR method.

AMS *Mathematics Subject Classification*(1980): 65H10
Key words and phrases: Iterative methods, nonlinear systems, relaxation methods.

1. Introduction

Let us consider the system of nonlinear equations

$$Fx = 0, \tag{1}$$

where $F : V \subset \mathbf{R}^n \to \mathbf{R}^n$, and suppose that F is F-differentiable and F' is continuous in an open neighbourhood $V_0 \subset V$ of a point x^* for which $Fx^* = 0$.

For the solution of the nonlinear system (1) one can give a direct extension of methods for solving linear systems. So, there are the following combinations: JOR-Newton, SOR-Newton, and many of their modifications, see [15] for instance. The AOR method, introduced in [11] for solving linear systems, which is a two-parameter generalization of the JOR and SOR methods, can be also extended to the nonlinear case. A modification of the AOR method is described in [2], [3], [5], [9], [12], and it is called the VAOR method. The extension of this method to the nonlinear case we have in [2], [3], [12], [14]. Our VAOR-Newton method is an extrapolated VSOR-Newton from [10], see also [4]. Hence, one can make the convergence statement using this fact in the same way for the extrapolation methods, see [15], [11]. However, it is possible to give sufficient conditions for the convergence of our method only by considering the structure of the mapping F.

Convergence studies of the extrapolation method for linear and nonlinear systems have been undertaken and reported on by various authors; see for instance, [2]-[5], [7]-[8], [11]-[14] and the books [15], [16], and references cited therein.

In this paper we considered the VAOR-Newton method in cases that $F'(x^*)$ belongs to some special subclasses of the H-matrix and give some sufficient conditions for its local convergence.

2. Notations and definitions

We denote by $\mathbf{R}^n$ the real n-dimensional linear space of column vectors $x = [x_1, x_2, \ldots, x_n]^T$ and by $L(\mathbf{R}^n)$ the linear space of real matrices $A = [a_{ij}]$ of order n with unit matrix E. We use the coordinatewise partial orderings on $L(\mathbf{R}^n)$ and $\mathbf{R}^n$; that is if $A, B \in L(\mathbf{R}^n)$ then $A \geq B (A > B)$ if and only if $a_{ij} \geq b_{ij}$ $(a_{ij} > b_{ij})$, for $i, j = 1, 2, \ldots, n$; and similarly for $\mathbf{R}^n$. Let

$$A = A_D - A_T - A_S$$

be the decomposition of $A \in L(\mathbf{R}^n)$ into its diagonal, strictly lower and strictly upper triangular parts, respectively.

Let $N = 1, 2, \ldots, n$, $N(i) = N \setminus \{i\}$ and for $A = [a_{ij}] \in L(\mathbf{R}^n)$ let

$$P_i(A) = \sum_{j \in N(i)} |a_{ij}|, \; P_i'(A) = \sum_{j=1}^{i-1} |a_{ij}|, \; Q_i(A) = \max_{j \in N(i)} |a_{ij}|, \; i \in N.$$

For any real or complex $n \times n$ matrix $A = [a_{ij}]$, we denote by

$$M(A) = [m_{ij}]$$

the $n \times n$ matrix defined by

$$m_{ij} = \begin{cases} |a_{ii}| & \text{if} \quad i = j, \\ -|a_{ij}| & \text{if} \quad i \neq j. \end{cases}$$

Definition 1. *A real matrix $A = [a_{ij}]$ is called an M-matrix if and only if $a_{ij} \leq 0$, $i \in N, j \in N(i)$, and A is nonsingular with $A^{-1} \geq 0$.*

Definition 2. *A real or complex matrix A is called an H-matrix if and only if $M(A)$ is an M-matrix.*

Definition 3. *An $n \times n$ complex matrix $A = [a_{ij}]$ is called lower semistrictly diagonally dominant if and only if*

$$|a_{ij}| \geq P_i(A), \quad i \in N,$$

$$|a_{ii}| > P_i'(A), \quad i \in N.$$

The matrix A is called semistrictly diagonally dominant if and only if there exists a permutation matrix Q such that QAQ^T is lower semistrictly diagonally dominant.

Definition 4. *An $n \times n$ complex matrix $A = [a_{ij}]$ is called generalized diagonally dominant if and only if there exists a regular diagonal matrix $W = diag(w_1, \ldots, w_n)$ such that AW is strictly diagonally dominant, that is $|a_{ii}w_i| > P_i(AW)$, $i \in N$.*

3. On the convergence of the VAOR–Newton - method

The SOR-Newton method, cf. [15], is defined by

$$x_i^{k+1} = x_i^k - \omega \frac{f_i(x^{k,i})}{f_{ii}'(x^{k,i})},$$

where, as usual, $f_1, \ldots, f_n$ are components of F, $\omega \in \mathbf{R}\setminus\{0\}$, $f'_{ii}(x) = \frac{\partial f_i}{\partial x_i}(x)$ and $f'_{ii}(x) \neq 0$, $x \in V$, $i \in N$.

Our generalization of the SOR-Newton method is the VAOR-Newton method, see [2] - [3], [12] - [14] , which reduces to it if $\omega = \sigma$ and $\varphi_i(x) = f'_{ii}(x)$:

$$\begin{aligned} x_i^{k+1} &= x_i^k - \omega \frac{f_i(z^{k,i})}{\varphi_i(z^{k,i})}, \ i \in N \\ z_i^{k+1} &= x_i^k - \sigma \frac{f_i(z^{k,i})}{\varphi_i(z^{k,i})}, \ i \in N \end{aligned} \tag{2}$$

where $\omega, \sigma \in \mathbf{R} \setminus \{0\}$, $\varphi_i(x) \neq 0$,and

$$z^{k,i} = [z_1^{k+1}, \ldots, z_{i-1}^{k+1}, x_i^k, \ldots, x_n^k]^T, \ i \in N.$$

Without loss of generality, we may assume that $f'_{ii}(x) > 0$, $x \in V$, $i \in N$. Because of that we assume now on that $\varphi_i(x) > 0$, $x \in V$, $i \in N$.

In case $\sigma = \omega$ our VAOR-Newton method reduces the SOR-Newton method from [10].

Let $\Phi(x) = diag(\varphi_1(x), \varphi_2, \ldots, \varphi_n(x))$. Our method (2) may be written in the form $x^{k+1} = G_{\Phi,\sigma,\omega}x^k$ although now mapping $G_{\Phi,\sigma,\omega}$ becomes rather complicated. If we denote with $G_{\Phi,\sigma}$ the iteration function of the SOR-Newton method, then for iteration function $G_{\Phi,\sigma,\omega}$ of the VAOR-Newton method we have

$$G_{\Phi,\sigma,\omega}x = \left(1 - \frac{\omega}{\sigma}\right) x + \frac{\omega}{\sigma} G_{\Phi,\sigma}x.$$

This relation shows that the VAOR-Newton method is the extrapolated VSOR-Newton method with extrapolation parameter $\frac{\omega}{\sigma}$. To prove the local convergence of the VAOR-Newton method it is sufficient to show that $G_{\Phi,\sigma,\omega}$ is differentiable at x^* and that $\rho(G'_{\Phi,\sigma,\omega}(x^*)) < 1$, see [15], the Ostrowski theorem. Here $\rho(G'_{\Phi,\sigma,\omega}(x^*))$ is the spectral radius of the matrix $G'_{\Phi,\sigma,\omega}(x^*)$.

In [15] we have $G'_{\Phi,\sigma}(x^*)$ for $\varphi_i(x) = f'_{ii}(x)$, $i \in N$, and in [10] is given

$$G'_{\Phi,\sigma}(x^*) = (\Phi(x^*) - \sigma F'_T(x^*))^{-1}(\Phi(x^*) - \sigma F'_D(x^*) + \sigma F'_S(x^*)),$$

where $F'(x^*) = F'_D(x^*) - F'_T(x^*) - F'_S(x^*)$ is the decomposition of $F'(x^*)$ into its diagonal, strictly lower and strictly upper triangular parts.

Thus, if $G_{\Phi,\sigma}$ is F-differentiable at $x^*, G_{\Phi,\sigma,\omega}$ is also F-differentiable at the same point and

$$\begin{aligned} G'_{\Phi,\sigma,\omega}(x^*) = \ & (\Phi(x^*) - \sigma F'_T(x^*))^{-1} (\Phi(x^*) - \omega F'_D(x^*) + \\ & + (\omega - \sigma) F'_T(x^*) + \omega F'_S(x^*)). \end{aligned} \tag{3}$$

Theorem 1. *Let $F : V \subset \mathbf{R}^n \to \mathbf{R}^n$ be F-differentiable in an open neighbourhood $V_0 \subset V$ of a point $x^* \in V$ at which F' is continuous and $Fx^* = 0$. If $F'(X^*)$ is an H-matrix and $\omega \in (o, q]$, $\sigma \in [o, q]$, where*

$$q = \min_{i \in N} \frac{\varphi_i(x^*)}{f'_{ii}(x^*)},$$

then the VAOR-Newton method is locally convergent.

Proof of this Theorem follows immediately from the corresponding theorem for the VAOR method from [3]. For technique see [2]- [5] , [12]- [14].

Theorem 2. *If $F'(x^*) = F' = [f_{ij}]$ and*

(4) *F' is lower semistrictly diagonally dominant, or*

(5) $f_{ii} f_{jj} > P_i(F')P_j(F'),\ i \in N,\ j \in N(i)$, *or*

(6) *there exists $i \in N$ such that*

$$f_{ii}(f_{jj} - P_j(F') + |f_{ji}|) > P_i(F')|f_{ji}|,\ j \in N(i), \text{ or}$$

(7) $f_{ii} > min\{P_i(F'), Q_i(F')\},\ i \in N$, *and*
$f_{ii} + f_{jj} > P_i(F') + P_j(F'),\ i \in N,\ j \in N(i)$,

then $F'(x^*)$ is an H-matrix.

Proof. If $F'(x^*)$ satisfies (4), then the statement follows immediately from [1] . In other cases proof is given in [6].

□

One can easily test conditions (4) - (7) and if $F'(x^*)$ satisfies one of them there follows the local convergence of the VAOR-Newton method for $\omega \in (0, q], \sigma \in [0, q]$.

It is known that a matrix A is an H-matrix if and only if it is generalized diagonally dominant, [3], [6], that is, there exists a regular diagonal matrix W such that AW is strictly diagonally dominant. In case that the matrix W is known, one can obtain a new area of convergence of the VAOR-Newton method using techniques from [2]- [5], [7]- [9], [12]- [14], for strictly diagonally dominant matrices and the following theorem, [3], [6].

Theorem 3. *Let $F'(x^*) = F' = [f_{ij}]$ be strictly diagonally dominant matrix. If $\varphi_i(x^*) > 0$, $i \in N$ and*

$$0 \le \sigma < t,\ 0 < \omega < \max\{t, \frac{2\sigma}{1+\rho(G'_{\Phi\sigma,\sigma})}\},\ \text{or}$$

$$\max_{i\in N} \frac{-\omega[f_{ii} - P_i(F')] + 2\max\{0, \omega f_{ii} - \varphi_i(x^*)\}}{2P_i(F'_T)} < \sigma < 0,\ 0 < \omega < t,\ \text{or}$$

$$t \le \sigma < \min_{i\in N} \frac{\omega[f_{ii} + P_i(F'_T) - P_i(F'_S)] + 2\min\{0, \varphi_i(x^*) - \omega f_{ii}\}}{2P_i(F'_T)},\ 0 < \omega < t,$$

where

$$t = \min_{i\in N} \frac{2\varphi_i(x^*)}{f_{ii} + P_i(F')},$$

then $\rho(G'_{\Phi,\sigma,\omega}) < 1$, that is, the VAOR-Newton method converges locally.

Our aim is to obtain in each of the cases (4) - (7), Theorem 2, a corresponding matrix $W = diag(w_1, w_2, w_3, \ldots, w_n)$ such that the matrix $F'(x^*)W$ is strictly diagonally dominant and $\varphi_i(x^*)w_i > 0$, $i \in N$. Then by using Theorem 3 we find wider than $\sigma \in [0,q], \omega \in (0,q]$ intervals of convergence for both σ and ω.

Theorem 4. *Let matrix $F'(x^*) = F' = [f_{ij}]$ be lower semistrictly diagonally dominant, and let $W = diag(w_1, w_2, \ldots, w_n)$, where*

$$1 > w_n > \frac{P'_n(F')}{f_{nn}}, \tag{8}$$

$$1 > w_i > \frac{P'_i(F') + \sum_{j=i+1}^{n} w_j|f_{ij}|}{f_{ii}},\ i = n-1, n-2, \ldots, 1. \tag{9}$$

Then the matrix $F'(x^*)W$ is strictly diagonally dominant.

Proof. Our assumptions are: $f_{ii} > 0$, $i \in N$, and

$$f_{ii} > P'_i(F') = \sum_{j=1}^{i-1} |f_{ij}|,\ i \in N,$$

$$f_{ii} \ge P'_i(F') = \sum_{j\in N} |f_{ij}|,\ i \in N.$$

So, we have that there exists w_n such that (8) is satisfied. Let us assume that there exists $w_{n-1}, w_{n-2}, \dots, w_{i+1} \in (0,1)$, such that (9) holds. Then

$$P_i(F') = P_i'(F') + \sum_{j=i+1}^{n} |f_{ij}| \geq P_i'(F') + \sum_{j=i+1}^{n} w_j |f_{ij}|,$$

where equality holds only if $\sum_{j=i+1}^{n} |f_{ij}| = 0$, that is, if $P_i'(F') = P_i(F')$. Therefore,

$$f_{ii} > P_i'(F') + \sum_{j=i+1}^{n} w_j |f_{ij}|,$$

and it follows that all w_i, $i \in N$ are well defined.

Let us consider the matrix AW and prove that it is strictly diagonally dominant, that is

$$f_{ii} w_i > P_i(F'W),\ i \in N.$$

From (8), (9) we have

$$f_{ii} w_i > P_i'(F'W) + \sum_{j=i+1}^{n} w_j |f_{ij}|,$$

and, since,

$$P_i'(F'W) \geq \sum_{j=1}^{i-1} w_j |f_{ij}|$$

we obtain

$$f_{ii} w_i > \sum_{j \in N(i)} w_j |f_{ij}| = P_i(F'W),\ i \in N.$$

□

Theorem 5. *Let the matrix* $F'(x^*) = F' = [f_{ij}]$ *be not strictly diagonally dominant and let it satisfy*

(10) $$f_{ii} > 0,\ f_{ii} f_{jj} > P_i(F') P_j(F'),\ i \in N,\ j \in N(i).$$

Then there exists exactly one $p \in N$ *such that* $f_{pp} \leq P_p(F')$.
Let $W = diag(w_1, w_2, \dots, w_n)$, *where* $w_i = 1$, $i \in N(p)$, *and* $w_p > \frac{P_p(F')}{f_{pp}}$, *if* $f_{ip} = 0$, $i \in N(p)$, *or*

$$w_p \in \left(\frac{P_p(F')}{f_{pp}},\ 1 + \min\left\{ \frac{f_{ii} - P_i(F')}{|f_{ip}|};\ i \in N(p),\ f_{ip} \neq 0 \right\} \right).$$

Then the matrix $F'(x^*)W$ *is strictly diagonally dominant.*

Proof. From (10) it is obviously that there is the most one $p \in N$ such that

$$f_{ii} > P_i(F'),\ i \in N(p)$$
$$f_{pp} \le P_p(F').$$

Since F' is not a strictly diagonally dominant matrix, it follows that there exists exactly one p with this property. If $f_{ip} = 0$ for all $i \in N(p)$, then

$$f_{ii}w_i = f_{ii} > P_i(F') = P_i(F'W),$$

and the matrix $F'W$ is strictly diagonally dominant.

Let us suppose that $f_{ip} \neq 0$ for some $i \in N(p)$, and let

$$\frac{f_{jj} - P_j(F')}{|f_{jp}|} = \min\left\{\frac{f_{ii} - P_i(F')}{|f_{ip}|}\ :\ i \in N(p),\ f_{ip} \neq 0\right\}.$$

Then $f_{jj} > P_j(F') \ge |f_{jp}| > 0$ and there follows $(f_{jj} - P_j(F'))(P_j(F') - |f_{jp}|) \ge 0$, that is,

$$\frac{f_{jj} - P_j(F')}{|f_{jp}|} + 1 \ge \frac{f_{jj}}{P_j(F')}.$$

Since, from (10), $\frac{f_{jj}}{P_j(F')} > \frac{P_p(F')}{f_{pp}}$, we now have that w_p is well defined.

Let us now consider the rows of the matrix $F'W$ for which $f_{ip} \neq 0$. Then we obtain

$$P_i(F'W) = P_i(F') + (w_p - 1)|f_{ip}| < P_i(F') + (f_{ii} - P_i(F')) \le f_{ii},$$

$$P_p(F'W) = P_p(F') < w_p f_{pp},$$

and conclude that $F'W$ is a strictly diagonally dominant matrix.

□

The proof of the following Theorem 6 is analogous to the proof of Theorem 5.

Theorem 6. *Let the matrix* $F'(x^*) = F' = [f_{ij}]$ *satisfy (6) and let* $W = diag(w_1, w_2, \ldots, w_n)$, *where* $w_j = 1$, $j \in N(i)$ *and*

$$w_i > \frac{P_i(F')}{f_{ii}} \text{ if } f_{ji} = 0,\ j \in N(i) \text{ or}$$

$$(11)\qquad w_i \in \left(\frac{P_i(F')}{f_{ii}},\ 1 + \min\{\frac{f_{jj} - P_j(F')}{|f_{ji}|} : j \in N(i),\ f_{ji} \neq 0\} \right).$$

Then the matrix $F'W$ is strictly diagonally dominant.

Proof. Let condition (6) be satisfied for some fixed $i \in N$. If $P_i(F') \geq f_{ii}$, then $P_j(F') < f_{jj}$, $j \in N(i)$, and the reduces to the proof of the previous Theorem. Indeed, from (6) we have $f_{jj} - P_j(F') > 0$ if $f_{ji} = 0$, and

$$f_{jj} - P_j > |f_{ji}| \left(\frac{P_i(F')}{f_{ii}} - 1 \right) \geq 0, \text{ if } f_{ji} \neq 0,\ j \in N(i).$$

If $P_i(F') < f_{ii}$, then

$$f_{jj} - P_j(F') > 0, \text{ if } f_{ji} = 0$$

and

$$\frac{f_{jj} - P_j(F')}{|f_{ji}|} + 1 > \frac{P_i(F)}{f_{ii}}, \text{ if } f_{ji} \neq 0.$$

So, w_i is well defined. Let us consider now the matrix $F'W$:

$$(F'W)_{jj} = \begin{cases} f_{jj}, & j \in N(i), \\ w_i f_{ii}, & j = i, \end{cases}$$

$$P_j(F'W) = \begin{cases} P_j(F') - |f_{ji}| + w_i|f_{ji}|, & j \in N(i), \\ P_i(F'), & j = i. \end{cases}$$

From (11) it follows that

$$(F'W)_{ii} = w_i f_{ii} > P_i(F') = P_i(F'W),$$

$$(F'W)_{jj} = f_{jj} > P_j(F') - |f_{ji}| + w_i|f_{ji}| = P_j(F'W),\ j \in N(i),$$

and we can conclude that $F'W$ is a strictly diagonally dominant matrix.

□

Theorem 7. *Let the matrix $F'(x^*) = F' = [f_{ij}]$ be not strictly diagonally dominant and let it satisfy (7).*

Then there exists exactly one $p \in N$ such that $f_{pp} \leq P_p(F')$.

Let $W = diag(w_1, w_2, \ldots, w_n)$, *where* $w_i = 1,\ i \in N(p)$, *and*

$$w_p > \frac{P_p(F')}{f_{pp}} \text{ if } Q_p(F') = 0, \text{ or}$$

$$w_p \in \left(\frac{P_p(F')}{f_{pp}}, 1 + \min\{ \frac{f_{ii} - P_i(F')}{Q_p(F')} : i \in N(p)\} \right) \text{ if } Q_p(F') \neq 0.$$

Then the matrix $F'(x^*)W$ *is strictly diagonally dominant.*

Proof. It is easy to see that there is exactly one $p \in N$ such that $f_{pp} \leq P_p(F')$. $Q_p(F') = 0$ implies $f_{ip} = 0$ for $i \in N(p)$, and, as in the proof of Theorem 5, we conclude that $F'(x)W$ is strictly diagonally dominant. If $Q_p(F') \neq 0$ we have $f_{pp} > Q_p(F')$ and

$$1 + \frac{f_{ii} - P_i(F')}{Q_p(F')} > 1 + \frac{f_{ii} - P_i(F')}{f_{pp}} > \frac{P_p(F')}{f_{pp}},\ i \in N(p)$$

because $f_{ii} - P_i(F') > P_p(F') - f_{pp}$, which follows from (7). So, w_p is well defined. As in the proof of Theorem 5, we can now obtain $P_i(F'W) < f_{ii}w_i$, $i \in N$, that is, $F'W$ is strictly diagonally dominant.

□

Now, we can easily prove the following theorem.

Theorem 8. *Let* $F'(x^*) = F' = [f_{ij}]$ *satisfy the conditions from one of the theorems 4,5,6,7 and let* W *be the diagonal matrix defined in the same theorem. If*

$$0 \leq \sigma < t,\ 0 < \omega < \max\{t, \frac{2\sigma}{1 + \rho_{\sigma,\sigma}}\}, \text{or}$$

$$\max_{i \in N} \frac{-\omega[f_{ii}w_i - P_i(F'W) + 2\max\{0, \omega f_{ii} - \varphi_i(x^*)\}w_i]}{2P_i(F'_T W)} < \sigma < 0,$$

$$0 < \omega < t, \text{or}$$

$$t \leq \sigma \leq \min \frac{\omega[f_{ii}w_i + P_i(F'_T W) - P_i(F'_S W)] + 2\min\{0, \varphi_i(x^*) - \omega f_{ii}\}w_i}{2P_i(F'_T W)}$$

$$0 < \omega < t,$$

where $\rho_{\sigma,\sigma} = \rho(H(\sigma,\sigma))$,

$$H(\sigma,\omega) = (\Phi(x^*)W - \sigma F'_T W)^{-1}\,(\Phi(x^*)W - \omega F'_D W + $$
$$+ (\omega - \sigma)F'_T W + \omega F'_S W)\,,$$

and

$$t = \min \frac{2\varphi_i(x^*)w_i}{f_{ii}w_i + P_i(F'W)},$$

then the VAOR-Newton method converges locally.

Proof. Under the conditions of Theorem 8, $F'W$ is strictly diagonally dominant, $\varphi_i(x^*)w_i > 0,\ i \in N$ and we can apply Theorem 3. Thus, we conclude that

$$\rho(H(\sigma,\omega)) < 1.$$

Since,

$$W^{-1}G'_{\Phi,\sigma,\omega}W = H,$$

the spectral radius of $G'_{\Phi,\sigma,\omega}$ is less than 1, and the VAOR-Newton method converges locally.

□

From the definition of t, we have $t \geq q$ because of $f_{ii}w_i > P_i(F'W)$. So, we have obtained new intervals for convergence σ and ω, which are wider than $[0,q]$ and $(0,q]$. At the same time matrix W in each of the considered cases is very simple and it can be calculated during test conditions (4)-(7).

If $\Phi = F_D$, we have the AOR-Newton method and Theorem 8 gives the area of convergence of this method. If F is a linear mapping and $\Phi = F_D$, the VAOR-Newton method reduces to the linear AOR method, and Theorem 8 describes the area of convergence for this method.

References

[1] Beauwens, R.: Semistrict diagonal dominance. SIAM, J. Numer. Anal., 13(1976), 109–112.

[2] Cvetković, Lj.: On a local convergence of the vAORN method. Univ. u Novom Sadu Zb. Rad. Prirod. Mat. Fak. Ser.Mat. 13(1983), 203–209.

[3] Cvetković,Lj.: Convergence theory of relaxation methods for systems of equations. Dissertation, Novi Sad, 1987.

[4] Cvetković,Lj., Herceg,D.: On a generalized vSOR-Newton method. IV Conference on Applied Mathematics (B. Vrdoljak,ed.), Faculty of Civil Engineering, Split, 1985, 99–103.

[5] Cvetković,Lj., Herceg,D.: Über die Konvergenz des VAOR-Verfahrens. Z. angew. Math. Mech. 66(1986), 405–406.

[6] Cvetković,Lj., Herceg,D.: Some results on M- and H-matrices. Univ. u Novom Sadu Zb. Rad. Prirod. Mat. Fak. Ser.Mat. 17,1(1987), 121–129.

[7] Cvetković,Lj., Herceg,D.: Convergence theory for AOR method. Journal of Computational Mathematics, 8(1990), 128–134.

[8] Cvetković,Lj., Herceg,D.: An improvement for the area of convergence of the AOR method. Anal. Numer. Theor. Approx., 16(1987), 109–115.

[9] Cvetković,Lj., Herceg,D.: Nonlinear AOR method. Z. angew. Math. Mech., 68(1968), 486–487.

[10] Gipser,M.: Untersuchungen an Relaxationsverfahren zur Lösung grösser nichtlineare Gleichungssysteme mit schwach besetzen Funktionalmatrizen. Dissertation, Darmstadt, 1980.

[11] Hadjidimos, A.: Accelerated overrelaxation method. Math. Comp., 32(1978), 149–157.

[12] Herceg,D.: Remarks on different splittings and associated generalized linear methods. Univ. u Novom Sadu Zb. Rad. Prirod. Mat. Fak. Ser. Mat. 13(1983), 177–186.

[13] Herceg,D.: Über die Konvergenz des AOR-Verfanrens. Z. angew. Math. Mech. 65(1985), 378–379.

[14] Herceg,D., Cvetković,Lj.: On the extrapolation method and USA algorithm. Journal of Economic Dynamics and Control 13(1989), 301–311.

[15] Ortega,J.M., Rheinboldt,W.C.: Iterative solution of nonlinear equations of several variables. Academic Press, New York, 1970.

[16] Young,D.M.: Iterative solution of large linear systems. Academic Press, New York, 1971.

REZIME

O MODIFIKACIJI NELINEARNIH METODA GORNJE RELAKSACIJE

Posmatra se komponovani VAOR-Njutnov iterativni postupak sa modifikovanom relaksacijom (VAOR) kao primarnim i Njutnovim postupkom kao sekundarnim postupkom. Dati su neki dovoljni uslovi za lokalnu konvergenciju ovog postupka. U linearnom slučaju ovi uslovi opisuju oblast konvergencije VAOR postupka i takodje, kao podslučaj, oblast konvergencije AOR postupka.

Received by the editors February 6, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 15-22

Review of Research
Faculty of Science
Mathematics Series

ON A METHOD FOR SOLVING AN ALMOST LINEAR SYSTEM OF EQUATIONS

Dragoslav Herceg and **Ljiljana Cvetković**
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In order to solve some special classes of nonlinear systems of equations the Modified Accelerated Overrelaxation - Newton (MAORN) method, introduced in [1], is used. The same nonlinear systems have been considered by Chernyak, [5] . For their numerical solution he has used the AORN (Accelerated Overrelaxation - Newton) method. By using a similar technique as in Chernyak's papers [4] and [5], some sufficient conditions are obtained for the global convergence. The modified method and the convergence result are simpler than the ones from [5] .

AMS *Mathematics Subject Classification* (1980): 65H10.
Key words and phrases: Nonlinear system, iterative method.

1. Introduction

A system of nonlinear equations

$$f_i(x_1, x_2, ..., x_n) = 0,\ i \in N := \{1, 2, ..., n\},$$

will be solved by using the MAORN (Modified Accelerated Overrelaxation - Newton) method: for $i \in N$, $k = 0, 1, 2, \ldots$

$$x_i^{k+1} = x_i^k - \omega\Delta_i^{(k)}(\sigma), \quad \bar{x}_i^{k+1} = x_i^k - \sigma\Delta_i^{(k)}(\sigma),$$

$$\Delta_i^{(k)}(\sigma) = \frac{f_i(\bar{x}_1^{k+1}, \ldots, \bar{x}_{i-1}^{k+1}, x_i^k, \ldots, x_n^k)}{d_i(\bar{x}_1^{k+1}, \ldots, \bar{x}_{i-1}^{k+1}, x_i^k, \ldots, x_n^k)},$$

where σ, $\omega \in \mathbf{R}$, $\omega \neq 0$, and $d_i(x)$, $i \in N, x \in \mathbf{R}^n$ are arbitrary real nonzero functions. This method was introduced in our paper [1].

In the case when $d_i(x) = \partial f_i / \partial x_i$, $i \in N$, ($x = [x_1, x_2, \ldots, x_n]^T$), the MAORN method reduces to an AORN (Accelerated Overrelaxation - Newton) one, investigated by Chernyak in [5].

It is obvious that the MAORN method has one very important advantage: we do not have to calculate the partial derivatives of the given nonlinear functions. Because of that we have started to investigate this method. But, in convergence analysis, until now, we have obtained sufficient conditions for local convergence only. Also, these sufficient conditions depend on the value of the Jacobian matrix at the exact solution of our nonlinear system.

The aim of this paper is to give some sufficient conditions for the global convergence of the MAORN method, at least for some specias nonlinear systems.

We shall analyse the same class of nonlinear systems as in [5], i.e. nonlinear systems of the following type:

$$(1) \qquad f_i(x_1, x_2, \ldots, x_n) \equiv \sum_{j=1}^{n} a_{ij} x_j + g_i(x_i) - b_i = 0, \; i \in N,$$

where a_{ij}, $b_i \in \mathbf{R}$, $i, j \in N$, $a_{ii} \neq 0$, $i \in N$, and $g_i(t)$, $t \in \mathbf{R}$, are real nonlinear differentiable functions. For nonzero functions $d_i(x)$ we shall choose

$$d_i(x) = a_{ii}, \; i \in N, \; x \in \mathbf{R}^n,$$

so that the MAORN method for solving system (1) will have the following form:

$$\text{for } i \in N, \; k = 0, 1, 2, \ldots$$

$$(2) \qquad x_i^{k+1} = x_i^k - \omega \Delta_i^{(k)}(\sigma), \quad \bar{x}_i^{k+1} = x_i^k - \sigma \Delta_i^{(k)}(\sigma),$$

$$\Delta_i^{(k)}(\sigma) = \frac{f_i(\bar{x}_1^{k+1}, \ldots, \bar{x}_{i-1}^{k+1}, x_i^k, \ldots, x_n^k)}{a_{ii}}.$$

Obviously, it can be applied without knowledge of partial derivatives of the nonlinear functions g_i, $i \in N$. Note that in paper [5] Chernyak used the

AORN method, which in this special case has the same form as (2), except that for a_{ii} in each iteration we should take

$$a_{ii} + g_i'(x_i^k).$$

In order to obtain some sufficient conditions for the convergence (global, if it is possible) of the MAORN method (2), we shall use the result of Theorem 1 from [4]. We are going to describe this result in the next section.

2. Preliminaries

Let $\Phi(x,y)$ be a continuous mapping from $X \times X$ to X (X is a Banach space). For the numerical solution of

$$\Phi(x,x) = 0, \tag{3}$$

the following iterative method is used:

$$x^{k+1} = x^k - (\Phi_x'(x^k,x^k))^{-1}\Phi(x^k,x^k), \quad k = 0,1,\ldots, \tag{4}$$

where $\Phi_x'(x,y)$ is an F-derivative of Φ with respect to the first argument.

Theorem 1. [4] *Let the following conditions be satisfied:*

1. *For each $x \in S := \{x \in X |\ \|x - x^0\| \le r\}$ there exists the linear inverse operator $\Gamma(x) = [\Phi_x'(x,x)]^{-1}$ and $\|\Gamma(x)\| \le B$,*

$$\|\Gamma(x^0)\Phi(x^0,x^0)\| \le \eta_0;$$

2. *For each $x, y \in S$ it holds that*

$$\|\Phi_x'(x,y) - \Phi_x'(y,y)\| \le M\|x - y\|,$$

$$\|\Gamma(x)[\Phi(x,x) - \Phi(x,y)]\| <$$

$$< \delta_1\|x - y\| + \delta_2\|\Gamma(x)\Phi(y,y)\|\ \|x - y\| + \delta_3\|x - y\|^2,$$

where $\delta_1,\ \delta_2,\ \delta_3 \ge 0$;

3. $h_1 = \delta_1 + \delta\eta_0 < 1$, *where* $\delta = 0.5BM + \delta_2 + \delta_3$;

4. $r \ge t^* = (1 + \sum_{k=1}^{\infty}\prod_{j=1}^{k} h_j)\eta_0$, *where* $h_j = \delta_1 + \delta\eta_0 \prod_{\nu=1}^{j-1} h_\nu$, $j = 2,3,\ldots$.

Then there exists $x^* \in S^* = \{x \in X \mid \|x - x^0\| \le t^*\}$, *so that* $\Phi(x^*, x^*) = 0$ *and the iterative method (4) converges to* x^*.

3. Convergence result

If we intend to apply Theorem 1, we have to choose the continuous nonlinear operator Φ , such that system (3) is equivalent to nonlinear system (1), and the MAORN method (2) is the same as method (4).

It is easy to prove that the nonlinear operator given by

$$(\Phi(x,y))_i = f_i\left(v^{(i)}(y)\right) + \frac{a_{ii}}{\omega}(x_i - y_i),$$

$$v^{(i)}(y) = [v_1(y), \ldots, v_{i-1}(y), y_i, \ldots, y_n]^T , \quad v_i(y) = y_i - \frac{\sigma}{a_{ii}} f_i\left(v^{(i)}(y)\right),$$

satisfies both of the previous conditions.

The following convergence result is some kind of generalization of the Sassenfeld criteria. In the linear case the same result for the AOR method was given in [2]. In the nonlinear case, for the same class of nonlinear systems and for the AORN method, the analogous result was given in [5].

Let us denote

$$p_{1i}(\sigma) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{ii}|}\left(|1-\sigma| + |\sigma| p_{1j}(\sigma)\right) + \sum_{j=i+1}^{n} \frac{|a_{ij}|}{|a_{ii}|},\ i \in N;$$

$$p_{2i}(\sigma) = |\sigma| \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{ii}|} p_{2j}(\sigma) + 1,\ i \in N.$$

Theorem 2. *Let*

$$|g_i'(t)| \le \gamma,\ t \in \mathbf{R},\ i \in N;$$

$$|a_{ii}| \ge a > 0,\ i \in N;$$

$$\delta_1 = \max_{i \in N}\{|1-\omega| + |\omega| p_{1i}(\sigma) + \frac{|\omega|}{a}\gamma p_{2i}(\sigma)\} < 1.$$

Then for each $x^0 \in \mathbf{R}^n$, *the MAORN method converges to the solution* x^* *of system (1) and*

$$\|x^* - x^k\|_\infty \le \frac{|\omega|\ \|\Phi(x^k, x^k)\|_\infty}{a(1-\delta_1)},\ k = 0, 1, 2, \ldots \tag{5}$$

Proof. It is easy to see that

$$\Phi'_x(x,y) \equiv diag\left(\frac{a_{11}}{\omega},\ldots,\frac{a_{nn}}{\omega}\right).$$

Hence, conditions 1) from Theorem 1 are satisfied with $B = |\omega|/a$ and arbitrary x^0 and r. Related η_0 always exists.
The first relation in conditions 2) is satisfied with $M = 0$. The second one, with $\delta_2 = \delta_3 = 0$ follows from relation

$$|\frac{\omega}{a_{ii}}(\Phi_i(x,x) - \Phi_i(x,y))| < \{|1-\omega| + |\omega|p_{1i}(\sigma) + \frac{|\omega|}{a}\gamma p_{2i}(\sigma)\}\|x-y\|_\infty,$$

which can be proved by using the mathematical induction.
Condition 3) reduces to $h_1 = \delta_1 < 1$, and condition 4)

$$t^* = \eta_0/(1-\delta_1) \le r$$

is always satisfied, because r is arbitrary.
From Theorem 1 we conclude that method (4), i.e. the MAORN method converges to x^*, for which is $\Phi(x^*,x^*) = 0$ and

$$\|x^0 - x^*\|_\infty \le t^* = \eta_0/(1-\delta_1) \le \frac{|\omega|\ \|\Phi(x^0,x^0)\|_\infty}{a(1-\delta_1)}.$$

Because all our considerations do not depend on start value x^0 and because each iteration x^k can be considered as a start one, we conclude that error estimation (5) is valid.

□

If we denote

$$p(\sigma) = \max_{i\in N}\{p_{1i}(\sigma) + \frac{\gamma}{a}p_{2i}(\sigma)\},$$

$$l_i = \sum_{j=1}^{i-1}\frac{|a_{ij}|}{|a_{ii}|},\quad u_i = \sum_{j=i+1}^{n}\frac{|a_{ij}|}{|a_{ii}|},\ \ i \in N,$$

after some calculations we shall obtain

$$p(\sigma) \le \max_{i\in N}\frac{|1-\sigma|l_i + u_i + \gamma/a}{1-|\sigma|l_i}.$$

Now, it is easy to see that the condition $\delta_1 < 1$ from Theorem 2 can be replaced by the following condition:

$$\max_{i \in N} \frac{|1-\omega| + (|\omega||1-\sigma| - |\sigma||1-\omega|)l_i + |\omega|u_i + |\omega|\gamma/a}{1 - |\sigma|l_i} < 1.$$

So, as a direct corollary of Theorem 2 we obtain the following theorem.

Theorem 3. *Let* $1 - |\sigma|l_i > 0,\ i \in N;$

$$|g_i'(t)| \le \gamma,\ t \in \mathbf{R},\ i \in N;$$

$$|a_{ii}| \ge a > 0,\ i \in N;$$

$$\delta^* = \max_{i \in N} \frac{|1-\omega| + (|\omega||1-\sigma| - |\sigma||1-\omega|)l_i + |\omega|u_i + |\omega|\gamma/a}{1 - |\sigma|l_i} < 1.$$

Then for each $x^0 \in \mathbf{R}^n$ *the MAORN method converges to solution* x^* *of the system (1) and*

$$\|x^* - x^k\|_\infty \le \frac{|\omega|\ \|\Phi(x^k, x^k)\|_\infty}{a(1-\delta^*)},\ k = 0, 1, 2, \dots.$$

Let us remark that in the linear case, when $\gamma = 0$, the MAORN method (2) reduces to the AOR method and δ^* is an upper bound for the maximum norm of the AOR matrix, obtained in [2].

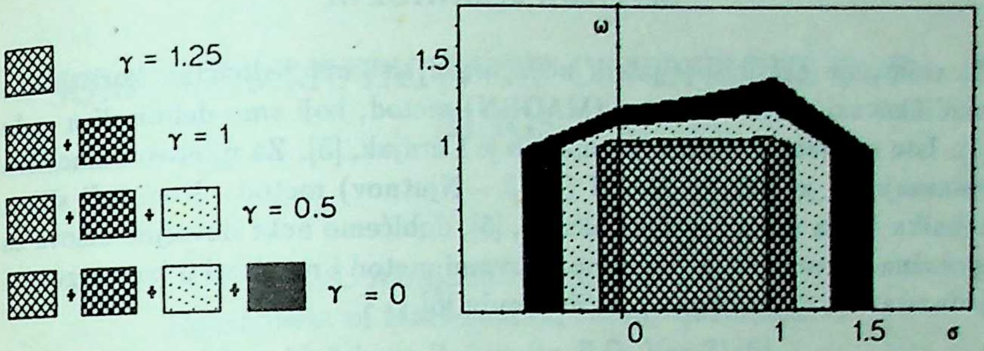
4. Numerical Example

We shall consider a system of nonlinear equations, for which there is

$$l_i = 1/3,\ i \in N \setminus \{1\};\ \ u_i = 1/4,\ i \in N \setminus \{n\};$$

$$l_1 = u_n = 0;\ \ a = 3.$$

For different nonlinearity (i.e. for different choices of the value γ), in the following picture we can see how to choose the parameters σ and ω, so that $\delta^* < 1$, i.e. the MAORN method converges (Theorem 3).



References

[1] Cvetković,Lj., Herceg,D.: On a generalized vSOR-Newton's method. IV Conference on Applied Mathematics, Vrdoljak,B., ed., Faculty of Civil Engineering, Split,1985, 99–102.

[2] Cvetković,Lj., Herceg,D.: An improvement for the area of convergence of the AOR method. Anal. Numer. Theor. Approx. 16(1987), 109–115.

[3] Cvetković,Lj., Herceg,D.: Nonlinear AOR method. Z. angew. Math. Mech. Bd. 68(1988), 486–487.

[4] Chernyak, V.Ya.: O skhodimosti metoda PVR-N'yutona. Prikladnaya matematika, Irkutsk, SÈI SO AN SSSR, 1978, 184–200.

[5] Chernyak, V.Ya.: O vliyanii nelineĭnosti reshaemoĭ sistemy na skhodimost' metodov Yakobi, PVR- i UVR-N'yutona. Chislenye metody analiza i ih prilozheniya. Akad. Nauk SSSR Sibirsk. Otdel., Energet. Inst., Irkutsk, 1983, 19–150.

REZIME

O JEDNOM POSTUPKU ZA REŠAVANJE SKORO LINEARNIH SISTEMA JEDNAČINA

Za rešavanje nekih specijalnih nelinearnih sistema jednačina koristićemo modifikovani AOR - Njutnov (MAORN) metod, koji smo definisali u radu [1]. Iste nelinearne sisteme posmatrao je Černjak, [5]. Za njihovo numeričko rešavanje on je koristio AORN (AOR - Njutnov) metod. Koristeći sličnu tehniku kao u radovima Černjaka [4], [5], dobićemo neke dovoljne uslove za globalnu konvergenciju. I naš modifikovani metod i rezultati o konvergenciji jednostavniji su od odgovarajućih u radu [5].

Received by the editors April 16, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 23-31

Review of Research
Faculty of Science
Mathematics Series

SOME COMMUTATIVITY CONDITIONS FOR RINGS

H.A.S. Abujabal and **M.S. Khan**
Department of Mathematics, Faculty of Science,
King Abdulazis University, P.O. Box 31464,
Jeddah, 21497, Saudi Arabia.

Abstract

Earlier results concerning the commutativity of rings with unity due to Quadri and Khan (Math. Japon., 33(2) (1988), 275-279) and Psomopoulos (Math. Japon., 29 (3) (1984), 371-373) have been obtained under a different set of conditions. The method of proof is based on an iteration technique.

AMS Mathematics Subject Classification (1980): Primary 16A70.
Key words and phrases: Rings, iteration, unity, commutativity.

1. Introduction

Recently, Psomopoulos [3], Quadri and Khan [4] proved some interesting commutativity theorems for rings with unity. While the work of Quadri and Khan [4] generalized a famous result due to Bell [1], the result of Psomopoulos [3] is new in its own right.

In this paper, we shall present some results on the commutativity of rings using polynomial identities as considered by the above authors but under a different set of conditions and employing an entirely new technique of proof which was given earlier by Tong [5].

Throught this note, R stands for a ring with unity 1. As usual, we write $[x, y] = xy - yx$, where x and y are arbitrary elements from R. The following well-known results will be frequently used in the sequel.

Lemma 1.1 *([2]). Let R be a ring with unity 1 and $f : R \to R$ be a function such that $f(x + 1) = f(x)$, for every x in R. If for some positive integer n, $x^n f(x) = 0$ for all x in R or $f(x)x^n = 0$ for all x in R, then necessarily $f(x) = 0$.*

Lemma 1.2 *([5]). Let R be a ring with unity 1. Let $I_0^r(x) = x^r$. If $k > 1$, let $I_k^r(x) = I_{k-1}^r(x+1) - I_{k-1}^r(x)$. Then, $I_{r-1}^r(x) = \frac{1}{2}(r-1)r! + r!x$; $I_r^r(x) = r!$, and $I_j^r(x) = 0$ for $j > r$.*

2. Results

In a recent paper, Wei Zong Xuan [6] proved that a semi prime ring R is commutative if for all x, y, z in R any one of the following conditions is satisfied

(i) $[x^2y^2 - xy^2x, z] = 0$,

(ii) $[x^2y^2 - yx^2y, z] = 0$.

Two of our results in this section are motivated by the above polynomial identities. Our conditions are in fact borrowed from a pre-print entitled " *On a commutativity condition for rings* " by M.A. Quadri et. al.

Theorem 2.1 *Let R be a ring with unity having the property: "there exist positive integers m and n such that*

$$(A) \qquad [x^m y^n - xy^n x, x] = 0$$

holds for all x, y in R." If every commutator in R is $2(n!)(m!)$ - torsion free, then R must be commutative.

Proof. We can write condition (A) of the theorem as

$$(1) \qquad x^m[x, y^n] - x[x, y^n]x = 0.$$

If $n = 1$, then (1) gives

(1') $$x^m[x,y] = x[x,y]x.$$

If $n > 1$, firstly, we shall apply iteration on y^n in (1). To do this, let $I_j(y) = I_j^n(y)$, for $j = 0, 1, 2, 3, \ldots$ Then (1) can be expressed as

(2) $$x^m[x, I_0(y)] - x[x, I_0(y)]x = 0.$$

Let us put $y = y + 1$ in (2). Then, we obtain

$$x^m[x, I_0(y+1)] - x[x, I_0(y+1)]x = 0.$$

Now,using Lemma 1.2, we have

$$x^m[x, I_1(y) + I_0(y)] - x[x, I_1(y) + I_0(y)]x = 0.$$

Using (2), we obtain

(3) $$x^m[x, I_1(y)] - x[x, I_1(y)]x = 0.$$

Again, letting $y = y + 1$ in relation (3) and using Lemma 1.2, we get

(4) $$x^m[x, I_2(y)] - x[x, I_2(y)]x = 0.$$

Finally, replacing y by $y + 1$ in (4) and then iterating $(n-1)$ times, we obtain

$$x^m[x, I_{n-1}(y)] - x[x, I_{n-1}(y)]x = 0.$$

But by Lemma 1.2, $I_{n-1}(y) = (n!)y + \frac{1}{2}(n-1)n!$. So,

(5) $$(n!)\{x^m[x,y] - x[x,y]x\} = 0,$$

and (1') follows immediately from (5).

If $m > 1$, we shall iterate x^m. To achieve it, we let $I_0(x) = x^m$ in (1') which then becomes

$$\{I_0(x)[x,y] - x[x,y]x\} = 0.$$

Let us replace x by $x+1$ in the above expression getting thereby

$$\{I_0(x+1)[x,y]-(x+1)[x,y](x+1)\}=0.$$

Then, as before, we get

$$\{I_1(x)[x,y]-[x,y]-[x,y]x-x[x,y]\}=0. \tag{6}$$

Now, replacing x by $x+1$ in (6) and using (6), we get

$$\{I_2(x)[x,y]-2[x,y]\}=0. \tag{7}$$

Again, putting $x=x+1$ in (7), we have

$$\{I_2(x+1)[x,y]-2[x,y]\}=0,$$

which when combined with (7) gives

$$\{I_3(x)[x,y]\}=0. \tag{8}$$

Similarly, we can get

$$\{I_4(x)[x,y]\}=0. \tag{9}$$

Finally, letting $x=x+1$ and iterating m times, we are left with

$$\{I_m(x)[x,y]\}=0.$$

But by Lemma 1.2, $I_m(x)=m!=I_m^m(x)$. So

$$(m!)[x,y]=0 \quad \text{i. e.} \quad [x,y]=0.$$

If $m=1$, from (1') by iteration on x we get

$$x[x,y]+[x,y]x=0,$$

and then

$$z[x,y]=0 \quad \text{i. e.} \quad [x,y]=0.$$

Therefore, R is commutative. This completes the proof.

Now, we shall present a short and easy proof of a result due to Quadri and Khan [4] under some extra torsion condition on commutators in the ring.

Theorem 2.2 *Let m and n be two fixed positive integers and R a ring with unity in which every commutator is $(m!)(n!)$ - torsion free. If the condition*

$$(B) \qquad [xy - y^m x^n, x] = 0$$

holds for all x, y in R, then R must be commutative for $m > 1$ and $n > 0$, and for $m > 0$ and $n > 1$.

Proof. Condition (B) is equivalent to

$$(1) \qquad x[x,y] = [x, y^m]x^n.$$

If $m > 1$, as in the proof of Theorem 2.1, we let $I_0(y) = y^m$.
Then (1) can be re-written as

$$(2) \qquad x[x,y] = [x, I_0(y)]x^n.$$

Now, letting $y = y + 1$ in the above expression and using (2), we get

$$[x, I_1(y)]x^n = 0.$$

A similar argument can be used to get

$$(3) \qquad [x, I_2(y)]x^n = 0.$$

Finally, replacing y by $y + 1$ in (3) and iterating $(m - 1)$ times, we obtain

$$(m!)[x,y]x^n = 0,$$

or equivalently

$$(m!)[x, yx^n] = 0.$$

As every commutator in R is $(m!)$ - torsion free, we get

$$[x, yx^n] = [x,y]x^n = 0,$$

and from there it follows that $[x, y] = 0$ because of Lemma 1.1

If $m = 1$ and $n > 1$, then (1) becomes

$$x[x,y] = [x,y]x^n,$$

and from there by iteration on x we get

$$(n!)[x,y] = 0 \quad \text{i. e.} \quad [x,y] = 0.$$

Thus R is commutative.

Following the idea developed in the proofs of our Theorem 2.1 and Theorem 2.2, one can supply another proof of a result due to Psomopoulos [3]. Here we shall only sketch of the proof for the sake of completeness.

Theorem 2.3 *Let $m > 1$ and $n > 0$ be two fixed positive integers and R a ring with unity in which every commutator is $(m!)$ - torsion free. If the condition*

$$(C) \qquad [x^n y - y^m x, x] = 0$$

holds for all x, y in R, then R must be commutative.

Proof. The equivalent form of condition (C) is given by

$$x^n[x,y] - [x, y^m]x = 0.$$

As in the proofs of the previous two results, firstly the iteration is applied to y^m by putting $y = y + 1$ to get

$$[x, I_1(y)]x = 0.$$

Now, as before, we get

$$(m!)[x,y]x = 0,$$

which along with Lemma 1.1 for $f(x) = [x, y]$ gives $[x, y] = 0$. Hence R is commutative.

Finally, we shall present a result which is similar to our Theorem 2.1.

Theorem 2.4 *Let $m > 1$ and $n > 0$ be two fixed positive integers and R a ring with unity in which every commutator is $2(n!)(m!)$ - torsion free and the condition*

$$[x^m y - xy^n x, y] = 0 \tag{D}$$

holds for all x, y in R. Then R must be commutative.

Proof. Let $m \neq 2$. We set $I_0(y) = yn$. Then (D) becomes

$$[x^m I_0(y) - xI_0(y)x, y] = 0.$$

Now, replacing y by $y + 1$ in the above expression and using the same expression again, we get

$$[x^m I_1(y) - xI_1(y)x, y] = 0.$$

Now, iterating the above expression n times we obtain

$$[x^m I_n(y) - xI_n(y)x, y] = 0.$$

Hence,

$$(n!)[x^m - x^2, y] = 0.$$

As every commutator in R is $(n!)$ - torsion free, we get

$$[x^m - x^2, y] = 0, \quad \text{i. e.} \quad [x^m, y] = [x^2, y].$$

It is not hard by iteration on x to have the last expression above reduced to

$$(m!)[x, y] = 0 \quad \text{if} \quad m > 2, \quad \text{and} \quad 2[x, y] = 0 \quad \text{if} \quad m = 1.$$

Once, again the hypothesis on commutators yields $[x, y] = 0$. For $m = 2$ and $n > 1$ from (D) we get by iteration on X:

$$[2xy^n - xy - yx, y] = 0,$$

i. e.

$$2 \cdot [x, y]y^n = [x, y]y + y[x, y],$$

and from there by iteration on y :

$$2 \cdot (n!)[x, y] = 0 \quad \text{i. e.} \quad [x, y] = 0.$$

Thus is all cases, R must be commutative.

Remarks.

i. Psomopoulos [3] assumed R to be s-unital ring. All previous theorems are still true for R as an s-unital ring. Namely, for arbitrary elements x, y of an s-unital ring R there exists $e = e(x, y)$ in R such that $ex = xe = x$ and $ey = ye = y$. With $e = e(x, y)$ instead of 1, Lemma 1.1 and Lemma 1.2 with $I_r^{r-1} = \frac{1}{2}(r-1)r!e + r!x$ and $I_r^r(x) = r!e$, obviously are still true for R as an s-unital ring. This permits our iteration on x or on y by setting $x + e$ or $y + e$ for x or y respectively.

ii. In the preprint of M. A. Quadri et. al, mentioned in the beginning of this section, the ring under consideration was semi-prime satisfying either of the conditions (A) or (D).

iii. Concering Theorem 2.1 and Theorem 2.4, we note that for $m > 1$ or $n > 1$, 2 divides $(m!)(n!)$ and then every commutator in R is $2 \cdot (m!)(n!)$ - torsion free if and only if every commutator in R is $(m!)(n!)$ - torsion free.

Acknowledgement.

We are grateful to Professor Dr. M. A. Khan and Professor E. Psomopoulos for providing us with their work which motivated the present study. Thanks is also due to the learned referee for several valuable comments.

References

[1] Bell, H.E.: A commutativity condition for rings, Canad. J. Math., 28 (1976), 986-991.

[2] Nicholson, W.K., Yagub, A.: A commutativity theorem for rings and groups, Canad. Math. Bull., 22 (1979), 419-423.

[3] Psomopoulos, E.: A commutativity theorem for rings, Math. Japon., 29 (3) (1984), 371-373.

[4] Quadri, M.A., Khan, M.A.: A commutativity theorem for associative rings, Math. Japon., 33 (2) (1988), 275-279.

[5] Tong, J.: On the commutativity of a ring with identity, Canad. Math. Bull., 27 (4) (1984), 456-460.

[6] Wei Zong Xuan: A note on the commutativity of semiprime rings, J. Math. Res. Exposition, 5 (1985), 109-110.

REZIME

NEKI USLOVI KOMUTATIVNOSTI ZA PRSTENE

Neki rezultati komutativnosti Quadrija i Khana (Math. Japon., 33(2)(1988), 275-279) i Psomopoulosa (Math. Japon., 29(3)(1984), 371-373) ovde su dobijeni pod drugim uslovima. Naš metod dokazivanja se zasniva na jednoj iteracionoj tehnici.

Recived by the editors March 6, 1989.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 33-38

Review of Research
Faculty of Science
Mathematics Series

SOME COMMUTATIVITY THEOREMS FOR RINGS VIA ITERATION

H.A.S. Abujabal and and **M.S. Khan**
Department of Mathematics, King Abdulazis University,
P.O. Box 31464 Jeddah-21497, Saudi Arabia

Abstract

Let m and n be fixed positive integers and R a ring with identity in which $x^m y^n = y^n x^m$ and $x^m y^{n+2} = y^{n+2} x^m$ hold for all $x, y \in R$. Then R is commutative provided it contains no non-zero elements x for which $p!x = 0$, where $p = \max\{m, n+1\}$. Another commutativity theorem for R under a different set of conditions is also obtained. The method of proof is based on an iteration technique.

AMS Mathematics Subject Classification (1980): Primary 16A70.
Key words and phrases: Ring, commutativity.

1. Introduction

Many sufficient conditions are well-known under which a given ring becomes commutative. Notable among them are some of those obtained by Jacobson, Kaplansky, Herstein, Bell, Kobayashi, Aby-Khuzam, Yagub, Tominaga, Hirano and several others. There are many commutativity theorems for rings satisfying polynomial identities involving consecutive integral powers of elements. The work of Abu-Khuzam [1], Abu-Khuzam - Yagub [2], Bell [3], Dong [4], Ligh - Richoux [5] and Tong [6] are worth mentioning.

In this note, we intend to prove some new commutativity theorems for a ring with identity in which a certain power of an element commutes with other elements having adjacent integral powers. We have used an iteration type method in the proof of these results.

The following results will be frequently used in the sequel. Throughout the rest of the paper, 1 stands for the indentity of a ring R.

Lemma 1.1 *(Bell [3]). Let x and y be elements of the ring R with 1. If there exists an integer $n \in Z^+$ for which $x^n y = (x+1)^n y = 0$, then $y = 0$.*

Lemma 1.2 *(Tong [6]). Let R be a ring with 1. Let $I_0^r(x) = x^r$. If $k > 1$, let $I_k^r(x) = I_{k-1}^r(1+x) - I_{k-1}^r(x)$. Then $I_{r-1}^r(x) = \frac{1}{2}(r-1)r! + r!x$; $I_r^r(x) = r!$, and $I_j^r(x) = 0$ for $j > r$.*

2. Results

The following is the main result of this paper.

Theorem 2.1 *Let m and n be fixed positive integers and R a ring with identity satisfyng the conditions*

$$x^m y^n = y^n x^m \text{ and } x^m y^{n+2} = y^{n+2} x^m,$$

for all x, y in R. Then R must be commutative provided it contains no non-zero elements x for which $p!x = 0$, where $p = \max\{m, n+1\}$.

Proof. Firstly, we note that the relations

$$x^m y^{n+2} = y^{n+2} x^2 = y^2(y^n x^m) = y^2(x^m y^n)$$

imply that

(1) $$[x^m, y^2]y^n = 0$$

for all x, y in R. Let $I_j(x) = I_j^m(x), \quad j = 0, 1, 2, 3, \ldots$ Then (1) can be written as

$$[I_0(x), y^2]y^n = 0.$$

Replacing x by $x+1$ in the above expression, we get

(2) $$[I_0(x+1), y^2]y^n = 0.$$

In view of Lemma 1.2, now equation (2) is reduced to

$$[I_1(x), y^2]y^n = 0. \tag{3}$$

Again, putting $x = x + 1$ in the above expression, we get

$$[I_1(x+1), y^2]y^2 = 0$$

for all x, y in R. Once again, an application of Lemma 1.2 yields for all x, y in R, the identity

$$[I_2(x), y^2]y^n = 0.$$

Finally, replacing x by $x + 1$ and iterating n times, we obtain

$$[I_{m-1}(x), y^2]y^n = 0$$

for all x, y in R. But, from Lemma 1.2, we known that $[I_{m-1}(x)] = \frac{1}{2}(m-1)m! + m!x$. So we are left with

$$m![x, y^2]y^n = 0.$$

Now, we shall iterate the power of y. For this, as above, we let $I_j(y) = I_j^n(y)$, $j = 0, 1, 2, \ldots$, so we get

$$m![x, y^2]I_0(y) = 0 \tag{4}$$

for all x, y in R. Let us replace y by $y + 1$ and then get from the above expression the identity.

$$m![x, 1 + 2y + y^2]I_0(y+1) = 0.$$

Using Lemma 1.2, we obtain

$$m!\{2[x, y](I_1(y) + I_0(y)) + [x, y^2](I_1(y) + I_0(y))\} = 0.$$

Then, in view of (4), we get

$$m!\{2[x, y]I_1(y) + 2[x, y]I_0(y) + [x, y^2]I_1(y)\} = 0.$$

Putting $y = y + 1$ again in the above expression, we have

$$m!\{2[x, y]I_1(y+1) + 2[x, y]I_0(y+1) + [x, (y+1)^2]I_1(y+1)\} = 0.$$

other elements having adjacent integral powers. We have used an iteration type method in the proof of these results.

The following results will be frequently used in the sequel. Throughout the rest of the paper, 1 stands for the indentity of a ring R.

Lemma 1.1 *(Bell [3]). Let x and y be elements of the ring R with 1. If there exists an integer $n \in Z^+$ for which $x^n y = (x+1)^n y = 0$, then $y = 0$.*

Lemma 1.2 *(Tong [6]). Let R be a ring with 1. Let $I_0^r(x) = x^r$. If $k > 1$, let $I_k^r(x) = I_{k-1}^r(1+x) - I_{k-1}^r(x)$. Then $I_{r-1}^r(x) = \frac{1}{2}(r-1)r! + r!x$; $I_r^r(x) = r!$, and $I_j^r(x) = 0$ for $j > r$.*

2. Results

The following is the main result of this paper.

Theorem 2.1 *Let m and n be fixed positive integers and R a ring with identity satisfyng the conditions*

$$x^m y^n = y^n x^m \text{ and } x^m y^{n+2} = y^{n+2} x^m,$$

for all x, y in R. Then R must be commutative provided it contains no non-zero elements x for which $p!x = 0$, where $p = \max\{m, n+1\}$.

Proof. Firstly, we note that the relations

$$x^m y^{n+2} = y^{n+2} x^2 = y^2 (y^n x^m) = y^2 (x^m y^n)$$

imply that

(1) $$[x^m, y^2] y^n = 0$$

for all x, y in R. Let $I_j(x) = I_j^m(x)$, $j = 0, 1, 2, 3, \ldots$ Then (1) can be written as

$$[I_0(x), y^2] y^n = 0.$$

Replacing x by $x+1$ in the above expression, we get

(2) $$[I_0(x+1), y^2] y^n = 0.$$

In view of Lemma 1.2, now equation (2) is reduced to

$$[I_1(x), y^2]y^n = 0. \tag{3}$$

Again, putting $x = x + 1$ in the above expression, we get

$$[I_1(x+1), y^2]y^2 = 0$$

for all x, y in R. Once again, an application of Lemma 1.2 yields for all x, y in R, the identity

$$[I_2(x), y^2]y^n = 0.$$

Finally, replacing x by $x + 1$ and iterating n times, we obtain

$$[I_{m-1}(x), y^2]y^n = 0$$

for all x, y in R. But, from Lemma 1.2, we known that $[I_{m-1}(x)] = \frac{1}{2}(m-1)m! + m!x$. So we are left with

$$m![x, y^2]y^n = 0.$$

Now, we shall iterate the power of y. For this, as above, we let $I_j(y) = I_j^n(y)$, $j = 0, 1, 2, \ldots$, so we get

$$m![x, y^2]I_0(y) = 0 \tag{4}$$

for all x, y in R. Let us replace y by $y + 1$ and then get from the above expression the identity.

$$m![x, 1 + 2y + y^2]I_0(y + 1) = 0.$$

Using Lemma 1.2, we obtain

$$m!\{2[x, y](I_1(y) + I_0(y)) + [x, y^2](I_1(y) + I_0(y))\} = 0.$$

Then, in view of (4), we get

$$m!\{2[x, y]I_1(y) + 2[x, y]I_0(y) + [x, y^2]I_1(y)\} = 0.$$

Putting $y = y + 1$ again in the above expression, we have

$$m!\{2[x, y]I_1(y + 1) + 2[x, y]I_0(y + 1) + [x, (y + 1)^2]I_1(y + 1)\} = 0.$$

Then, as above after simplication, we get

$$m!\{4[x,y](I_2(y) + 4[x,y]I_1(y) + [x,y^2]I_2(y)\} = 0.$$

Replacing y by $y+1$, in the above relation we get

$$m!\{6[x,y]I_3(y) + 6[x,y]I_2(y) + [x,y^2](I_3(y)\} = 0.$$

Finally, putting $y = y+1$ in the last expression and iterating n times, we obtain

$$m!\{2(n+1)[x,y]I_{n+1}(y) + 2(n+1)[x,y]I_n(y) + [x,y^2]I_{n+1}(y)\} = 0.$$

But Lemma 1.2 suggests that $I_{n+1}(y) = 0$ and $I_n(y) = n!$ giving thereby

$$2m!(n+1)![x,y] = 0$$

for all x, y in R. Now, from the hypothesis of the theorem, the commutativity of R follows immediately. This completes the proof.

Remark. For $m > n$, $p = m$. But we can always suppose that $m > n$, since for $m = 1$ (res. $m > 1$), the first assumption of Theorem 2.1 is fulfilled for $n+1$ (resp. m^k, $k = 1, 2, 3, \ldots$) instead of m.

The next result provides another criterion for a ring to be commutative.

Theorem 2.2 *Let R be a ring with identity and suppose that a positive integer m exists such that for any x and y in R there is a positive integer $n = n(x,y)$ with $x^m y^n = y^n x^m$ and $x^m y^{n+1} = y^{n+1} x^m$. Then, R is necessarily commutative, whenever $m!x \neq 0$ except for $x = 0$.*

Proof. We have

$$y^{n+1}x^m = y^{n+1} = (x^m y^n)y = (y^n x^m)y,$$

for all x, y in R. This implies

$$y^n[y, x^m] = 0 \tag{1}$$

for all x, y in R. Now, replacing y by $(y+1)$ similarly we get

$$(y+1)^{n'}[y, x^m] = 0 \tag{2}$$

for all x, y in R, where $n' = n(x, y+1)$. Then, Lemma 1.1 yields

$$[y, x^m] = 0 \tag{3}$$

for all x, y in R. Now we shall apply the iteration method.

Let $I_j(x) = I_j^m(x)$, for $j = 0, 1, 2, \ldots$
Then (3) can be re-written as

$$[y, I_0(x)] = 0 \tag{4}$$

for all x, y in R. Now, replacing x by $x+1$ in (4), we obtain for all x, y in R, the identity

$$[y, I_0(x+1)] = [y, I_1(x)] = 0 \tag{5}$$

Again, replacing x by $x+1$, (5) gives

$$[y, I_2(x)] = 0 \tag{6}$$

for all x, y in R.

Finally, after putting $x = x+1$ in (6) and iterating (m_1) times, we get

$$[y, I_{m-1}(x)] = 0. \tag{7}$$

But, from Lemma 1.2, $I_{m-1}(x) = \frac{1}{2}(m-1)m! + m!x$, so (7) is reduced to

$$m![y, x] = 0,$$

which, along with the hypothesis $m!x \neq 0$ except for $x = 0$, forces R to be commutative. This ends the proof.

Remarks.

i. In view of (3), our Theorem 2.2 can be derived from Theorem 3 of Tong [6]. However, we have include the proof of Theorem 2.2 just for the sake of completeness as Theorem 3 in [6] was stated without proof.
ii. A result similar to our Theorem 2.2 was obtained by Dong [4] for a semi prime ring. Our result is, however, proved under less stringent conditions on the ring.
iii. Only from (3) of the proof of Theorem 2.2, one can not conclude that R is necessarily commutative. To support this, we quote the famous example of the Corbas $(2, 2, \phi)$ - ring due to Bell [3]. In fact, one needs some torsion condition on the integer occuring as the power in (3) in order to establish the commutativity of R.

Acknowledgement

We are greatful to Professor H. E. Bell and Professor H. Abu-Khuzam for providing us with their reprints which have motivated the present study. Thanks is also due to the learned referee for most valuable comments on the earlier version of our paper.

References

[1] Abu-Khuzam, H.: n-torsion free rings with commuting powers, Math. Japon., 25 (1980), 37-42.

[2] Abu-Khuzam, H., Adil Yagub: Rings and groups with commuting powers, Internat. J. Math. and Math. Sci., 4 (1981), 101-107.

[3] Bell, H.E.: On rings with commuting powers, Math. Japon., 24 (1979), 473-478.

[4] Dong, N.C.: Commutativity of some non-associative rings with identity, Acta Sci. Natur. Univ. Jilin, 1 (1983), 1-9.

[5] Ligh, S., Richuox, A.: A commutativity theorem for rings, Bull. Austral. Math. Soc., 16 (1977), 75-77.

[6] Tong, J.: On the commutativity of a ring with identity, Canad. Math. Bull., 27 (4) (1984), 456-460.

REZIME

NEKE TEOREME O KOMUTATIVNOSTI PRSTENA PREKO ITERACIJE

Neka su m i n fiksni pozitivni celi brojevi i R prsten sa jedinicom u kome $x^m y^n = y^n x^m$ and $x^m y^{n+2} = y^{n+2} x^m$ važi za sve x, y. Tada je R komutativan ako ne sadrži ni jedan ne-nula element x za koji $p!x = 0$, gde je $p = \max\{m, n+1\}$. Druga teorema o komutativnosti za R je takodje dobijena, pod drugim uslovima. Metod dokazivanja je zasnovan na jednoj iteracionoj tehnici.

Recived by the editors August 8, 1989.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 39-56

Review of Research
Faculty of Science
Mathematics Series

A SPACE OPTIMAL ALGORITHM FOR ENUMERATING SPANNING TREES OF A CONNECTED GRAPH

Ladislav Novak, Žarko Karadžić
Faculty of Technical Sciences, University of Novi Sad
Trg Dositeja Obradovića 6, 21000 Novi Sad, Yugoslavia

and

Dragan M. Acketa
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

A simple space optimal algorithm is introduced for generating all the spanning trees of a connected graph G. The algorithm is based on a backtracking on the family F, which contains all the stars of G, with one single exception, sorted by non-decreasing cardinality. The efficiency of the algorithm is due to a local test, which gives the answer to the question: "Will the addition of a new edge x to a cycle-free subset S of edges of G make the set $S \cup x$ contain a cycle of G?"

AMS Mathematics Subject Classification (1980): Primary: 05C05 Secondary: 05C30, 68E10
Key words and phrases: Spanning trees, enumeration, backtracking

1. Introduction

We assume familiarity with some very basic notions of graph theory.

Let there be given a connected unoriented graph G by its standard representation, that is, by a collection of (un)ordered pairs of its vertices. We shall give an algorithm for an efficient enumeration of all the spanning subtrees of G.

Let the vertices of G be denoted by $1, 2, ..., n$. A *star* associated to a vertex i (denotation: $Star[i]$) is a set of edges of G, which are incident to i. The vertex-set of G and the edge-set of G will be denoted by $V(G)$ and $E(G)$ respectively.

A *spanning tree* T of G is a cycle-free subgraph T of G (= a tree), which is incident with all the vertices of G. Thus, if $|V(G)| = n$, then $|E(T)| = n - 1$.

This paper introduces a simple space optimal algorithm for generating all the spanning trees of a connected graph G. The algorithm is illustrated by an example, which includes the trace. Some experimental results concerning the speed of the algorithm on some classes of graphs, as well as the influence of sorting to this speed, are also discussed.

2. Algorithm

2.1 Structure of the algorithm

We are going to primarily identify the stars associated to the vertices of the input graph G. The family of $V(G) - 1$ stars of G may be implemented as an array of linked lists, the elements of which are associated to edges.

It turns out (this will be explained later) that the sorting of the identified stars w.r.t. their non-decreasing cardinalities is a useful operation in the stage of preprocessing.

The algorithm uses a standard backtracking on the stars, by keeping a current candidate for tree (a cycle-free set of edges), which is expanded whenever the addition of a new edge to the candidate would not produce a cycle within the enlarged candidate and reduced whenever there are no new possibilities for expansion:

The main shell BACKTRACK of the backtracking process seems as follows:

```
PROCEDURE BACKTRACK;
  BEGIN
    (* Let $C$ denote a (current) tree candidate
        (a subgraph of $G$ which has no cycles), and
        let $x$ denote a current edge of $G$,
        which we will try to add to $C$. *)

    C := the empty set ; (* Initialization *)
    x := the first edge from the first star ;

    REPEAT
        IF the set $C \cup x$ does not contain a cycle of G THEN
            IF $|C \cup x| = |V(G)| - 1$ THEN BEGIN
                output ( $C \cup x$ ); (* new tree is generated *)
                BACKWARDS
            END
            ELSE
                FORWARDS
        ELSE
            BACKWARDS
    UNTIL stop
    (* the TRUE value of the global boolean variable "stop"
          should be set during the last call of BACKWARDS *)
  END; (* BACKTRACK *)
```

Thus the backtracking is directed backwards in the cases when the addition of new edge x closes a cycle, as well as in the cases of a final success, that is, in the cases when a new spanning tree is produced by addition of x. In both cases the edge x is deleted from the current candidate and a new attempt should be made; that is, a new edge x should be found.

```
PROCEDURE BACKWARDS;
  BEGIN
    Give up from the addition of the new edge and choose the following
    edge $x$, for which a new addition atempt will be made
  END;

PROCEDURE FORWARDS;
  BEGIN
```

```
        Add the new edge to the current candidate
    END;
                                            (* Main program *)
BEGIN
    Identify the stars of the input graph $G$;
    (* Generate their associated linked lists *)

    Sort the identified stars by their non-decreasing cardinalities
    and delete one star with the maximal cardinality;

    BACKTRACK
END.
```

2.2 Local test and conrol connectivity structure

The key idea of the algorithm is to introduce a local test, which decides whether the addition of a new edge to a cycle-free set of edges will close a cycle or not (answer to the question posed in the first IF-statement within the main loop of the backtracking).

For this purpose, we would by no means construct the whole family of cycles of G. That family would be an unbearably large database with large memory requirements, which would be time-consuming even in the construction stage, but much more time-consuming in the stage of its use at each step of the backtracking.

Instead, we shall use a control structure, which would give exact answers to the very question that we ask, but which would not recognize the cycles of G themselves (such a recognition would be beyond our requirements). This structure will be adjusted at each step of the backtracking and will consist of a vector indexed by vertices of G, the components of which are the labels of connected components, which correspond to the current candidate.

The crucial idea of the local test is the following:

An edge $x = \{u, v\}$ is allowed to be added to a candidate C (and a step forwards should be made) if and only if the vertices u and v belong to different connected components of the graph $G(C)$, which is defined in the following way:

$$V(G(C)) = V(G) \quad \text{and} \quad E(G(C)) = V(C)\,.$$

Namely, if the vertices u and v belong to the same connected component of $G(C)$, then there already exists a path P connecting them. The addition of the edge $\{u,v\}$ would close the cycle $P + \{u,v\}$, which is a forbidden situation.

On the other hand, if the vertices u and v belong to different connected components , then the addition of $\{u,v\}$ would join two connected components (trees) into one (larger tree); $n-1$ consecutive joins of this type are sufficient to produce a spanning tree.

Initially, each vertex i ($1 \leq i \leq |V(G)|$) is a separate connected component labelled with the integer i. Connected components can be implemented as linked lists, the elements of which are associated to the vertices of G. Each one of these lists can be reached by two pointers, which are attached to the head and to the tail of the list, respectively.

It is essential to note that a new control vector is produced whenever the procedure FORWARDS is called and that the control vectors associated to lower levels of backtracking are kept in memory, in order to make possible their restoration in cases of back flow during the backtracking process. We shall put these comments more precisely:

We need to have exactly $n-2$ control vectors. Each control vector $v[i]$ is associated to Star[i] $(1 \leq i \leq n-2)$, the star ordering after sorting being assumed.

Suppose that the candidate C has j edges $(1 \leq i \leq n-2)$ and that the addition of a new edge $\{u,v\}$ is considered, where the vertices u and v belong to different connected components $Comp_1$ and $Comp_2$ of the graph $G(C)$. Let

- $head_1$ and $tail_1$ be the pointers showing to $Comp_1$;
- $head_2$ and $tail_2$ be the pointers showing to $Comp_2$.

The control vector $v[j+1]$ is generated (this is an additional task performed by the procedure FORWARDS) from the vector $v[j]$ by application of the following two commands:

(1) $v[j+1] := v[j]$; (* copy the previous control vector *)

(2) Replace in $v[j+1]$ the labels (associated to $Comp_2$) of all the vertices from $Comp_2$ by the label associated to $Comp_1$. (* This last label will become the common label of the joined tree *).

Remark. Note that the labels themselves are generated at random; in fact, they are highly dependent on the order of input data. However, their role will be successfully performed regardless of their values; all that the algorithm requires to know at some moment is whether the labels of the corresponding two vertices are equal or not.

Step 2) (the label replacement) is performed during the passage through the linked list associated to $Comp_2$. The new linked list, which corresponds to the joined tree, is generated by the following three operations:

- $newhead := head_1$;
- $newtail := tail_2$;
- $tail_1\hat{}.next := head_2$.

The procedure BACKWARDS requires restoration of the previous control vector solely in cases when the current star is exhausted. Then the last control vector, say $v[j+1]$, is replaced by the already existing control vector $v[j]$. Such a replacement will be impossible solely in the case when $j = 0$. However, the last condition denotes that the whole backtracking is completed (the value of the boolean variable stop in the above shell should be set to TRUE at that moment).

Remark. Note that the control vector $v[n-1]$ is not necessary. Those calls of the procedure FORWARDS, which complete a spanning tree, are necessarily followed by the deletion of the last added edge, and by coming back to the previous state, which is controlled by the vector $v[n-2]$.

2.3 Review of the algorithm

We shall proceed with a complete and detailed review of the above described algorithm:

ALGORITHM
for enumeration of all the spanning trees of a connected graph

```
ALGORITHM List-all-spanning-trees;

Input: Connected unoriented graph $G$ represented by its edges
Output: All the spanning trees of $G$, without repetitions

PROCEDURE Choice-of-the-following edge;
    BEGIN (* Choice-of-the-following edge *)

        IF there exists the next edge in the j-th list THEN choose it
        ELSE WHILE (end of the $j$-th list) AND ($j > 0$) DO BEGIN

            Make the current edge of the $j$-th list be equal
            to the first edge of that list;
            (* this is in accordance with the lexicographic principle;
                later visits to the $j$-th list should start from the
                beginning, because some move to the right
                has already been made on a lower level *)
            $j := j - 1$;

            Restore the previous state of the control structure
            (* this state has been kept since the previous visit *)
        END
    END; (* Choice-of-the-following edge *)
```

Remark. Global variable j is used within the main loop by means of a "side effect" (the same holds for the most of the variables used).

```
BEGIN (* List-all-spanning-trees *)

    Identify each one of the $n$ stars of $G$;
    Determine the degrees of these stars;
    Sort the degrees into a non-decreasing sequence;
    Delete the star with a maximal degree;
    $j := 1$;  (* the serial number of the current star *)
    REPEAT (* main loop of the backtracking *)

        IF the vertices of the current edge of the j-th
            list belong to two different connected components THEN

            IF $j = |V(G)| - 1$ THEN BEGIN
                (* the last edge ; there is no need to
                adjust the control structure *)
            Output tree;
            Choice-of-the-following edge
            END
            ELSE BEGIN
                Adjust the control structure;
                $j := j + 1$
            END

        ELSE
            Choice-of-the-following edge

    UNTIL $j = 0$

END ; (* List-all-spanning-trees *)
```

3. Proof of validity of the algorithm

In this section we shall prove that the algorithm given in Section 2. is correct. In addition, we shall show that the algorithm is space optimal.

We are primarily going to prove the following lemma:

Lemma 1. *Let G be a connected graph on n vertices, let S be a subset of $V(G)$ consisting of $n-1$ vertices, and let T be a spanning tree of G. Then there exists a bijection $f : E(T) \rightarrow S$, such that each edge x of T belongs to the star determined by the vertex $f(x)$.*

Proof. We label the vertices of T in the following way: the unique vertex z from $V(G) - S$ is labelled by 0, while the other vertices are labelled by the distance (within T) from z. Each edge x of T connects two vertices with different labels $k-1$ and k and the required bijection f can be defined with $f(x)=k$. □

Now we are able to prove the validity of the algorithm :

Theorem 1. *The algorithm given in Section 2. is correct, that is, it produces exactly all the spanning trees of the input connected graph G without duplications.*

Proof. We shall separately prove three facts concerning the presented algorithm. Two independent proofs will be given concerning the third of them:

(1) All the spanning trees of G are generated by our algorithm; that is. no spanning tree of G can be missed.

(2) The only sets which are output by our algorithm are the edge-sets of spanning trees of G .

(3) Each spanning tree of G is generated only once by our algorithm; that is, the algorithm does not produce duplicates.

Proof of (1): The above proved lemma implies that the set of edges of each spanning tree T is a transversal (= a system of distinct representatives) of the family F of stars indexed by vertices of S. More precisely, each edge x of T should be represented as a member of $Star(f(x))$. Such a representation guarantees that an algorithm which passes through all the transversals of the family F will not miss any spanning tree of T.

Proof of (2): Each output set has cardinality $n-1$ and has no cycles by construction, that is, it IS a spanning tree of G.

The first proof of (3): Suppose that there exists a spanning tree T, which has two distinct associated transversals $a_1,...,a_{n-1}$ and $b_1,...,b_{n-1}$. Thus both sets $\{a_1,...,a_{n-1}\}$ and $\{b_1,...,b_{n-1}\}$ are equal to the set $E(G)$ and there exists an index i, such that $a_i \neq b_i$ for some i from the set $\{1,2,...,n-1\}$.

It is easy to conclude that there exists a sequence $i=i_1,...,i_k$ of indices such that

$$a_{i1}=b_{i2},\ a_{i2}=b_{i3}\ ,\ \dots\ ,\ a_{ik}=b_{i1}\ .$$

Note that, e.g., $a_{i1} = b_{i2}$ implies that $Star(i1)$ and $Star(i2)$ have a common edge of G. It follows that the edges $\{i1, i2\}, \{i2, i3\}, ..., \{ik, i1\}$ of T constitute a cycle, which contradicts the assumption that T is a tree.

The second proof of (3): It suffices to show that the bijection f introduced in the above lemma is unique. In other words, we should show that the transversal representation of a tree T w.r.t. the stars associated to the vertices of S is unique.

Let z denote the only vertex in the difference of the sets $V(G)$ and S. Any edge of T, which is of the form $\{z, y\}$, must be considered as the member of $Star(y)$, because $Star(z)$ is not included in our representation of spanning trees. In a similar way, by expanding the process started in the "root" z, we can uniquely determine the star to which each of the remaining edges of T should be attached. □

There remains to show that our algorithm is space optimal :

Theorem 2. *The given algorithm is space optimal; more precisely, its worst-case space complexity is equal to the size of the input; that is to* $O(n^2)$, *where* n *is the number of vertices of* G.

Proof. The structures that consume most of space in our algorithm are arrays of linked lists. Their particular lists are associated to vertices of G; that is, to their stars. However, the space necessary to store them is obviously of an $O(n^2)$ size. □

4. An illustrative example

We are going to give the trace of the above described algorithm , which is applied to a small-size example :

Let the connected graph G be given on 4 vertices (numerated 1 thru 4) and 6 edges (denoted $e1, ..., e6$) as follows:

$e1 = (1, 2)$; $e2 = (2, 3)$; $e3 = (2, 3)$;
$e4 = (3, 4)$; $e5 = (4, 1)$; $e6 = (4, 2)$;

The linked lists which corresponding to particular connected stars seem after sorting as follows :

$Star[1]: \to e5 \to e1$
$Star[3]: \to e4 \to e3 \to e2$
$Star[4]: \to e6 \to e5 \to e4$
$Star[2]: \to e6 \to e3 \to e2 \to e1$

We shall proceed with a trace of the backtracking in this example:

Applications of the procedure FORWARDS are denoted by "ADD", while the applications of the procedure BACKWARDS are denoted by "TRY" or "DELETE". "TRY" denotes an unsuccessful attempt of attaching new edge on the same level , while "DELETE" reduces the candidate after all the possibilities for augmentation on the previous level are exhausted. The vector of incidencies of the vertices 1,2,3,4,5,6 (in order) to the labelled connected components. The labels of which are the elements of the vector are provided at each step of the algorithm.

Note that the current candidate determines uniquely the corresponding incidence vector. However, both of them are not sufficient to determine uniquely the position within the backtracking . Therefore, an additional piece of information is also added (in brackets) with each position: the next edge from the considered star, the addition of which should be attempted at the next step; if such an edge does not exist, then the algorithm deletes the last edge and comes back to the previous star in search of a new edge to be added. Such a situation is denoted by "—-" in the column (next).

TRANSFORMATION			CURRENT CANDIDATE	(next)	INCIDENCE VECTOR
			- - - - -	(*e*5)	1 2 3 4
ADD	e5		e5	(*e*4)	4 2 3 4
ADD	e4		e5 e4	(*e*6)	3 2 3 3
ADD	e6	*TREE*	e5 e4 e6	- - - -	- - - - -
DELETE	e6		e5 e4	(*e*5)	3 2 3 3
TRY	e5		e5 e4	(*e*4)	3 2 3 3
TRY	e4		e5 e4	- - - -	3 2 3 3
DELETE	e4		e5	(*e*3)	4 2 3 4
ADD	e3		e5 e3	(*e*6)	4 2 2 4
ADD	e6	*TREE*	e5 e3 e6	- - - -	- - - - -
DELETE	e6		e5 e3	(*e*5)	4 2 2 4
TRY	e5		e5 e3	(*e*4)	4 2 2 4

TRANSFORMATION	CURRENT CANDIDATE	(next)	INCIDENCE VECTOR
ADD e4	*TREE* e5 e3 e4	– – – –	– – – – – –
DELETE e4	e5 e3	– – – –	4 2 2 4
DELETE e3	e5	(e2)	4 2 3 4
ADD e2	e5 e2	(e6)	4 2 2 4
ADD e6	*TREE* e5 e2 e6	– – – –	– – – – – –
DELETE e6	e5 e2	(e5)	4 2 2 4
TRY e5	e5 e2	(e4)	4 2 2 4
ADD e4	*TREE* e5 e2 e4	– – – –	– – – – – –
DELETE e4	e5 e2	– – – –	4 2 2 4
DELETE e2	e5	– – – –	4 2 3 4
DELETE e5	– – – – –	(e1)	1 2 3 4
ADD e1	e1	(e4)	1 1 3 4
ADD e4	e1 e4	(e6)	1 1 3 3
ADD e6	*TREE* e1 e4 e6	– – – –	– – – – – –
DELETE e6	e1 e4	(e5)	1 1 3 3
ADD e5	*TREE* e1 e4 e5	– – – –	– – – – – –
DELETE e5	e1 e4	(e4)	1 1 3 3
TRY e4	e1 e4	– – – –	1 1 3 3
DELETE e4	e1	(e3)	1 1 3 4
ADD e3	e1 e3	(e6)	1 1 1 4
ADD e6	*TREE* e1 e3 e6	– – – –	– – – – – –
DELETE e6	e1 e3	(e5)	1 1 1 4
ADD e5	*TREE* e1 e3 e5	– – – –	– – – – –
DELETE e5	e1 e3	(e4)	1 1 1 4
ADD e4	*TREE* e1 e3 e6	– – – –	– – – – –
DELETE e4	e1 e3	– – – –	1 1 1 4
DELETE e3	e1	(e2)	1 1 3 4
ADD e2	e1 e2	(e6)	1 1 1 4
ADD e6	*TREE* e1 e2 e6	– – – –	– – – – –
DELETE e6	e1 e2	(e5)	1 1 1 4
ADD e5	*TREE* e1 e2 e5	– – – –	– – – – –
DELETE e5	e1 e2	(e4)	1 1 1 4
ADD e4	*TREE* e1 e2 e6	– – – –	– – – – –
DELETE e4	e1 e2	– – – –	1 1 1 4
DELETE e2	e1	– – – –	1 1 3 4
DELETE e1	– – – – –	– – – –	

Total number of trees : 13

Remark. The steps backwards in the above trace are always preceded by either producing a tree or by an attempt to add a new edge to a candidate which already contains that edge (the last situation is equal to the occurrence of a cycle of size 2 within the candidate). In the general case the attempts will also be made to close some cycles of a larger size. These attempts will be prevented in principally the same manner by use of the control vector.

5. Experimental results on some graph classes

In this section we are going to give a number of examples, with which the tests on the action of the above described algorithms were performed. The obtained experimental results include the total number of trees with graphs of a specified type, the total time (in seconds), which was used by a certain PC AT 386 (in all cases the same one) to enumerate these trees, as well as the average time (in 10^{-6} seconds) of the calculation per tree.

Example 1. Let K_n denote the complete graph on n vertices. It is well-known that the total number of spanning trees of Kn is n^{n-2}.

Experimental results for the class K_n:

n	5	6	7	8	9	10
num.of trees	125	1296	16807	262144	4782969	100000000
time(sec)	0.06	0.11	1.10	16.15	281.27	5664.85
10^{-6} s p.t.	480	84.9	65.4	61.6	58.8	56.6

Conclusion: It seems that the average speed per tree does note an increase with n. In fact, it slowly decreases (perhaps towards a limit ?), with the exception of the first few values, where there is a great decrease. Such a behaviour might be explained by some constant time (probably spent with the input-output business), which does not depend on the size of the problem.

This example suggests that the complexity of the algorithm is equal to the size of the output (that is, that a constant time is needed for each particular generated tree).

Example 2. Let A_n denote the graph on $2 \cdot n$ vertices, numerated by

$1, 2, ..., 2 \cdot n$, and the following $3 \cdot n - 2$ edges:

$$\begin{array}{ll} [i, i+1] & \text{for } i = 1, 2, ..., 2 \cdot n - 1 \\ [i, 2 \cdot n + 1 - i] & \text{for } i = 1, 2, ..., n - 1 \end{array}$$

Experimental results for the class A_n:

n	1	2	3	4	5	6	7	8	9
num. of trees	1	4	15	56	209	780	2911	10864	40545
time(sec)	–	–	–	–	0.11	0.50	3.57	14.94	45.37
10^{-6} s p.t	–	–	–	–	526.3	641.0	1226.4	1375.2	1119.0

n	10	11	12	13
num. of trees	151316	564719	2107560	7865521
time(sec)	322.02	1248.34	3528.20	25076.45
10^{-6}s p/t	2128.1	2210.6	1674.1	3188.1

Example 3. Let B_n denote the graph on $4 \cdot n$ vertices, numerated by $1, 2, ..., 4 \cdot n$, and the following $8 \cdot n - 2$ edges:

$$\begin{array}{ll} [i, i+1] & \text{for } i = 1, 2, ..., 4 \cdot n - 1 \\ [i, 4 \cdot n + 1 - i] & \text{for } i = 1, 2, ..., 2 \cdot n - 1 \\ [2 \cdot i - 1, 2 \cdot i] & \text{for } i = 1, 2, ..., n \\ [2 \cdot i, 4 \cdot n + 2 - 2 \cdot i] & \text{for } i = 1, 2, ..., n\,. \end{array}$$

Experimental results for the class B_n:

n	1	2	3	4
num. of trees	13	533	21801	891709
time(sec)	–	0.22	12.79	904.57
10^{-6} s p/t	–	412.8	586.7	279.5

Remark. Note that the graphs B_n are obtained by inserting some edges in the corresponding graphs A_n. In such a case the average time per tree decreases. It seems as if the graphs with a "higher edge density" have smaller average time per tree and conversely.

Example 4. Let P_n denote the path consisting of n edges. The time (in hds/s = sec/100) necessary to output the only spanning subtree of P_n by our algorithm is given below for sime values of n:

n	40	50	60	70	80	90	100	110	120	130	140	150
	5	11	16	22	27	39	50	55	66	77	88	104

n	160	170	180	190	200	210	220	230	238
	115	132	148	165	187	208	225	247	263

This time has an almost linear increase, which is quite an expectable behaviour, since a linear growth of the structure is also present.

6. On the influence of star sorting

We have made an extensive study of the dependence between the ordering of stars (by their cardinalities) and the speed of our algorithm for generating all the spanning trees in the following example:

Let the connected graph G be given on 7 vertices (numerated 1 thru 7) and 10 edges (denoted e1,...,e10) as follows:

$$e1 = (1,2);\quad e2 = (1,3);\quad e3 = (2,3);\quad e4 = (2,5);$$
$$e5 = (3,4);\quad e6 = (3,5);\quad e7 = (3,6);\quad e8 = (4,5);$$
$$e9 = (5,6);\quad e10 = (1,7);$$

We observe that the degrees of the vertices 1,2,3,4,5,6,7 are 3,3,5,2,4,2,1 respectively. Of course, the same holds for the cardinalities of the corresponding stars. On the other hand, 7 is the only vertex of degree 1, 4 and 6 are the vertices of degree 2, 5 is the only vertex of degree 4 and 3 is the only vertex of degree 5.

We have compared 480 runnings of our algorithm on the graph G (which has 52 spanning trees), which differed from each other solely in the ordering of the stars (equivalently, the corresponding vertices). These orderings were chosen as follows:

For each one of the 120 permutations of the degree set 1,2,3,4,5, we have introduced four corresponding orderings. We have adjoined to the degrees (in order) either the unique corresponding vertex or (in the case of degrees 2 and 3) two corresponding vertices (there are four possible arrangements in this last case).

For example, given the permutation 21435, we can primarily adjoin the auxiliary sequence 2214335 to it, which further leads to the following four orderings:

4,6,7,5,1,2,3; 4,6,7,5,2,1,3; 6,4,7,5,1,2,3; 6,4,7,5,2,1,3.

With each one of the 480 orderings, we have counted the number of steps down (= the number of steps up) in the backtracking across the list of stars.

Observations:

1) There is a considerable difference between the number of counted steps, as well as with the runtime, with distinct orderings, with this particular graph G. It seems that the same is generally valid for the graphs with sufficiently diverse vertex degrees.

2) There is a strong correlation between the number of counted steps, although there is also some alteration w.r.t. the expected strict direct proportion.

Example. The first and the second ordering associated to the permutation 41235, that is, 5746123 and 5746213 respectively, have the respective numbers of steps 51 and 59, but the respective runtimes are 11 hds/s and 6 hds/s.

Example. As two antipodes, we might consider the ordering 7461253, obtained lexicographically with the first permutation 12345, which has 35 steps and 5 hds/s, and the ordering 3512647, obtained lexicographically with the last permutation 54321, which has 286 steps and 44 hds/s. All the four figures (35, 5, 286 and 44) are the absolute extremes within this test.

3) Given a quadruple of orderings, note that one ordering can be transformed to another one according to the following scheme:

first → third , by transposition (46)
second → fourth , by transposition (46)
first → second , by transposition (12)
third → fourth , by transposition (12) .

With all the 120 quadruples of orderings, the number of steps with the first and the third (respectively, with the second and the fourth) ordering have the same numbers of steps (although in many cases all the four numbers are equal). This seemingly interesting feature is valid regardless of whether 2 is in front of 3, or conversely, in the initial permutation.

The question is: Why does the permutation (12) , of vertices of degree 3, have stronger consequences (on the number of steps) than the permutation (46), of vertices of degree 2 ?

4) In many cases we have coinciding runtimes with the first and third, respectively with the second and the fourth ordering associated to a permutation. However, in many other cases, the coincidence relates to the first and the fourth, respectively the second and the third ordering. At last, there are many cases with which there is one outstanding (mostly greater) runtime with respect to the other three. Adding the cases with four coincidences, these are the four by far most common behaviours of the runtimes.

Remark. Observe that some other elementary steps should be counted as well. Besides up and down, it would be interesting to count steps to the right, within the particular stars.

5) It would be of extreme interest to study the relationships between the initial permutation and the (average) number of steps and runtime associated to the corresponding orderings. It seems that the last values are proportional to a specific function defined on permutations p_1, p_2, p_3, p_4, p_5:

$$\sum_{i=1}^{5} |i - p_i| \ ,$$

although some refinement is also necessary. (Look to the permutations with the same value of this function!)

6) It seems reasonable to study the effect of sorting with different types of examples. Thus after the sorting is completed, it is important which of the following monotonously increasing behaviours is present:
a) concave **b)** linear **c)** convex **d)** constant

Paper [1] contains another algorithm for the enumeration of all the spanning trees of a connected graph. The worst-case time complexity of the algorithm is $O(n + m + n \cdot t)$, where n and m denote respectively the number of vertices and the number of edges, while t denotes the number of spanning trees. This paper also contains a detailed comparative study of the existing algorithms for spanning tree enumeration, as well as an extensive list of references on the subject.

References

[1] Winter, P.: An algorithm for the enumeration of spanning trees, BIT 26(1986), 44-62.

REZIME

JEDAN MEMORIJSKI OPTIMALAN ALGORITAM ZA NABRAJANJE PREPOKRIVAJUĆIH STABALA POVEZANOG GRAFA

Opisan je jedan jednostavan, i u pogledu memorijskih zahteva optimalan, algoritam za generisanje svih prepokrivajućih stabala datog povezanog grafa G. Izbor grana se vrši bektrekingom po familiji koja sadrži sve zvezde grafa, sa izuzetkom jedne (proizvoljne). Efikasnost algoritma se zasniva na *lokalnosti* testa, kojim se ispituje da li će dodavanje jedne nove grane x skupu grana S, koji ne sadrži konturu grafa G, – dovesti do toga da $S \cup x$ sadrži takvu konturu.

Received by the editors March 12, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 57-64

Review of Research
Faculty of Science
Mathematics Series

CONNECTEDNESS OF THE GENERALIZED DIRECT PRODUCT OF REGULAR DIGRAPHS

Milenko V. Petrić
Advanced Technical School,
Školska 1, 21000 Novi Sad, Yugoslavia

Abstract

Using the spectral method a theorem is proved giving the necessary and sufficient conditions for the generalized direct product (GDP) of regular (di)graphs to be connected (di)graph.

AMS Mathematics Subject Classification (1980): Primary 05C40, Secondary 05C20.
Key words and phrases: Connectedness, product of digraphs.

1. Introduction

Let B be a set of n-tuples $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ of symbols 1,0,-1 which does not contain an n-tuple (0,0,…,0). If G is a digraph with at most ν parallel arcs (if there are no parallel arcs then $\nu = 1$) between any two vertices or loops of a vertex in G, then complement $\bar{G}$ of G is a digraph which has the same set of vertices as G and for any ordered pair (u, v) of vertices u and v of $\bar{G}$ (if loops are not allowed then $u \neq v$) from u to v lead $\nu - a$ arcs, where a is the number of arcs leading from u to v in G.

The following definition is introduced in [5] and [6] and represent a generalization of the definition of the GDP of graphs [7] to digraphs (digraphs can have multiple arcs and or loops).

Definition 1. *The generalized direct product with a basis B of digraphs G_1, G_2 ,*
$\ldots, G_n$ is the digraph G=GDP $(B; G_1, \ldots, G_n)$ whose set of vertices is the Cartesian product of the sets of vertices of digraphs $G_1, G_2, \ldots, G_n$.For two vertices $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ of G construct all the possible arc selections of the following type. For each $(\beta_1, \ldots, \beta_n)$ from B, for which $u_k = v_k$ holds whenever $\beta_k = 0$, and for any i ($i = 1, 2, \ldots, n; i \neq k$) select an arc going from u_i to v_i in G_i if $\beta_i = 1$ and an arc going from u_i to v_i in $\bar{G}_i$

mboxif $\beta_i = -1$. The number of arcs going from u to v in G is equal to the number of such selections.□

If B consists of n-tuples of symbols 1 and 0 the resulting operation is called a non-complete extended p-sum (NEPS). Conditions, under which this operation on arbitrary digraphs is a strongly connected digraph, have been investigated in [3].

It is clear that the GDP $(B; G_1, \ldots, G_n)$ can be connected if any one of the digraphs $G_1, \ldots, G_n$ is not connected. On the other hand, if $G_i (i = 1, \ldots, n)$ has at least two vertices, then the GDP $(B; G_1, \ldots, G_n)$ is not connected if basis B has not the property (D): that for every $j \in \{1, \ldots, n\}$ there exist in B at least one n-tuple $(\beta_1, \ldots, \beta_n)$ with $\beta_j \neq 0$. (This cond

ition implies that the GDP, effectively, depends on each G_i.) Further, if $G_j(\bar{G}_j)$, for some $j \in \{1, \ldots, n\}$, is not connected, then the GDP $(B; G_1, \ldots, G_n)$ is not connected, if each n-tuple $(\beta_1, \ldots, \beta_n)$ in B has $\beta_j \neq -1$ $(\beta_j \neq 1)$.

In order to get conditions for $B, G_1, \ldots, G_n$ being GDP $(B; G_1, \ldots, G_n)$ strongly connected, we shall investigate the connectivity of the isomorphic GDP given from this one by the replacement of each non-connected digraph G_j by its complement $\bar{G}_j$ together with the replacement in each $\beta \in B$ j-th coordinate 1 by -1 and vice versa (Proposition 3 in [5]). So we suppose that in GDP $(B; G_1, \ldots, G_n)$ all the $G_1, \ldots, G_n$ are strongly connected. In the same manner, we shall consider the GDP whose basis B has property (E): if for any $j \in \{1, \ldots, n\}$ all β_j (j-th coordinate of $\beta \in B$) take only one value among 1 and -1, then $\beta_j \in \{1, 0\}$.In this case if G_j is not connected, then GDP $(B; G_1, \ldots, G_n)$ is not connected.

A digraph is called a regular of degree r if each indegree and each outdegree equals r. The cycle, denoted by $\vec{C}_p$, is a connected regular digraph of degree 1 with p verices. Digraph G is called bicomplete if G is a complete bipartite graph (symmetric). Notice that the strong components of a regular digraph are its components too.

We shall investigate the connectedness of the GDP of regular digraphs by using Theorems 0.3 and 0.4 from [2].

For this purpose we need the following results of [5].

Theorem 1. *([5]). The GDP of regular digraphs is a regular digraph.*

Theorem 2. *([5]). Let G be a regular digraph of order p, degree r and maximum number of parallel arcs or loops of a vertex equal to ν and let $S = \{\lambda_1 = r, \lambda_2, \dots, \lambda_n\}$ be the spectrum of G. The complement $\bar{G}$ of the digraph G has the spectrum given by*
$S_1 = \{\bar{\lambda}_1 = \nu \cdot p - r, \bar{\lambda}_2 = -\lambda_2, \dots, \bar{\lambda}_p = -\lambda_p\}$, *if loops are allowed,*
$S_2 = \{\bar{\lambda}_1 = \nu \cdot p - \nu - r, \bar{\lambda}_2 = -\nu - \lambda_2, \dots, \bar{\lambda}_p = -\nu - \lambda_p\}$, *if loops are not allowed.*

The eigenvectors correspoding to λ_i and $\bar{\lambda}_i$ are the same and the eigenvector belonging to the eigenvalue λ distinct from r in G is orthogonal to the eigenvector (1,...,1) belonging to the r. □

Theorem 3. *([5]). For i=1,2,...,n let G_i be a regular digraph with p_i vertices, degree r_i and let λ_{ij_i} $(\bar{\lambda}_{ij_i})$ $j_i = 1, 2, \dots, p_i$ be the spectrum of $G_i(\bar{G}_i$ determined by Theorem 2). Then, the spectrum of the GDP $(B; G_1, \dots, G_n)$ consists of all the possible values of $\Lambda_{j_1,\dots,j_n}$ where*

(1)
$$\Lambda_{j_1,\dots,j_n} = \sum_{\beta \in B} \lambda^{[\beta_1]}_{1_{j_1}} \lambda^{[\beta_2]}_{2_{j_2}} \dots \lambda^{[\beta_n]}_{n_{j_n}},$$

$$\lambda^{[1]}_{i_{j_i}} = \lambda_{i_{j_i}}, \lambda^{[0]}_{i_{j_i}} = 1, \lambda^{[-1]}_{i_{j_i}} = \bar{\lambda}_{i_{j_i}}, j_i = 1, 2, \dots, p_i;\ i = 1, 2, \dots, n$$

The eigenvector $x_{j_1, \dots, j_n} = x_{1_{j_1}} \otimes x_{2_{j_2}} \otimes \dots \otimes x_{n_{j_n}}$ *corresponds to the eigenvalue* $\Lambda_{j_1, \dots, j_n}$, *where* $x_{i_{j_i}}$ *is the eigenvector belonging to* $\lambda_{i_{j_i}}$ *in* G_i *and* $\otimes$ *denotes the Kronecker product of matrices.* □

2. Main theorems

Let h be the greatest common divisor of the lengths of all the cycles in a digraph G. The digraph G is called primitive if it is strongly connected and $h = 1$ [4,p.210], and imprimitive if it is strongly connected and $h > 1$. In the second case, h is called the index of imprimitivity (h is the index of imprimitivity of the adjacency matrix of the digraph G as well [1,p.183].

Theorem 4. *Let* $G_i, i = 1, 2, \dots, n$ *be a regular,connected, non-complete symmetric digraph of degree* r_i *containing* $p_i (p_i \geq 2)$ *vertices. Suppose also that loops are not allowed in* G_i *and* $\bar{G}_i$ $(i = 1, 2, \dots, n)$. *Further, let digraphs* $G_{i_1}, G_{i_2}, \dots, G_{i_s}(\{i_1, i_2, \dots, i_s\} \subset \{1, 2, \dots, n\})$ *be imprimitive with the imprimitivity indices* $h_{i_1}, h_{i_2}, \dots, h_{i_s}$ *respectively. The GDP with the basis B satisfying conditions (D) and (E) of digraphs* $G_1, G_2, \dots, G_n$ *is a connected digraph if and only if for every non-empty subset* $\{j_1, j_2, \dots, j_k\}$ *of* $\{i_1, i_2, \dots, i_s\}$ *and every choice of integers* $\ell_{j_1}, \ell_{j_2}, \dots, \ell_{j_k}, 1 \leq \ell_{j_t} \leq h_{j_t} - 1, t = 1, 2, \dots, k$ *there exist* $\beta \in B$ *such that* $\{j_1, j_2, \dots, j_k\} \cap \{\nu \mid \beta_\nu \neq 0\} = I_\beta \neq 0$ *and the following holds: for at least one* $\nu \in I_\beta$, *for which* $\beta_\nu = -1, G_\nu$ *is neither bicomplete nor* $\vec{C}_3$ *or otherwise*

$$\sum_{\nu \in I_\beta} \left(\frac{1}{2}(1 + \beta_\nu)\frac{\ell_\nu}{h_\nu} + \frac{\ell_\nu}{3}(1 - \beta_\nu)(p_\nu - 2r_\nu)\right)$$

is not an integer.

Proof. According to Theorems 0.3 and 0.4 from [3] a digraph G, with an adjecency matrix A, is strongly connected if and only if its index r is a simple eigenvalue and if the positive eigenvectors belong to r both in A and A^T.

Without loss of generality we may suppose that the digraphs $G_1, G_2, \dots, G_n$ are without multiple arcs.

By Theorem 3 the index of G=GDP $(B; G_1, \dots, G_n)$ is $\Lambda = \sum_{\beta \in B} r_1^{[\beta_1]} r_2^{[\beta_2]} \dots r_n^{[\beta_n]}$ and $(1, 1, \dots, 1)$ is the positive eigenvector belonging to the index Λ in the adjacency matrix A of G and A^T, where $r_i^{[1]} = r_i, r_i^{[0]} = 1$ and $r_i^{[-1]} = p_i - r_i - 1$.

By the same theorem, if none of the G_i is complete, the index of GDP can be obtained only from those eigenvalues of the digraphs $G_i(\bar{G}_i), i =$

$1,2,\dots,n$ which have a modulus equal to $r_i(p_i-r_i-1)$. All these eigenvalues of G_j can be written in the form $r_j\exp(l_j\frac{2\pi}{h_j})$, $0\le l_j\le h_j-1$ ($\exp(t)=e^{ti}$, $i^2=-1$) (Theorem of Frobenius). Therefore, from (1) we have

$$\Lambda=\sum_{\beta\in B}\prod_{i=1}^{n}(\frac{1}{2}(\beta_i^2+\beta_i)r_i\exp(\frac{\ell_i}{h_i}2\pi)+(1-\beta_i^2)+ \tag{2}$$

$$+\frac{1}{2}(\beta_i^2-\beta_i)(\bar{sg}(\ell_i)p_i-1-r_i\exp(\frac{l_i}{h_i}2\pi))),$$

where $\bar{sg}(0)=1$ and $\bar{sg}(x)=0$ for $x>0$. From (2) it follows that the index Λ is a simple eigenvalue if for each choice of integers $\ell_{i_1},\ell_{i_2},\dots,\ell_{i_s}$, $0\le \ell_{i_t}\le h_{i_t}-1(t=1,2,\dots,s)$ with at least one $\ell_{i_t}>0$, at least one summand in Λ is differen from $r_1^{[\beta_1]},r_2^{[\beta_2]}\dots r_n^{[\beta_n]}$.

For any choice of integers $\ell_{j_1},\ell_{j_2},\dots,\ell_{j_k}, 1\le \ell_{j_t}\le h_{j_t}-1, \{j_1,j_2,\dots,j_k\}\subset \{i_1,i_2,\dots,i_s\}$ and any $\beta\in B$ let $I_\beta=\{j_1,j_2,\dots,j_k\}\cap\{\nu\mid\beta_\nu\ne 0\}$. Then, from (2) we have ($I=\{1,2,\dots,n\}$):

$$\Lambda=\sum_{\beta\in B}(\prod_{i\in I\setminus I_\beta}r_i^{[\beta_i]})(\prod_{\nu\in I_\beta}(\frac{1}{2}(1+\beta_\nu)r_\nu\exp(\frac{\ell_\nu}{h_\nu}2\pi)+$$

$$+\frac{1}{2}(1-\beta_\nu)(-1-r_\nu\exp(\frac{\ell_\nu}{h_\nu}2\pi)))),$$

or

$$\Lambda=\sum_{\beta\in B}(\prod_{i\in I\setminus I_\beta}r_i^{[\beta_i]})(\prod_{\nu\in I_\beta}(r_\nu^2+\frac{1}{2}(1-\beta_\nu)(2r_\nu\cos\frac{\ell_\nu}{h_\nu}2\pi+1))^{\frac{1}{2}}\times \tag{3}$$

$$\times\exp(\frac{1}{2}(1+\beta_\nu)\frac{\ell_\nu}{h_\nu}2\pi+\frac{1}{2}(1-\beta_\nu)\Theta_\nu)).$$

Hence, index Λ is a simple eigenvalue of GDP if and only if for at least one $\beta\in B$ one of the following conditions is satisfied : (a) there exists $\nu\in I_\beta$ such that $\beta_\nu=-1$ and $(r_\nu^2+2r_\nu\cos\frac{\ell_\nu}{h_\nu}2\pi+1)^{\frac{1}{2}}\ne p_\nu-r_\nu-1$ or (b) the argument of the operator exp of the corresponding summand in Λ is different from $2k\pi, k\in Z$. From (a) the following two cases arise: 1°$p_\nu-r_\nu-1=(r_\nu^2+2r_\nu\cos\frac{\ell_\nu}{h_\nu}2\pi+1)^{\frac{1}{2}}=r_\nu-1$ and 2°$p_\nu-r_\nu-1=(r_\nu^2+2r_\nu\cos\frac{\ell_\nu}{h_\nu}2\pi+1)^{\frac{1}{2}}=r_\nu$. The case $(r_\nu^2+2r_\nu\cos\frac{\ell_\nu}{h_\nu}2\pi+1)^{\frac{1}{2}}=r_\nu+1$ is impossible, because $\ell_\nu>0$. If 1° holds then $\cos\frac{\ell_\nu}{h_\nu}2\pi=-1$, consequently G_ν

is bicomplete ($p_\nu = 2r_\nu, h_\nu = 2l_\nu$) and $\Theta_\nu = 0$. If 2° holds, then $\cos\frac{\ell_\nu}{h_\nu}2\pi = -\frac{1}{2r_\nu}$, which is possible only for $r_\nu = 1$, when $p_\nu = 2r_\nu + 1 = 3$ and $h_\nu = 3$ i.e. G_ν is $\vec{C}_3$. In this case $\Theta = \frac{4\pi}{3}$ if $\ell_\nu = 1$ and $\Theta_\nu = \frac{2\pi}{3}$ if $\ell_\nu = 2$. From these facts, the first part of the theorem follows.

Using condition (b) and supposing that 1° or 2° of (a) is satisfied for each $\nu \in I_\beta$ for which $\beta_\nu = -1$, we get the following condition

$$\sum_{\nu \in I_\beta} \left(\frac{1}{2}(1+\beta_\nu)\frac{\ell_\nu}{h_\nu}2\pi + \frac{1}{2}(1-\beta_\nu)\frac{4\pi}{3}\ell_\nu(p_\nu - 2r_\nu)\right) \neq 2k\pi,$$

which completes the proof of the theorem. □

Theorem 5. *Let* $G_i, i = 1, 2, \dots, n$ *be a regular, connected, non-complete symmetric digraph with at least two vertices. Suppose also that loops are permitted in the digtaphs. Further, let digraphs* $G_{i_1}, G_{i_2}, \dots, G_{i_s}$ $(\{i_1, i_2, \dots, i_s\} \subset \{1, 2, \dots, n\})$ *be imprimitive with the imprimitivity indices* $h_{i_1}, h_{i_2}, \dots, h_{i_s}$ *respectively. The GDP with the basis* B *satisfying conditions (D) and (E) of digraphs* $G_1, G_2, \dots, G_n$ *is a connected digraph if and only if for every non-empty subset* $\{j_1, j_2, \dots, j_k\}$ *of* $\{i_1, i_2, \dots, i_s\}$ *and for every choice of integers* $\ell_{j_1}, \ell_{j_2}, \dots \ell_{j_k}, 1 \leq \ell_{j_t} \leq h_{j_t} - 1, t = 1, 2, \dots, k$ *there exists* $\beta \in B$, *such that* $\{j_1, j_2, \dots, j_k\} \cap \{\nu \mid \beta_\nu \neq 0\} = I_\beta \neq \emptyset$ *and the following holds: for at least one* $\nu \in I_\beta$, *for which* $\beta_\nu = -1, G_\nu$ *is not bicomplete or otherwise*

$$\sum_{\nu \in I_\beta} \left(\frac{\ell_\nu}{h_\nu} + \frac{1}{4}(1-\beta_\nu)\right)$$

is not an integer.

Proof. We use the same proof method as in Theorem 4.We also suppose that the digraphs $G_1, G_2, \dots, G_n$ are without multiple arcs.

If $p_i(r_i)$ denotes the number of vertices (degres-index) of $G_i (i = 1, 2, \dots, n)$, then by Theorem 3 the index of $G = \mathrm{GDP}(B; G_1, \dots, G_n)$ is $\Lambda = \sum_{\beta \in B} r_1^{[\beta_1]} r_2^{[\beta_2]} \dots r_n^{[\beta_n]}$ and $(1, 1, \dots, 1)$ is the positive eigenvector belonging to the index Λ in the adjacency matrix A of G and A^T, where $r_i^{[1]} = r_i, r_i^{[0]} = 1$ and $r_i^{[-1]} = p_i - r_i$.

Similarly as in the previous theorem index Λ of GDP $(B; G_1, \dots, G_n)$ can be obtained from those eigenvalues of the digraphs $G_i(\bar{G}_i)$, $i = 1, 2, \dots, n$

which have a modulus equal to $r_i(p_i - r_i)$ i.e. from the exspresion

$$\Lambda = \sum_{\beta\in B} \prod_{i=1}^{n} (\frac{1}{2}(\beta_i^2 + \beta_i) r_i \exp(\frac{\ell_i}{h_i} 2\pi) + (1-\beta_i^2) + \tag{4}$$
$$+ \frac{1}{2}(\beta_i^2 - \beta_i)(\bar{sg}(\ell_i) p_i - r_i \exp(\frac{\ell_i}{h_i}2\pi))).$$

For any choice of integers $\ell_{j_1}, \ell_{j_2}, \ldots, \ell_{j_k}$, $1 \le \ell_{j_t} \le h_{j_t} - 1$, $\{j_1.j_2, \ldots, j_k\} \subset \{i_1, i_2, \ldots, i_s\}$ and any $\beta \in B$ let $\{j_1, j_2, \ldots, j_k\} \cap \{\nu | \beta_\nu \neq 0\} = I_\beta$. Then from (4) we have $(I = \{1, 2, \ldots, n\})$:

$$\Lambda = \sum_{\beta\in B} (\prod_{i \in I\setminus I_\beta} r_i^{[\beta_i]})(\prod_{\nu\in I_\beta} (\frac{1}{2}(1+\beta_\nu) r_\nu \exp(\frac{\ell_\nu}{h_\nu}2\pi) +$$
$$+ \frac{1}{2}(1-\beta_\nu)(-r_\nu \exp(\frac{\ell_\nu}{h_\nu}2\pi))))$$

or

$$\Lambda = \sum_{\beta\in B} (\prod_{i \in I\setminus I_\beta} r_i^{[\beta_i]})(\prod_{\nu\in I_\beta} r_\nu \exp(\frac{\ell_\nu}{h_\nu}2\pi + \frac{1}{2}(1-\beta_\nu)\pi)).$$

Hence, index Λ is a simple eigenvalue of GDP if and only if for at least one $\beta \in B$ one of the following conditions is satisfied: (a) there exists $\nu \in I_\beta$ such that $\beta_\nu = -1$ and $r_\nu \neq p_\nu - r_\nu$ i.e. G_ν is not bicomplete or (b) the argument of the operator exp of the corresponding summand in Λ is different from $2k\pi$, $k \in Z$. From these facts the statement of this theorem follows. □

The following theorem is a specialization of the previous once to undirected graphs.

Theorem 6. *Let* G_i, $i = 1, 2, \ldots, n$ *be a connected regular, non complete, graph with at least two vertices and let* $G_{i_1}, G_{i_2}, \ldots, G_{i_s}$, $\{i_1, i_2, \ldots, i_s\} \subset \{1, 2, \ldots, n\}$ *be bipartite. The GDP with the basis B satisfying conditions (D) and (E) of graphs* $G_1, G_2, \ldots, G_n$ *is a connected graph if and only if for every non-empty subset* $\{j_1, j_2, \ldots, j_k\}$ *of* $\{i_1, i_2, \ldots, i_s\}$, *there exists* $\beta \in B$ *such that* $\{j_1, j_2, \ldots, j_k\} \cap \{\nu | \beta_\nu \neq 0\} = I_\beta \neq \emptyset$ *and one of the following conditions is satisfied:*

(i) For at least one $\nu \in I_\beta$, $\beta_\nu = -1$ *and* G_ν *is not bicopmlete.*

(ii) The number of $\nu \in I_\beta$ *for which* $\beta_\nu = 1$ *is odd.* □

This theorem holds that either loops are or are not allowed in graphs.

References

[1] Cvetković, D.M.: Kombinatorna teorija matrica. Naučna knjiga, Beograd 1980.

[2] Cvetković, D.M., Doob, M., Sachs, H.: Spectra of graphs-theory and application. Deutscher Verlag der Weissenschaften-Academic Press, Berlin-New York 1980;second edition 1982.

[3] Cvetković, D.M., Petrić, M.V.: Connectedness of the non-complete extended p-sum of graphs. Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat, 13(1983), 345-351.

[4] Harary, F.: Graph Theory. Addison-Wesley, Reading 1972.

[5] Petrić, M.V.: On the generalized direct product of graphs. Proceedings of the Eighth Yugoslav Seminar on Graph Theory (Novi Sad 1987), Novi Sad 1989, 99-105.

[6] Petrić, M.V.: On some operations on normal digraphs. Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., 18,1(1988), 57-67.

[7] Šokarovski, R.: A generalized direct product of graphs. Publ. Inst. Math. (Beograd) 22 (36) (1977), 267-269.

REZIME

POVEZANOST GENERALISANOG DIREKTNOG PROIZVODA REGULARNIH DIGRAFOVA

Korištenjem spektralnog metoda dobijeni su potrebni i dovoljni uslovi (teoreme 4 i 5) da generalisani direktan proizvod (GDP) regularnih digrafova bude povezan digraf.

Recived by the editors January 11, 1989.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 65-74

Review of Research
Faculty of Science
Mathematics Series

MANY DIMENSIONAL LYAPUNOV CONVEXITY THEOREM FOR AN E_λ-DECOMPOSABLE MEASURE

Endre Pap
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

A Lyapunov type theorem on the convexity of the range for a many dimensional E_λ-decomposable measure with respect to a non-strict t-conorm $E_\lambda, 0 < \lambda \leq 1$, is proved.

AMS Mathematics Subject Classification (1980): 28B10
Key words and phrases: t-conorm, strongly continuous, E_λ-convex subset

1. Introduction

One of the important results in the investigation of ranges of measures is Lyapunov's theorem on compactness and convexity. This theorem has many generalizations ([2], [3], [4], [6], [13]) and many applications, specially in the mathematical economy ([3], [5]).

We shall continue in this paper investigations on E_λ -decomposable measures ([9] - [12], see also [1], [4], [14] - [17]). As a direct consequence of our earlier results [10] we shall formulate in this paper a one-dimensional version of Lyapunov 's theorem for an atomless $\sigma - \perp$- decomposable measure. M.Fedrizzi, M.Squllante and A.G.S.Ventre [4] have proved a version of a one- dimensional Lyapunov theorem for strongly continuous $\perp$-decomposable measures with respect to the Archimedean t-conorm. $\perp$. The closedness part of Lyapunov's theorem cannot be generalized for the many dimensional case even for additive set functions (charges)-see [13], Example 11.4.8. In this paper we shall generalize the convexity part of Lyapunov's theorem for many dimensional $E_\lambda-$ decomposable measures with respect to the non-strict t-conorm $E_\lambda, 0 < \lambda \leq 1$.

2. One dimensional Lyapunov theorems

An operation $\perp : [0,1] \times [0,1] \to [0,1]$ is said to be a t-conorm if it is commutative, associative,nondecreasing in each argument and has zero as a neutral element.

An operation $\top : [0,1] \times [0,1] \to [0,1]$ is said to be a t-norm if it is commutative, associative, nondecreasing and has 1 as a unit.

A t-conorm $\perp$ is said to be partially continuous if it satisfies condition

(F) For each $\varepsilon > 0$ and each $x \in [0,1], x < \varepsilon$, there exists $y \in (0,1]$ such that $x \perp y < \varepsilon$ (see E.Pap [10]).

A t-conorm $\perp$ is said to be Archimedean if $\perp$ is continuous and $\perp(x,x) > x$ for all $x \in (0,1)$. An Archimedean t-conorm is said to be strict if $\perp$ is strictly increasing in $(0,1) \times (0,1)$.

Ling's representation theorem [7]: An operation $\perp$ is an Archimedean t-conorm if and only if there exists an increasing and continuous function $g : [0,1] \to [0,1]$ with $g(0) = 0$, such that

$$\perp(x,y) = g^{(-1)}(g(x) + g(y)),$$

where $g^{(-1)}$ is the pseudoinverse of g, given by

$$g^{(-1)}(y) := \begin{cases} g^{(-1)}(y) \text{ for } y \in [0, g(1)] \\ 1, \text{ for } y \in [g(1), \infty] \end{cases}$$

Function g is called an additive generator of $\perp$ and it is unique up to a positive constant factor. If $\perp$ is a non- strict t-conorm with the additive generator g such that $g(1) = 1$, then we shall call g a normed generator.

For any t-conorm $\perp$ we define (S.Weber [16]):

$$b -_{\perp} a := \inf\{y : a \perp y \geq b\}.$$

For any non-strict Archimedean t-conorm $\perp$ with additive generator g

$$b -_{\perp} a = g^{-1}(g(b) - g(a))$$

for $a \leq b$ holds.

Let Σ be a σ-algebra of subsets of a non empty set X. A set function $m : \Sigma \to [0,1]$ with $m(\emptyset) = 0$ will be called a $\perp$-decomposable measure, if

$$m(A \bigcup B) = m(A) \perp m(B)$$

holds for all $A, B \in \Sigma$ such that $A \bigcap B = \emptyset$. A set function $m : \Sigma \to [0,1]$ with $m(\emptyset) = 0$ will be called a $\sigma - \perp -$ decomposable measure if it holds that

$$m(\bigcup_{n=1}^{\infty} A_n) = \perp_{n=1}^{\infty} m(A_n)$$

for each sequence (A_n) of pairwise disjoint elements from Σ.

A set function $m : \Sigma \to [0,1]$ is order continuous if $\lim_{n\to\infty} m(E_n) = 0$ for any sequence $(E_n), E_n \in \Sigma \quad (n \in N)$, such that $E_n \searrow \emptyset$.

A set $A \in \Sigma$ is an atom of a $\perp$-decomposable measure m iff $m(A) > 0$ and either $m(A \bigcap B) = 0$ or $m(A \setminus B) = 0$ for any $B \in \Sigma$. A set function $m : \Sigma \to [0,1]$ is atomless, if it is without atoms.

As a consequence of Theorem 1.3.4. (the Darboux proporty) from [10], we obtain the following version of Lyapunov's theorem.

Theorem 1. *Let $m : \Sigma \to [0,1]$ be an atomless order continuous $\sigma - \perp$-decomposable measure with respect to a t-conorm $\perp$, which is partially continuous. Then the range of m is the interval $[0, m(X)]$.*

For the second version of Lyapunov's theorem we need the following definition: a $\perp$-decomposable measure $m : \Sigma \to [0,1]$ is strongly continuous on Σ if for every $\varepsilon > 0$ there exits a finite partition

$$\{A_1, A_2, ..., A_n\}, A_i \in \Sigma \quad (i = 1, 2, ...n),$$

of X, such that

$$m(A_i) < \varepsilon$$

for $i = 1, 2, ..., n$. Then we have, according to [4](Theorem 4.1)

Theorem 2. *Let $m : \Sigma \to [0,1]$ be a $\perp$- decomposable measure with respect to the Archimedean t-conorm $\perp$. If m is strongly continuous, then range $R(m)$ of m is the interval $[0, m(X)]$.*

3. The many dimensional case

Let E_λ for a fixed $\lambda, 0 < \lambda \leq 1$, be defined in the following way

$$xE_\lambda y := \min((x^\lambda + y^\lambda)^{1/\lambda}, 1) \quad (x, y \in [0,1]).$$

Then E_λ is a non-strict t-conorm with the normed additive generator $g(x) = x^\lambda$.

The restricted distribution law

$$c.(xE_\lambda y) = (c.x)E_\lambda(c.y)$$

is valid for all $c \in [0,1]$ and for all $(x, y) \in D$, where

$$D = \{(x, y) : x^\lambda + y^\lambda \leq 1\}$$

(sec. S.Weber [17] and Schwyhla [15]).

A subset C of $[0,1]^n$ is E_λ- convex if for each $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in C$ such that $((x_1, y_1), (x_2, y_2), ..., (x_n, y_n)) \in D^n$ and each $t \in [0,1]$, we have $(z_1, z_2, ..., z_n) \in C$, where

$$z_i = (c.x_i)E_\lambda((1 -_\perp t).y_i)$$

for $i = 1, 2..., n$. The operation $-_\perp$ is with respect to E_λ. The E_λ-dyadic numbere from [0,1] is of the form

$$e = \frac{a_1}{2}E_\lambda\frac{a_2}{2^2}E_\lambda...E_\lambda\frac{a_k}{2^k},$$

where $a_i, i = 1, 2, ..., k$, are zero or one.

Theorem 3. *Let $m_1, m_2, ..., m_n$ be strongly continuous E_λ-decomposable measures for a fixed real number $\lambda, 0 < \lambda \leq 1$ defined on a σ-algebra Σ of subset of a set X. Then for the range*

$$R(m_1, m_2, ..., m_n) = \{(m_1(A), m_2(A), ..., m_n(A)) : A \in \Sigma\}$$

of $m_1, m_2, ..., m_n$ the set $D_n \bigcap R(m_1, m_2, ..., m_n)$ is an E_λ-convex subset of the n-dimensional interval $[0,1]^n$, where $D_n = \{(x_1, x_2, ..., x_n) : x_1^\lambda + x_2^\lambda + ... + x_n^\lambda \leq 1\}$.

Proof. Let

$$\mu_i = m_i E_\lambda m_{i+1} E_\lambda ... E_\lambda m_n$$

for $i = 1, 2, ..., n$. We have that the set $D_n \cap R(m_1, m_2, ..., m_n)$ is E_λ - convex if and only if the range $(\mu_1, \mu_2, ..., \mu_n)$ of $\mu_1, \mu_2, ..., \mu_n$ is E_λ− convex. So we shall prove that $R(\mu_1, \mu_2, ..., \mu_n)$ is E_λ− convex.

For that purpose we shall show by inducition the following proprty of $\mu_1, \mu_2, ..., \mu_n$:

(a) For every $A \in \Sigma$ there exists a set B in Σ, such that $B \subset A$ and $\mu_i(B) = \frac{1}{2}\mu_i(A)$ for $i = 1, 2, ..., n$.

Then (a) will imply that for any arbitrary but fixed set A from Σ and every E_λ- dyadic number e from the interval $[0,1]$, there exists a set A_e such that:

$$A_o = \emptyset,\ \ A_1 = A,\ \ A_{e_1} \subset A_{e_2}$$

whenever for the E_λ− dyadic numbers e_1 and $e_2, 0 \leq e_1 \leq 1$ holds and

(1)
$$\mu_i(A_e) = e\mu_i(A)$$

for $i = 1, 2, ..., n$.

Namely, by repeated application of (a) for $A \in \Sigma$ and $k \in N \bigcup \{0\}$, there exists the set $A_{\frac{1}{2^k}}$ such that

$$\mu_i(A_{\frac{1}{2^k}}) = \frac{1}{2^k}\mu_i(A)$$

for $i = 1, 2, ..., n$. Using the sets $A_{\frac{1}{2^k}}$ we can construct the set A_e for the E_λ - dyadic number $e = \frac{a_1}{2} E_\lambda \frac{a_2}{2^2} E_\lambda ... E_\lambda \frac{a_s}{2^s}$, where $a_i, i = 1, 2, ..., s$, is zero or one. For example, if

$$e = \frac{1}{2} E_\lambda \frac{1}{2^2},$$

then $A_e = A \setminus A_{\frac{1}{2^2}}$ and

$$\mu(A \setminus A_{\frac{1}{2^2}}) = (1 -_{\perp} \frac{1}{2^2}) \mu_i(A) = (\frac{1}{2} E_\lambda \frac{1}{2^2}) \mu_i(A)$$

for $i = 1, 2, ..., n$.

Further, for any real number t from $[0, 1]$ we take all the E_λ-dyadic numbers e such that $e \leq t$. Then we have $t = \sup\{e : e \leq t\}$. Therefore, if we take

$$A_t = \bigcup_{e \leq t} A_e, \tag{2}$$

where the union runs through the E_λ - dyadic numbers $e \leq t$, we have $A_t \in \Sigma$ and

$$\mu_i(A_t) = t.\mu_i(A) \tag{3}$$

for $i = 1, 2, ..., n$.

Now we shall prove (a) by induction. For $n = 1, (a)$ follows by Theorem 2.2. Let us assume (a) to be true for $n = k$ and prove (a) for $n = k + 1$. Let

$$\mu_i = m_i E_\lambda m_{i+1} E_\lambda ... E_\lambda m_{k+1}$$

for $i = 1, 2, ..., k + 1$, where $m_1, m_2, ..., m_{k+1}$ are strongly continuous E_λ-docomposable measures.

Let $B' \in \Sigma$ be a set obtained by the induction step for $n = k$, i.e. B' is a subset of A such that

$$\mu_i(B') = \frac{1}{2} \mu_i(A)$$

for $i = 1, 2, ..., k$. If it is

$$\mu_{k+1}(B') = \frac{1}{2} \mu_{k+1}(A),$$

then we have proved the desired equality for $n = k + 1$. If not, then it is either.

$$\mu_{k+1}(B') < \frac{1}{2}\mu_{k+1}(A) < \mu_{k+1}(A \setminus B')$$

or

$$\mu_{k+1}(A \setminus B') < \frac{1}{2}\mu_{k+1}(A) < \mu_{k+1}(B').$$

We shall suppose the first case. For the second case we proceed in an analogous way. By the induction hypothesis we take for $\mu_1, \mu_2, ..., \mu_k$ the sets B'_t $(0 \leq t \leq 1)$ and $(A \setminus B')_t$ $(0 \leq t \leq 1)$ which are corresponding sets to the sets B' and $A \setminus B'$, respectively, and defined by (2) such that satisfy (3). For any two real numbers a and b such that $b \leq a$, we have

$$\mu_{k+1}(B'_a) -_\perp \mu_{k+1}(B'_b) = \mu_{k+1}(B'_a \setminus B'_b) \leq \mu_k(B'_a \setminus B'_b) = \mu_k(B'_a -_\perp \mu_k(B'_b) =$$

$$(a -_\perp b)\mu_k(B'),$$

and analogously

$$\mu_{k+1}((A \setminus B')_a) -_\perp \mu_{k+1}((A \setminus B'_b) \leq (a -_\perp b)\mu_k(A \setminus B').$$

Since the operation $-_\perp$ is continuous, we obtain by the preceding inequalities that the functions $\mu_{k+1}(B'_t)$ and $\mu_{k+1}((A \setminus B')_t)$ are continuous with respect to t.

Namely, if an arbitrary but fixed sequence (x_n) from $[0,1]$ converges to $c \epsilon [0,1]$, then $x_n - c \to 0$ implies $x_n -_\perp c \to 0$. The last inequalities imply

$$\mu_{k+1}(B'_{x_n}) -_\perp \mu_{k+1}(B'_c) \to 0,$$

i.e.

$$\mu_{k+1}(B'_{x_n})^\lambda -_\perp \mu_{k+1}(B'_c)^\lambda \to 0.$$

Hence,

$$\mu_{k+1}(B'_{x_n}) \to \mu_{k+1}(B'_c),$$

i.e. $\mu_{k+1}(B'^t)$ is continuous. This implies that the function

$$F(t) := \mu_{k+1}(B'_t \bigcup (A \setminus B')_{1 -_\perp t} = \mu_{k+1}(B'_t) E_\lambda \mu_{k+1}((A \setminus B')_t)$$

is also a continuous function of t for $0 \leq t \leq 1$. Then F takes all the values between $F(1) = \mu_{k+1}(B')$ and $F(0) = \mu_{k+1}(A \setminus B')$. So there exists $t_o \in [0,1]$ such that

$$F(t_o) = \frac{1}{2}\mu_{k+1}(A).$$

Taking $B = B'_{t_o} \bigcup (A \setminus B')_{1-_{\perp} t}$, we find the desired set in (1) for $n = k +$ 1.Namely, we also have for each i between 1 and k

$$\mu_i(B) = \mu_i(B'_{t_o}) E_\lambda \mu_i((A \setminus B')_{1-_{\perp} t_o}) =$$

$$= t_o \mu_i(B') E_\lambda (1 -_{\perp} t_o) \mu_i(A \setminus B') =$$

$$= t_o, \frac{1}{2}.\mu_i(A) E_\lambda (1 -_{\perp} t_o).\frac{1}{2}\mu_i(A) =$$

$$= (t_o E_\lambda (1 -_{\perp} t_o)).\frac{1}{2}.\mu_i(A) = \frac{1}{2}\mu_i(A).$$

Now, we can prove that $R(\mu_1, \mu_2, ..., \mu_{k+1})$ is E_λ-convex. Taking $T, S \in \sum$ and $0 \leq t \leq 1$ we have

$$t.\mu_i(T) E_\lambda (1 -_{\perp} t)\mu_i(S) =$$

$$= (t.\mu_i(T \setminus S)) E_\lambda (t.\mu_i(T \bigcap S)) E_\lambda ((1 -_{\perp} t)\mu_i(S \setminus T)) E_\lambda ((1 -_{\perp} t).\mu_i(S \setminus T)) =$$

$$= (\mu_i((T \setminus S)_t)) E_\lambda (\mu_i((S \setminus T)_{1-_{\perp} t})) E_\lambda (\mu_i(S \cap T)) =$$

$$\mu_i((T \setminus S)_t \cup (S \cap T) \cup (S \setminus T)_{1-_{\perp} t})$$

for every $i = 1, 2, ..., k+1$.

Therefore we have proved our theorem by induction.

References

[1] D' Apuzzo, L., Squillante, M., Ventre, A.G.S.: A survey on decomposable measures, Control and Cybernetics (to appear).

[2] Arbib, M.A.: Generalizations of Lyapunov's theorem on the range of a vector measure, J. Math. Mech. 15, 4 (1966), 609-616.

[3] Aumann, R.J., Shapley,L.S.: Values of non-atomic games, Princeton University Press, Princeton, 1974.

[4] Fedrizzi, M., Squillante, M., Ventre, A.G.S.: On the range of ⊥-decomposable measures, Rivista di Matematica, Universita di Parma (to appear).

[5] Hildenbrand,W.: Core of an economy, Handbook of mathematical economies Vol.2 (K.J. Arrow and M.D.Inriligator, eds), North-Holland, 1982.

[6] Lindenstrauss, J.: A short proof of Lyapunov's convexity theorem, J.Math. 15, 6 (1966), 971-972.

[7] Ling, C.H.: Representation of associative function, Publ. Math. Debrecen 12 (1965), 189-212.

[8] Lyapunov, A.A.:Sur les fonctions-vecteurs complétement additives, Bull. Acad. Sci. URSS Math. 4 (1940), 465-478.

[9] Pap, E.: Lebesgue and Saks decomposition of ⊥ - decomposable measures, Fuzzy Sets and Systems 38(1990), 345-353.

[10] Pap, E.: On non-additive set functions, Att Sem. Mat. Fis. Universita Modena , 39(1991), 345-360.

[11] Pap, E.: Extension of continuous t-conorm decomposable measures, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. (to appear).

[12] Pap, E.: Regular Borel t-conorm decomposable measures, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. (to appear).

[13] Rao, K.P.S.B., Rao, M.B.: Theory of charges, Academic Press, London, 1983.

[14] Schweizer, B., Sklar, A.: Probabilistic metric spaces, North-Holland, Amsterdam, 1983.

[15] Schwyhla, W.: About the isomorphism between some Sugeno measures and classical measures, in E.P. Klement, Ed., Proc. 2nd Int. Sem. on Fuzzy Set Theory, Linz (1980), 143-176.

[16] Weber, S.: ⊥ - decomposable measures and integrals for Archimedean t-conorms, J. Math. Anal. Appl. 101 (1984), 114-138.

[17] Weber, S.: Two integrals and some modified versions-c ritical remarks, Fuzzy Sets and Systems 20 (1986), 97-105.

REZIME

VIŠEDIMENZIONALNA TEOREMA LJAPUNOVA O KONVEKSNOSTI ZA E_λ - DEKOMPOZABILNU MERU

U radu se dokazuje teorema tipa Ljapunova o konveksnosti skupa vrednosti višedimenzionalne E_λ-dekompozabilne mere u odnosu na t-konormu $E_\lambda, 0 < \lambda \leq 1$.

Received by the editors May 17, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 75-81

Review of Research
Faculty of Science
Mathematics Series

THE BROOKS-JEWETT THEOREM FOR NON-ADDITIVE SET FUNCTIONS

Endre Pap
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In this paper a version of the Brooks-Jewett theorem on the convergence of sequences of set functions which have ranges in an arbitrary uniform space (without considering any algebraic operation on it) is proved.

AMS Mathematics Subject Classification (1980): 28B99
Key words and phrases: Uniform space, x_0–exhaustive set function

The Brooks-Jewett theorem [1], as an additive version of the Nikodym convergence theorem, was considered and generalized also for non-additive set functions [3], [4], [5], [6], [7], [9], [10], [11], [12]. In all these papers the ranges of the considered set functions (additive, subadditive, k–subadditive, k–triangular, etc.) were endowed with some algebraic operation.

In our paper [11] we have proved a Nikodym uniform boundedness type theorem for set functions with the values in an arbitrary uniform space. Using the approach from paper [11] we shall obtain a version of the Brooks-Jewett theorem for set functions defined on a quasi-σ-ring and with the values in a uniform space.

Let Y be a uniform space endowed with the uniformity $\mathcal{U}$. We shall use the following definition of the boundedness given by J. Hejcman [8].

Definition 1. *A subset B of Y is bounded ($\mathcal{U}$-bounded) if for every $U \in \mathcal{U}$ there exist a finite set $K \subset B$ and a natural number n such that*

$$B \subset U^n[K],$$

where $U^1 = U$, $U^n = U \circ U^{n-1}$ ($\circ$ is the composition of the relations) and $U[K]$ is the set of all $x \in Y$ such that $(x, y) \in U$ for some $y \in K$.

We shall use following characterization of the $\mathcal{U}$-boundedness (Theorem 1.12 from [8]).

Theorem 1. *A set $B \subset Y$ is $\mathcal{U}$-bounded if and only if it is $d-$bounded for every uniformly continuous pseudo-metric d defined on Y.*

Using this theorem we shall denote with $\mathcal{D}$ the family of all the uniformly continuous pseudometrics defined on $(Y, \mathcal{U})$.

Definition 2. *A ring of sets $\sum$ is called a quasi-σ-ring if any disjoint sequence in $\sum$ posseses a subsequence which belongs to the family of disjoint sequences $\{A_n\}$ in $\sum$ for which*

$$\left\{ \bigcup_{n \in M} A_n : M \subset \mathrm{N} \right\} \subset \sum .$$

In the whole paper $\sum$ will always denote a quasi-σ-ring.

Definition 3. *A set function $\mu : \sum \to Y$ is said to be x_0-exhaustive, for $x_0 \in Y$, if for each $d \in \mathcal{D}$*

$$\lim_{n \to \infty} d(\mu(E_n), x_0) = 0$$

for each sequence $\{E_n\}$ of pairwise disjoint sets from $\sum$.

For $d \in \mathcal{D}$ the $d-$semivariation ([11]) of the set function μ, $\mu : \sum \to Y$, with respect to a point $x_0 \in Y$ is

$$\widetilde{\mu}_d^{x_0}(B) := \sup\{d(\mu(C), x_0) : C \subset B,\ C \in \sum\}\ (B \in \sum).$$

We shall need following (Lemma 2.2.2 from [11])

Lemma 1. *Let $\mu : \sum \to Y$ be an x_0-exhaustive set function, and let $\{A_n\}$ be a sequence of pairwise disjoint elements from $\sum$. Then, for each $d \in \mathcal{D}$ and each $\varepsilon > 0$, there exists a subsequence $\{A_{n_i}\}$ of $\{A_n\}$ such that*

$$\widetilde{\mu}_d^{x_0}\left(\bigcup_{i\in I} A_{n_i}\right) < \varepsilon,$$

for any $I \subset \mathbf{N}$.

We introduced in [11] for $d \in \mathcal{D}$ and $x_0 \in Y$

$$\alpha_d^{x_0}(A,\mu) := \lim_{n\to\infty} \sup\{d(\mu(A\cup B), x_0) : \widetilde{\mu}_d^{x_0}(B) < \frac{1}{n},\ B \in \sum\}$$

$$(A \in \sum,\ \mu : \sum \to Y).$$

Now we have the Brooks-Jewett type theorem.

Theorem 2. *Let $\{\mu_n\}$ be a sequence of set functions μ_n, $\mu_n : \sum \to Y$, such that they satisfy the following conditions for an arbitrary but fixed $x_0 \in Y$*

(i) for each $d \in \mathcal{D}$ and for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d(\mu_n(A), x_0) < \delta$ and $d(\mu_n(B), x_0) < \delta$ for $B \subset A$, $A, B \in \sum$, $n \in \mathbf{N}$ implies

$$d(\mu_n(A\backslash B), x_0) < \varepsilon,$$

(ii) for each $d \in \mathcal{D}$ and for each $\delta > 0$, there exists $\Theta > 0$ such that $d(\mu_n(A), x_0) < \Theta$, $A \in \sum$, $n \in \mathbf{N}$ implies

$$\alpha_d^{x_0}(A, \mu_n) < \delta, \quad n \in \mathbf{N},$$

(iii) for each $d \in \mathcal{D}$

$$\lim_{n\to\infty} d(\mu_n(E),\ \mu(E)) = 0$$

for each $E \in \sum$.

Then μ is x_0-exhaustive if and only if μ_n, $n \in \mathbf{N}$, are uniformly x_0-exhaustive.

Proof. Suppose that μ is x_0–exhaustive,but the sequence $\{\mu_n\}$ is not uniformly x_0–exhaustive. Then there exists $\varepsilon > 0$, d from $\mathcal{D}$ and a sequence $\{E_k\}$ of pairwise disjoint sets from $\sum$ and a subsequence $\{\mu_{n_k}\}$ such that

$$d(\mu_{n_k}(E_k), x_0) > \varepsilon. \tag{1}$$

We choose $\delta > 0$ by (i) corresponding to $\varepsilon > 0$. Now we choose $\Theta > 0$ by (ii) corresponding to $\delta > 0$. By the x_0–exhaustivity of μ and the Lemma 1 there exists a subsequence $\{E_{k_i}\}$ of the sequence $\{E_k\}$ such that

$$\widetilde{\mu}_d^{x_0}\left(\bigcup_{i\in I} E_{k_i}\right) < \frac{\Theta}{2} \tag{2}$$

for any $I \subset \mathrm{N}$. Now, we shall take $m_i = \mu_{n_{k_i}}$ and $A_i = E_{k_i}$ $(i \in \mathrm{N})$ and $i_1 = 1$. Since we have

$$d(m_i(A_{i_1}), \mu(A_{i_1})) \geq d(m_i(A_{i_1}), x_0) - d(\mu(A_{i_1}), x_0)$$

and by (iii) there exists an index i_2 such that

$$d(m_{i_2}(A_{i_1}), \mu(A_{i_1})) < \frac{\Theta}{2},$$

using also (2) we obtain

$$d(m_{i_2}(A_{i_1}), x_0) < \Theta. \tag{3}$$

By the x_0–exhaustivity of m_{i_2} we have by the Lemma that there exists a subsequence $\{A_i^2\}$ of the sequence $\{A_i\}_{i=i_1+1}^{\infty}$ such that

$$(\widetilde{m}_{i_2})_d^{x_0}\left(\bigcup_{i\in I} A_i^2\right) < \frac{\Theta}{2}.$$

Hence by (3) and (ii), we have

$$\alpha_d^{x_0}\left(A_{i_1} \bigcup \bigcup_{i\in I} A_i^2, m_{i_2}\right) < \delta \tag{4}$$

for any $I \subset \mathrm{N}$. By (2) $d(\mu(A_{i_1} \cup A_k^2), x_0) < \frac{\Theta}{2}$ for each $k \in \mathrm{N}$ and by (iii) there exists an index i_3 such that

$$d(m_{i_3}(A_{i_1} \cup A_k^2),\ \mu(A_{i_1} \cup A_k^2)) < \frac{\Theta}{2}.$$

Then, the inequality

$$d(m_i(A_{i_1} \cup A_k^2), \mu(A_{i_1} \cup A_k^2)) \geq d(m_i(A_{i_1} \cup A_k^2), x_0) - d(\mu(A_{i_1} \cup A_k^2), x_0)$$

with the preceding two facts implies

$$d(m_{i_3}(A_{i_1} \cup A_{i_2}), x_0) < \Theta, \tag{5}$$

where A_{i_2} is taken from the sequence $\{A_k^2\}$.

By x_0−exhaustivity of m_{i_3}, we have by the Lemma 1 that there exists a subsequence $\{A_i^3\}$ of the sequence $\{A_i^2\}_{i=i_2+1}^{\infty}$ such that

$$(\widetilde{m}_{i_3})_d^{x_0}\left(\bigcup_{i \in I} A_i^3\right) < \frac{\Theta}{2}$$

for any $I \subset \mathbf{N}$. Hence by (5) and (ii), we have

$$d(m_{i_3}\left(A_{i_1} \bigcup A_{i_2} \bigcup \bigcup_{i \in I} A_i^3\right), x_0) < \delta$$

for any $I \subset \mathbf{N}$.

Continuing this procedure we obtain two sequences $\{m_{i_k}\}$ and $\{A_{i_k}\}$. If we take $A_0 = \bigcup_{k=1}^{\infty} A_{i_k}$, then by (iii) there exists an index k_0 such that

$$d(m_{i_{k_0}}(A_0), x_0) < \eta < \delta. \tag{6}$$

Namely, this follows by (2) and the inequality

$$d(m_{i_{k_0}}(A_0), \mu(A_0)) \geq d(m_{i_{k_0}}(A_0), x_0) - d(\mu(A_0), x_0).$$

From the procedure of the whole previous construction, it follows that

$$d(m_{i_{k_0}}(A_0 \backslash A_{i_{k_0}}), x_0) < \delta.$$

Hence, using (i), we have by (6)

$$\varepsilon > d(m_{i_{k_0}}(A_0 \backslash (A_0 \backslash A_{i_{k_0}}), x_0) = d(m_{i_{k_0}}(A_{i_{k_0}}), x_0),$$

which is in contradiction with (1).

Now, if we suppose that $\mu_n (n \in \mathbf{N})$ are uniformly x_0−exhaustive, then by (iii) it follows that μ is x_0−exhaustive.

Corollary 1. *(Theorem 1. from [9]) Let $\{\mu_n\}$ be a sequence of $k-$triangular exhaustive set functions $\mu_n : \sum \to R^+$. If there exists*

$$\lim_{n\to\infty} \mu_n(E) = \mu(E)$$

for each $E \in \sum$ and μ is exhaustive, then $\{\mu_n\}$ is uniformly exhaustive and μ is $k-$triangular.

Proof. Conditions (i) and (ii) follow by $k-$triangularity.

References

[1] Brooks, J.K., Jewett, R.S.: On finitely additive vector measures, Proc. Nat. Acad. Sci. U.S.A. 67 (1970), 1294-1298.

[2] Constantinescu, C.: Spaces of Measures, Walter de Gruyter, Berlin-New York, 1984.

[3] Dobrakov, I.: On submeasures I, Dissertationes Math. 112, Warszawa, 1974.

[4] Dobrakov, I., Farkova, J.: On submeasures II, Math. Slovaca 30, (1980), 65-81.

[5] Dobrakov, I.: Uniform boundedness principle for exhaustive set functions, Annal. Soc. Math. Polon., Ser. 1; Comm. Math., 24 (1984), 201-205.

[6] Drewnowski, L.: On the continuity of certain non-additive set functions, Colloquium Math., 38 (1978), 243-253.

[7] Guariglia, E.: $k-$triangular functions on an orthomodular lattice and the Brooks-Jewett theorem (to appear).

[8] Hejcman, J.: Boundedness in uniform spaces and topological groups, Czehosl. Mat. Journal, 9 (84) (1959), 544-562.

[9] Pap, E.: The Vitali-Hahn-Saks theorems for $k-$triangular set functions, Atti Sem. Mat. Fis. Univ. Modena, 26 (1987), 21-32.

[10] Pap, E.: Nikodym type theorem for metric valued x_0-exhaustive set functions, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 18(1988), 101-109.

[11] Pap, E.: On non-additive set functions, Atti Sem. Mat. Fis. Univ. Modena, 39 (1991), 377-392.

[12] Weber, H.: Compactness in spaces of group-valued contents, the Vitali-Hahn-Saks theorem and Nikodym boundedness theorem, Rocky Mountain J. Math., 16 (2) (1986), 253-275.

REZIME

BROOKS-JEWETTOVA TEOREMA ZA NEADITIVNE SKUPOVNE FUNKCIJE

U radu se dokazuje teorema tipa Brooksa-Jewetta o konvergenciji niza skupovnih funkcija sa vrednostima u proizvoljnom uniformnom prostoru (u opštem slučaju bez algebarske operacije). U tu svrhu se koriste rezultati iz ranijeg rada [11].

Received by the editors September 20, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 83-93

Review of Research
Faculty of Science
Mathematics Series

FIXED POINTS ON TWO COMPLETE METRIC SPACES

Valeriu Popa
Department of Mathematics, Higher Education Institute
5500 Bacău, Romania

Abstract

Let (X, d) and (Y, e) be two complete metric spaces. It is proved that if T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalites

$$e^2(Tx, TSy) \leq$$
$$c_1 \max\{d(x, Sy)e(y, Tx), d(x, Sy)e(y, TSy), e(y, Tx)e(y, TSy)\}.$$
$$d^2(Sy, STx) \leq$$
$$c_2 \max\{e(y, Tx)d(x, Sy), e(y, Tx)d(x, STx), d(x, Sy)d(x, STx)\}.$$

for all x in X and y in Y, where $0 \leq c_1 \cdot c_2 < 1$ or the inequalities

$$e(Tx, TSy) \cdot \max\{e(y, Tx), e(TSy, y)\}$$
$$\leq c_1 d(x, Sy) \cdot \max\{d(x, Sy), e(y, TSy)\}$$
$$d(Sy, STx) \cdot \max\{d(x, Sy), d(x, STx)\} \leq$$
$$\leq c_2 e(y, Tx) \cdot \max\{e(y, Tx), d(x, STx)\}$$

for all x in X and y in Y , where $0 \leq c_1, c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Ams Mathematics Subject Classification (1980): Primary 54H25, Secondary 54E40, 54E45, 54E50.
Key words and phrases: Complete metric space, fixed point.

In a recent paper [1], the following theorem was proved

Theorem 1. *Let (X,d) and (Y,e) be complete metric spaces. If T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities*

$$e(Tx,TSy) \leq c\ \max\{d(x,Sy), e(y,Tx), e(y,TSy)\} \tag{1}$$

$$d(Sy,STx) \leq c\ \max\{e(y,Tx), d(x,Sy), d(x,STx)\} \tag{2}$$

for all x in X and y in Y, where $0 \leq c < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

In [2] and [3] the other fixed point theorems on two metric spaces are proved. Now, we shall prove two fixed point theorems involving two metric spaces.

Theorem 2. *Let (X,d) and (Y,e) be complete metric spaces. If T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities*

$$\begin{gathered} e^2(Tx,TSy) \leq \\ c_1 \max\{d(x,Sy)e(y,Tx), d(x,Sy)e(y,TSy), e(y,Tx)e(y,TSy)\} \end{gathered} \tag{3}$$

$$\begin{gathered} d^2(Sy,STx) \leq \\ c_2 \max\{e(y,Tx)d(x,Sy), e(y,Tx)d(x,STx), d(x,Sy)d(x,STx)\} \end{gathered} \tag{4}$$

for all x in X and y in Y, where $0 \leq c_1 \cdot c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Proof. Let x be an arbitrary point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively by

$$(ST)^n x = x_n,\ \ T(ST)^{n-1}x = y_n$$

for $n = 1,2,\ldots$. Using inequality (4) we have

$$d^2(x_n, x_{n+1}) \leq$$

$$\leq c_2 \max\{e(y_{n+1}, y_n)d(x_n, x_n), d(x_n, x_n)d(x_n, x_{n+1}),$$

$$e(y_n, y_{n+1})d(x_n, x_{n+1})\} = c_2 e(y_n, y_{n+1})d(x_n, x_{n+1}),$$

which implies

$$d(x_n, x_{n+1}) \leq c_2\, e(y_n, y_{n+1})$$

if $d(x_n, x_{n+1}) \neq 0$ and by using inequality (3), we have

$$e^2(y_n, y_{n+1}) \leq$$

$$c_1 \max\{d(x_{n-1}, x_n)e(y_n, y_{n+1}), e(y_n, y_n)e(y_{n+1}, y_n), d(x_{n-1}, x_n)e(y_n, y_{n+1})\} =$$

$$c_1 d(x_{n-1}, x_n)e(y_n, y_{n+1}),$$

which implies

$$e(y_n, y_{n+1}) \leq c_1 d(x_{n-1}, x_n)$$

if $e(y_n, y_{n+1}) \neq 0$. It follows that

$$d(x_n, x_{n+1}) \leq c_2 e(y_n, y_{n+1}) \leq$$

$$\leq c_1 c_2 d(x_{n-1}, x_n) \leq \ldots \leq (c_1 c_2)^n\, d(x, x_1)$$

and since $0 \leq c_1 \cdot c_2 < 1$, $\{x_n\}$ is a Cauchy sequence with a limit z in X and $\{y_n\}$ is a Cauchy sequence with a limit w in Y.

Now, by using inequality (3), we have

$$e^2(Tz, y_n) \leq$$

$$c_1 \max\{d(z, x_{n-1})e(y_{n+1}, Tz), d(z, x_{n-1})e(y_{n-1}, y_n),$$

$$e(y_{n-1}, y_n)e(y_{n-1}, Tz)\}.$$

Letting n tend to infinity we have $e^2(Tz, w) \leq 0$ and so $Tz = w$. Similarly, we can prove that $Sw = z$ and so $STz = Sw = z, TSw = Tz = w$. Thus ST has a fixed point z and TS has a fixed point w. Now suppose that ST has a second fixed point z'. Then by using inequality (4), we have

$$d^2(z, z') = d^2(STz', STz) \leq$$

$$c_2 \max\{e(Tz', Tz)d(z, STz'), e(Tz', Tz), d(z, STz),$$

$$d(z, STz')d(z, STz)\} = c_2 e(Tz', Tz)d(z, z')$$

which implies $d(z',z) \le c_2 e(Tz',Tz)$. But by using inequality (3),

$$e^2(Tz,Tz') = e^2(Tz',TSTz) \le$$
$$c_1 \max\{d(z',STz)e(Tz,Tz'), d(z',STz)$$
$$e(Tz,TSTz), e(Tz,z)e(Tz,TSTz)\} = c_1 d(z',STz)e(Tz',Tz),$$

which implies

$$e(Tz',Tz) \le c_1 d(z',STz) = c_1 d(z,z')$$

and so

$$d(z,z') \le c_2 e(Tz',Tz) \le c_1 c_2 d(z,z').$$

Since, $0 \le c_1 c_2 < 1$, the uniqueness of z follows. Similarly, w is the unique fixed point of TS. If $\exists n \in N$ that $d(x_n, x_{n+1}) = 0$ or $e(y_n, y_{n+1}) = 0$ the theorem is evident.

Corollary 1. *Let (X,d) be a complete metric space. If S and T are mappings of X into itself satisfying the inequalities*

$$d^2(Tx,TSy) \le \tag{5}$$
$$c_1 \max\{d(x,Sy)d(y,Tx), d(x,Sy)d(y,TSy), d(y,Tx)d(y,TSy)\}$$

$$d^2(Sy,STx) \le \tag{6}$$
$$c_2 \max\{d(y,Tx)d(x,Sy), d(y,Tx)d(y,STx), d(x,Sy)d(x,STx)\}$$

for all x,y in X, where $0 \le c_1, c_2 < 1$, then ST has a unique fixed point z and TS has a unique fixed point w. Further $Tz = w$ and $Sw = z$ and if $z = w$, z is the unique fixed point of S and T.

Proof. The existence of z and w follows from Theorem 2. If $z = w$, then z is of course a common fixed point of S and T.

Now suppose that T has a second fixed point z'. Then, by using inequality (5), we have

$$d^2(z,z') = d^2(Tz',TSz) \le$$
$$c_1 \max\{d(z',Sz)d(z,Tz'), d(z',Sz)d(z,TSz),$$
$$d(z,Tz')d(z,TSz)\} = c_1 d^2(z',z).$$

Since $0 \le c_1 < 1$, the uniqueness of z follows. Similarly, z is the unique fixed point of S.

The proofs of Corollaries 2 and 3 follow easily.

Corollary 2. *Let (X,d) and (Y,e) be complete metric spaces. If T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities*

$$e^2(Tx,TSy) \leq$$

$$\leq a_1 d(x,Sy)e(y,Tx) + b_1 d(x,Sy)e(y,TSy) + c_1 e(y,Tx)e(y,TSy)$$

$$d^2(Sy,STx) \leq$$

$$\leq a_2 e(y,Tx)d(x,Sy) + b_2 d(x,STx)e(y,Tx) + c_2 d(x,Sy)d(x,STx)$$

for all x in X and y in Y, where $a_1, b_1, c_1, a_2, b_2, c_2 \geq 0$ and $(a_1+b_1+c_1)\cdot(a_2+b_2+c_2) < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Corollary 3. *Let (X,d) be a complete metric space. If S and T are mappings of X into itself, satisfying the inequalities*

$$d^2(Tx,TSy) \leq$$

$$\leq a_1 d(x,Sy)d(y,Tx) + b_1 d(x,Sy)d(y,TSy) + c_1 d(y,Tx)d(y,TSy)$$

$$d^2(Sy,STx) \leq$$

$$\leq a_2 d(y,Tx)d(x,Sy) + b_2 d(y,Tx)d(x,STx) + c_2 d(x,Sy)d(x,STx)$$

for all x,y in X, where $a_1, b_1, c_1, a_2, b_2, c_2 \geq 0$ and $a_1+b_1+c_1 < 1$, $a_2+b_2+c_2 < 1$, then ST has a unique fixed point z and TS has a unique fixed point w. Further, $Tz = w$ and $Sw = z$ and if $z = w$, z is the unique fixed point of S and T.

Theorem 3. *Let (X,d) and (Y,e) be complete metric spaces. If T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities*

$$e(Tx,TSy)\cdot\max\{e(y,Tx),e(TSy,y)\} \tag{7}$$

$$\leq c_1 d(x,Sy)\cdot\max\{d(x,Sy),e(y,TSy)\}$$

$$d(Sy,STx)\cdot\max\{d(x,Sy),d(x,STx)\} \tag{8}$$

$$\leq c_2 e(y,Tx)\cdot\max\{e(y,Tx),d(x,STx)\}$$

for all x in X and y in Y, where $0 \leq c_1, c_2 < 1$ then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Proof. Let x be an arbitrary point in X.Define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively by

$$x_n = (ST)^n x,\ y_n = T(ST)^{n-1}x$$

for $n = 1, 2 \ldots$. Using inequality (8), we have

$$d(x_n, x_{n+1}) \cdot \max\{d(x_n, x_{n+1}), d(x_n, x_n)\} \leq$$

$$\leq c_2 \cdot e(y_n, y_{n+1}) \cdot \max\{d(x_n, x_{n+1}), e(y_n, y_{n+1}\}.$$

and hence

$$d^2(x_n, x_{n+1}) \leq c_2 e(y_n, y_{n+1}) d(x_n, x_{n+1}),$$

which implies

$$d(x_n, x_{n+1}) \leq c_2 e(y_n, y_{n+1})$$

if $d(x_n, x_{n+1}) \neq 0$ or $d^2(x_n, x_{n+1}) \leq c_2 e^2(y_n, y_{n+1})$, which implies

$$d(x_n, x_{n+1}) \leq \sqrt{c_2} \cdot e(y_n, y_{n+1}).$$

Since $c_2 \leq \sqrt{c_2}$, we have

$$d(x_n, x_{n+1}) \leq \sqrt{c_2} \cdot e(y_n, y_{n+1}).$$

Using inequality (7), we have

$$e(y_n, y_{n+1}) \cdot \max\{e(y_n, y_{n+1}), e(y_n, y_n)\} \leq$$

$$\leq c_1 d(x_{n-1}, x_n) \cdot \max\{d(x_{n-1}, x_n), e(y_n, y_{n+1})\}$$

and $e^2(y_n, y_{n+1}) \leq c_1 d(x_n, x_{n-1}) e(y_n, y_{n+1})$, which implies

$$e(y_n, y_{n+1}) \leq c_1 d(x_n, x_{n+1})$$

if $e(y_n, y_{n+1}) \neq 0$ or $e^2(y_n, y_{n+1}) \leq c_1 d^2(x_{n-1}, x_n)$, which implies

$$e(y_n, y_{n+1}) \leq \sqrt{c_1} \cdot d(x_{n-1}, x_n).$$

Since $c_1 \leq \sqrt{c_1}$, we have

$$e(y_n, y_{n+1}) \leq \sqrt{c_1} \cdot d(x_{n-1}, x_n).$$

It follows that

$$d(x_n, x_{n+1}) \leq \sqrt{c_2} \cdot e(y_n, y_{n+1}) \leq$$

$$\sqrt{c_1c_2}\,d(x_n,x_{n-1}) \le \ldots \le$$
$$\le (\sqrt{c_1c_2})^n d(x,x_1)$$

and since $0 \le \sqrt{c_1c_2} < 1$, $\{x_n\}$ is a Cauchy sequence with a limit z in X and $\{y_n\}$ is a Cauchy sequence with a limit w in Y.

Now, by using inequality (7), we have

$$e(Tz,y_n)\cdot\max\{e(y_{n-1},Tz),e(y_{n-1},y_n)\} \le$$
$$\le c_1\cdot d(z,x_{n-1})\cdot\max\{d(z,x_{n-1}),e(y_n,y_{n+1})\}.$$

Letting n tend to infinity, we have $e^2(Tz,w)\le 0$ and so $Tz=w$. Similarly, we can prove that $Sw=z$ and so

$$STz = Sw = z,\ TSw = Tz = w.$$

Thus ST has a fixed point z and TS has a fixed point w.

Now, suppose that ST has a second fixed point z'. Then by using inequality (8), we have

$$d(z,z')\cdot\max\{d(z,STz'),d(z,STz)\} = d(STz',STz)\cdot$$
$$\cdot\max\{d(z,STz'),d(z,STz)\} \le c_2e(Tz,Tz')\cdot$$
$$\cdot\max\{e(Tz,Tz'),d(z,STz)\} = c_2e^2(Tz,Tz'),$$

which implies

$$d^2(z,z') \le c_2\cdot e^2(Tz,Tz').$$

But, by using inequality (7), we have

$$e(Tz,Tz')\cdot\max\{e(Tz,Tz'),e(Tz,TSTz)\} =$$
$$= e(Tz',TSTz)\cdot\max\{e(Tz,Tz'),e(TSTz,TSTz)\} \le c_1d(z',STz)\cdot$$
$$\cdot\max\{d(z',STz),e(Tz,TSTz)\} = c_1d^2(z',STz) = c_1d^2(z,z'),$$

which implies

$$e^2(Tz',Tz) \le c_1\cdot d^2(z,z').$$

It follows that

$$d^2(z,z') \le c_2e^2(Tz,Tz') \le c_1c_2d^2(z,z')$$

and so

$$d^2(z,z') \leq c_1c_2d^2(z,z').$$

Since $0 \leq c_1c_2 < 1$, the uniqueness of z follows. Similarly, w is the unique fixed point of TS.

If $\exists n \in N$ that $d(x_n, x_{n+1}) = 0$ or $e(y_n, y_{n+1}) =$ the theorem is evident.

Corollary 4. *Let (X,d) be a complete metric space. If S and T are mappings of X into itself satisfying the inequalities*

$$(9) \qquad d(Tx,TSy)\max\{d(y,Tx),d(y,TSy)\} \leq$$

$$c_1d(x,Sy)\max\{d(x,Sy),d(y,TSy)\}$$

$$(10) \qquad d(Sy,STx)\max\{d(x,Sy),d(x,STx)\} \leq$$

$$c_2d(y,Tx)\max\{d(y,Tx),d(x,STx)\}$$

for all x,y in X, where $0 \leq c_1,c_2 < 1$, then ST has a unique fixed point z and TS has a unique fixed point w. Further, $Tz = w$ and $Sw = z$ and if $z = w$, z is the unique fixed point of S and T.

Proof. The existence of z and w follows from Theorem 3. If $z = w$, the z is of course a common fixed point of S and T.

Now, suppose that T has a second fixed point z'. Then, by using inequality (9), we have

$$d(z,z') \cdot \max\{d(z',Sz),d(z,TSz)\} =$$

$$= d(Tz',TSz) \cdot \max\{d(z',Sz),d(z,TSz)\} \leq$$

$$\leq c_1d(z,Tz') \cdot \max\{d(z,Tz'),d(z,TSz)\} = c_1d^2(z,z'),$$

which implies

$$d^2(z,z') \leq c_1d^2(z,z').$$

Since, $0 \leq c_1 < 1$, the uniqueness of z follows. Similarly, z is the unique fixed point of S.

The proofs of Corollaries 5 and 6 follow easily.

Corollary 5. *Let (X,d) and (Y,e) be complete metric spaces. If T is a mapping of X into Y and S is a mapping of Y into X, satisfying the inequalities*

$$e(Tx,TSy)\cdot\max\{e(y,Tx),e(y,TSy)\}$$

$$\leq d(x,Sy)[a_1d(x,Sy)+b_1e(y,TSy)]$$

$$d(Sy,STx)\cdot\max\{d(x,Sy),d(x,STx)\}\leq e(y,Tx)$$

$$[a_2e(y,Tx)+b_2d(x,STx)]$$

for all x in X and y in Y, where $0\leq a_i+b_i<1$, $i=1,2$, then TS has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz=w$ and $Sw=z$.

Corollary 6. *Let (X,d) be a complete metric space. If S and T are mappings of X into itself, satisfaying the inequalities*

$$d(Tx,TSy)\cdot\max\{d(y,Tx),d(y,TSy)\}\leq$$

$$d(x,Sy)[a_1d(x,Sy)+b_1d(y,TSy)]$$

$$d(Sy,STx)\cdot\max\{d(x,Sy),d(x,STx)\}\leq$$

$$d(y,Tx)[a_2d(y,Tx)+b_2d(x,STx)]$$

for all x,y in X, where $0\leq a_i+b_i<1$, $i=1,2$, then ST has a unique fixed point z and TS has a unique fixed point w. Further, $Tz=w$ and $Sw=z$ and if $z=w$, z is the unique fixed point of S and T.

Remark. A condition of the following type was used in [4]:

$$d(Tx,Sy)\leq f(d(x,y),d(x,Sx),d(y,Ty),d(x,Sy),d(y,Tx))$$

with a semi-homogeneous function f, $f:R_+^5\to R_+$, satisfying some additional assumption.Similarly, conditions (3) and (4) could be written down in the same form with

$$f(s,t,r,m,p,q)=\max\{st,sm,pq\}/q,\ q\neq 0.$$

A similar connection is between [5] and conditions(7) and (8).

References

[1] Fisher B.: Fixed point on two metric spaces, Glasnik matem. 16 (36) (1981), 333-337

[2] Fisher B.: Related fixed points on two metric spaces, Mathematics Seminar Notes, 10 (1982), 17-26.

[3] Hadžić O.: Fixed point theorems for multivalued mapping in metric spaces, Anal. Univ. Bucureşti (1984), 30-37.

[4] Tasković M.: On a family of contractive maps, Bull.Austral. Math. Soc. 13 (1975), 301-308.

[5] Tasković M.: On the convergence of certain sequences and some applications - II, Publ. Inst. Math. 35 (1977), 271-281.

REZIME

NEPOKRETNE TAČKE NA DVA KOMPLETNA METRIČKA PROSTORA

Neka su (X, d) i (Y, e) dva kompletna metrička prostora. Dokazano je da ako je T preslikavanje X u Y i ako je S preslikavanje Y u X tako da su zadovoljene nejednakosti

$$e^2(Tx, TSy) \leq$$

$$c_1 \max\{d(x, Sy)e(y, Tx), d(x, Sy)e(y, TSy), e(y, Tx)e(y, TSy)\}.$$

$$d^2(Sy, STx) \leq$$

$$c_2 \max\{e(y, Tx)d(x, Sy), e(y, Tx)d(x, STx), d(x, Sy)d(x, STx)\}.$$

za sve x u X i y u Y, gde je $0 \leq c_1 \cdot c_2 < 1$ ili nejednakosti

$$e(Tx, TSy) \cdot \max\{e(y, Tx), e(TSy, y)\}$$

$$\leq c_1 d(x, Sy) \cdot \max\{d(x, Sy), e(y, TSy)\}$$

$$d(Sy, STx) \cdot \max\{d(x, Sy), d(x, STx)\} \leq$$

$$\leq c_2 e(y, Tx) \cdot \max\{e(y, Tx), d(x, STx)\}$$

za sve x u X i y u Y , gde je $0 \leq c_1, c_2 < 1$, onda ST ima jedinstvenu nepokretnu tačku z u X i TS ima jedinstvenu nepokretnu tačku w u Y. Dalje, $Tz = w$ i $Sw = z$.

Received by the editors April 13, 1987.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 95-104

Review of Research
Faculty of Science
Mathematics Series

THE SEQUENCE SPACE $Ces(p, s)$ AND RELATED MATRIX TRANSFORMATIONS

F.M. Khan and **M.A. Khan**
Department of Mathematics, Aligarh Muslim University,
Aligarh, India, 202002

Abstract

In the paper, the main purpose is to define and to investigate the sequence space *Ces(p,s)* and to determine the matrices of classes $(Ces(p,s), \ell_\infty)$ and $(Ces(p,s), c)$ where ℓ_∞ and c are respectively the space of bounded and convergent complex sequences and for $p = (p_r)$ with inf $p_r > 0$, the space $Ces(p,s)$ is defined for $s \geq 0$ by

$$Ces(p,s) = \{x = (x_k) : \sum_{r=0}^{\infty} (2^r)^{-s} (\frac{1}{2^r} \sum_r |x_k|)^{p_r} < \infty\},$$

where $\sum_r$ denotes a sum over the ranges $2^r \leq k < 2^{r+1}$.

These spaces i.e. $Ces(p,s)$ can be viewed as $Ces(p)$ spaces with weights, generalizing $Ces(p)$ spaces.

AMS Subject classification (1980): 46A45
Key words and phrases: Köthe-Toeplitzdual, $Ces^*(p,s)$, Matrix transformations.

1.

Let $A = (a_{n,k})$ be an infinite matrix of complex numbers $a_{n,k}(n, k = 1,2,\ldots)$ and V, W be two subsets of the space of complex sequences. We say that the matrix A defines a matrix transformations from V into W and denote it by $A \in (V, W)$, if for every sequence $x = (x_k) \in V$ the sequence $A(x) = A_n(x)$ is in W, where $A_n = \sum_{k=1}^{\infty} a_{n,k} x_k$.

The main purpose of this note is to define and to investigate the sequence space $Ces(p,s)$ and to determine the matrices of classes $(Ces(p,s), \ell_\infty)$ and $(Ces(p,s), c)$, where ℓ_∞ and c are respectively the space of bounded and convergent complex sequences and for $p = (p_r)$ with inf $p_r > 0$, the space $Ces(p,s)$ is defined for $s \geq 0$ by

$$Ces(p,s) = \{x = (x_k) : \sum_{r=0}^{\infty} (2^r)^{-s} (\frac{1}{2^r} \sum_r |x_k|)^{p_r} < \infty\},$$

where $\sum_r$ denotes a sum over the ranges $2^r \leq k < 2^{r+1}$.

Obviously, the sequence space

$$Ces(p) = \{x = (x_k) : \sum_{r=0}^{\infty} (\frac{1}{2^r} \sum_r |x_k|)^{p_r} < \infty\}, \text{ where inf } p_r > 0,$$

which has been investigated by K.P. Lim ([3], [4]) is a special case of $Ces(p,s)$ which corresponds to $s = 0$. And $Ces(p,s) \supset Ces(p) \supset \ell(p)$ for $p_r \geq 1$.

With regard to notation, the dual space of $Ces(p,s)$, i.e. the space of all the continuous linear functionals on $Ces(p,s)$, will be denoted by $Ces^*(p,s)$. We write $A_r(n) = \max_r |a_{n,k}|$ where for each n the maximum is taken for k in $[2^r, 2^{r+1}]$.

Troughout the paper the following well-known inequality (See[1]) will be used frequelntly.

For any $C > 0$ and any two complex numbers a, b:

$$(1) \qquad |ab| \leq C(|a|^q C^{-q} + |b|^p) \text{ where } p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

To begin with we can show that the space $Ces(p,s)$ is paranormed by

$$(2) \qquad g(x) = (\sum_{r=0}^{\infty} (2^r)^{-s} (\frac{1}{2^r} \sum_r |x_k|)^{p_r})^{1/M}$$

if $H = \sup_r p_r < \infty$ and $M = \max(1, H)$. Clearly $g(\Theta) = 0$ and $g(x) = g(-x)$, where $\Theta = (0, 0, \ldots\ldots)$. Take any $x, y \in Ces(p,s)$. Since

$p_r \le M$ and $M \ge 1$, using Minkowski's inequality we obtain that g is subadditive. Finally, we have to check the continuity of scalar multiplication. For any complex λ, we have

$$g(\lambda x) = [\sum_{r=0}^{\infty}(2^r)^{-s}(\frac{1}{2^r}|\lambda x_k|)^{p_r}]^{1/M}$$

$$\le \sup_r |\lambda|^{p_r/M} \cdot g(x).$$

Now let $\lambda \to 0$ for any fixed x with $g(x) \ne 0$. Since $\sum_{r=0}^{\infty}(2^r)^{-s}(\frac{1}{2^r}\sum_r |x_k|)^{p_r} < \infty$, there exists an integer $m_0 > 1$, for $|\lambda| < 1$ and $\epsilon > 0$, such that

$$\sum_{r=m_0}^{\infty}(2^r)^{-s}(\frac{1}{2^r}\sum_r |\lambda x_k|)^{p_r} < \epsilon. \tag{3}$$

Taking $|\lambda|$ sufficiently small such that $|\lambda|^{p_r} < \epsilon/g(x)$ for $r = 0, 1, \ldots\ldots m_0 - 1$, we then have

$$\sum_{r=0}^{m_0-1}(2^r)^{-s}(\frac{1}{2^r}\sum_r |\lambda x_k|)^{p_r} < \epsilon. \tag{4}$$

(3) and (4) together implies that $g(\lambda x) \to 0$ as $\lambda \to 0$.

It is quite routine to show that $(Ces(p,s), d)$ is a metric space with the metric $d(x,y) = g(x-y)$ provided that $x, y \in Ces(p,s)$, where g is defined by (2). And using a similar method to that in [2] one can show that $Ces(p,s)$ is complete under the metric mentioned above.

2.

Now we are going to give the following theorem by which the Köthe-Toeplitz dual of $Ces(p,s)$ will be determined.

Theorem 1. *If* $1 < p_r \le \sup_r p_r < \infty$ *and* $\frac{1}{p_r} + \frac{1}{q_r} = 1$, $r = 0, 1, 2, \ldots$ *then*

$$Ces^{+}(p,s) = \{a = (a_k) : \sum_{r=0}^{\infty}(2^r)^{s(q_r-1)}(2^r \max_r |a_k|)^{q_r} E^{-q_r} < \infty$$

$$\textit{for some integer } E > 1\}, \ s \ge 0.$$

Proof. Let $1 < p_r \leq \sup_r p_r < \infty$ and $\frac{1}{p_r} + \frac{1}{q_r} = 1$, for $r = 0, 1, 2, \ldots$. Then take

$$\mu(q,s) = \{a = (a_k) : \sum_{r=0}^{\infty} (2^r)^{s(q_r-1)} (2^r \max_r |a_k|)^{q_r} E^{-q_r} < \infty \tag{5}$$

$$\text{for some integer } E > 1\}, \quad s \geq 0.$$

We now want to show that $Ces^+(p,s) = \mu(q,s)$. Let $x \in Ces(p,s)$, $a \in \mu(q,s)$. Therefore using inequality (1), we get

$$\begin{aligned}
\sum_{k=1}^{\infty} |a_k x_k| &= \sum_{r=0}^{\infty} \sum_r |a_k x_k| \\
&\leq \sum_{r=0}^{\infty} 2^r \max_r |a_k| (2^r)^{s/p_r} \cdot \frac{1}{2^r} (2^r)^{-s/p_r} \sum_r |x_k|
\end{aligned}$$

$$\leq E\Big(\sum_{r=0}^{\infty} (2^r \max_r |a_k|)^{q_r} (2^r)^{sq_r/p_r} E^{-q_r} + \sum_{r=0}^{\infty} (2^r)^{-s} \Big(\frac{1}{2^r} \sum_r |x_k|\Big)^{p_r}\Big).$$

So $\sum |a_k x_k|$ is convergent, which implies that $\sum a_k x_k$ is convergent i.e. $a \in Ces^+(p,s)$. In other words $Ces^+(p,s) \supset \mu(q,s)$. Conversely, let us suppose that $\sum a_k x_k$ is convergent and $x \in Ces(p,s)$, but $a \notin \mu(q,s)$. Then we write that

$$\sum_{r=0}^{\infty} (2^r)^{s(q_r-1)} (2^r \max_r |a_k|)^{q_r} E^{-q_r} = \infty \text{ for each } s \geq 0$$

and for every $E > 1$. So we can find a sequence $0 = n(0) < n(1) < n(2) < \ldots$ such that for $\nu = 0, 1, 2, \ldots$

$$M_\nu = \sum_{r=n(\nu)}^{n(\nu+1)-1} (2^r)^{s(q_r-1)} (2^r \max_r |a_k|)^{q_r} (\nu+2)^{-q_r/p_r} > 1.$$

Now define a sequence $x = (x_k)$ as follows: for each ν,

$$x_{N(r)} = 2^{rq_r} |a_{N(r)}|^{q_r-1} \, sgn \, a_{N(r)} (2^r)^{s(q_r-1)} (\nu+2)^{-q_r} M_\nu^{-1},$$

for $n(\nu) \leq r \leq n(\nu+1) - 1$, and $x_k = 0$ for $k \neq N(r)$ where $N(r)$ is such that $|a_{N(r)}| = \max_r |a_k|$, the maximum is taken for k in $[2^r, 2^{r+1})$.

Therefore,

$$\sum_{\substack{n(\nu)\\ k=2}}^{\substack{n(\nu+1)-1\\ 2}} a_k x_k = \sum_{r=n(\nu)}^{n(\nu+1)-1} (2^r|a_{N(r)}|)^{q_r}(2^r)^{s(q_r-1)}(\nu+2)^{-q_r}M_\nu^{-1} =$$

$$= M_\nu^{-1}(\nu+2)^{-1}\sum_{r=n(\nu)}^{n(\nu+1)-1} (2^r|a_{N(r)}|)^{q_r}\cdot(2^r)^{s(q_r-1)}(\nu+2)^{-q_r/p_r} =$$

$$= (\nu+2)^{-1}.$$

It follows that $\sum_{k=1}^{\infty} a_k x_k = \sum_{\nu=0}^{\infty}(\nu+2)^{-1}$ diverges. Moreover

$$\sum_{r=n(\nu)}^{n(\nu+1)-1} (2^r)^{-s}\cdot(\frac{1}{2^r}\sum_r |x_k|)^{p_r} =$$

$$= \sum_{r=n(\nu)}^{n(\nu+1)-1} (2^r)^{-s}(2^{r(q_r-1)p_r}|a_{N(r)}|^{(q_r-1)p_r}(\nu+2)^{-q_r p_r}.$$

$$\cdot M_\nu^{-p_r}(2^r)^{s(q_r-1)p_r})$$

$$\leq (\nu+2)^{-2}M^{-1}\sum_{r=n(\nu)}^{n(\nu+1-1)} (2^r)^{s(q_r-1)}2^{rq_r}|a_{N(r)}|^{q_r}(\nu+2)^{-q_r/p_r}$$

$$= (\nu+2)^{-2}$$

Hence, $\sum_{r=0}^{\infty}(2^r)^{-s}(\frac{1}{2^r}\sum_r |x_k|)^{p_r} \leq \sum_{\nu=0}^{\infty}(\nu+2)^{-2} < \infty$ i.e. $x \in Ces(p,s)$. And this contradicts our assumption. So $a \in \mu(q,s)$ i.e. $\mu(q,s) \supset Ces^+(p,s)$. Then combining these two results we get

$$Ces^+(p,s) = \mu(q,s).$$

Let us now determine the continuous dual of $Ces(p,s)$ by the following theorem.

Theorem 2. *Let $1 < p_r \leq \sup_r p_r < \infty$. Then $Ces^*(p,s)$ is isomorphic to $\mu(q,s)$ which is defined by (5).*

Proof. It is easy to check that each $x \in Ces(p,s)$ can be written as $x = \sum_{k=1}^{\infty} x_k e_k$ where $e_k = (0,0,\ldots,1,0,0\ldots)$, where 1 appears at k-th place.

Then for any $f \in Ces^*(p,s)$

$$f(x) = \sum_{k=1}^{\infty} x_k f(e_k) = \sum_{k=1}^{\infty} x_k a_k,$$

where $f(e_k) = a_k$.

By theorem 1, the convergence of $\sum_{k=1}^{\infty} a_k x_k$ for every x in $Ces(p,s)$ implies that $a \in \mu(q,s)$.

If $x \in Ces(p,s)$ and if we take $a \in \mu(q,s)$ taken by Theorem 1, $\sum_{k=1}^{\infty} a_k x_k$ converges and clearly defines a linear functional on $Ces(p,s)$. Using the same kind of argument to that in Theorem 1, it is easy to check that

$$\sum_{k=1}^{\infty} |a_k x_k| \le E(\sum_{r=0}^{\infty} (2^r)^{s(q_r-1)} (2^r \max_r |a_k|)^{q_r} E^{-q_r} + 1) g(x)$$

whenever $g(x) \le 1$. Hence $\sum_{k=1}^{\infty} a_k x_k$ defines an element of $Ces^*(p,s)$. Obviously, the map $T : Ces^*(p,s) \to \mu(q,s)$ given by $T(f) = (a_1, a_2, \cdots)$ is linear and bijective. Hence $Ces^*(p,s)$ is isomorphic to $\mu(q,s)$.

3.

In the following theorems we are going to characterize the matrix classes $(Ces(p,s), \ell_\infty)$ and $(Ces(p,s), c)$.

Theorem 3. *Let* $1 < p_r \le \sup_r p_r < \infty$. *Then* $A \in (Ces(p,s), \ell_\infty)$ *iff there exists an integer* $E > 1$ *such that* $U(E) > \infty$ *where*

$$U(E) = \sup_n \sum_{r=0}^{\infty} (2^r A_r(n))^{q_r} (2^r)^{s(q_r-1)} E^{-q_r} < \infty$$

and

$$\frac{1}{p_r} + \frac{1}{q_r} = 1, \quad r = 0, 1, 2, \ldots.$$

Proof. Sufficiency. Suppose there exists an integer $E > 1$ and consider the hypothesis. Then by inequality (1), we have

$$\sum_{k=1}^{\infty}|a_{n,k}x_k| = \sum_{r=0}^{\infty}\sum_{r}|a_{n,k}x_k|$$

$$\le E(\sum_{r=0}^{\infty}(2^r A_r(n))^{q_r}(2^r)^{sq_r/p_r}$$

$$E^{-q_r} + \sum_{r=0}^{\infty}(2^r)^{-s}(\frac{1}{2^r}\sum_{r}{}^{n}x_k|)^{p_r}) < \infty.$$

Therefore $A \in (Ces(p,s), \ell_\infty)$.

Necessity. Suppose that $A \in (Ces(p,s), \ell_\infty)$ but that

$$\sup_n \sum_{r=0}^{\infty}(2^r A_r(n))^{q_r}(2^r)^{s(q_r-1)}E^{-q_r} = \infty \text{ for every integer } E > 1.$$

Then $\sum_{k=1}^{\infty} a_{n,k}x_k$ convergences for every n and for every $x \in Ces(p,s)$, whence $(a_{n,k})_{k=1,\ldots} \in Ces^{+}(p,s)$ for every n.

By Theorem 1, it follows that each A_n defined by $A_n(x) = \sum_{k=1}^{\infty} a_{n,k}x_k$ is an element of $Ces^{*}(p,s)$. Since $Ces(p,s)$ is complete and since $\sup_n |A_n(x)| < \infty$ on $Ces(p,s)$, by the uniform boundedness principle there exists a number L independent of n and x and a number $\delta < 1$ such that

(6) $$|A_n(x)| \le L$$

for every $x \in S[\Theta, \delta]$ and every n, where $S[\Theta, \delta]$ is the closed sphere in $Ces(p,s)$ with centre the origin Θ and radius δ.

Now choose an integer $Q > 1$ such that

$$Q\delta^M > L.$$

Since

$$\sup_n \sum_{r=0}^{\infty}(2^r A_r(n))^{q_r}(2^r)^{s(q_r-1)}Q^{-q_r} = \infty,$$

there exists an integer $m_0 > 1$ such that

(7) $$R = \sum_{r=0}^{m_0}(2^r A_r(n))^{q_r}(2^r)^{s(q_r-1)}Q^{-q_r} > 1.$$

Define a sequence $x = (x_k)$ as follows:

$$x_k = 0 \text{ if } k \geq 2^{m_0+1}$$

and

$$x_{N(r)} = 2^{rq_r}\delta^{M/p_r}(sgn\ a_{n,N(r)})$$

$$|a_{n,N(r)}|^{q_r-1}R^{-1}Q^{-q_r/p_r}(2^r)^{s(q_r-1)}$$

$x_k = 0$ ($k \neq N(r)$, for $0 \leq r \leq m_0$) where $N(r)$ is the smallest integer such that $|a_{n,N(r)}| = \max_r |a_{n,k}|$. Then one can easly show that $g(x) \leq \delta$ but $|A_n(x)|^r > L$, which contradicts to (6).

This completes the proof of the Theorem.

Theorem 4. *Let* $1 < p_r \leq \sup_r p_r < \infty$. *Then* $A \in (Ces(p,s), c)$ *if*

(4.1) $a_{n.k} \rightarrow \alpha_k (n \rightarrow \infty, k$ *fixed*)

(4.2) *there exists an integer* $E > 1$ *such that* $U(E) < \infty$, *where* $U(E)$*is defined as in Theorem 3.*

Proof. Suppose $A \in (Ces(p,s), c)$. Then $A_n(x)$ exists for each $n \leq 1$ and $\lim_n A_n(x)$ exists, for every $x \in Ces(p,s)$. Therefore, by a similar argument to that in Theorem 3, we have condition (4.2). The condition (4.1) is obtained by taking $x = e_k \in Ces(p,s)$, where $e_k = (0, 0, \ldots 1, 0, 0, 0, \ldots)$ where 1 appearsat k-th place.

For the sufficiency the conditions of the Theorem imply that

$$(8) \qquad \sum_{r=0}^{\infty}(2^r \max_r |\alpha_k)^{q_r}(2^r)^{s(q_r-1)}$$

$$E^{-q_r} \leq U(E) < \infty.$$

By using (8), it is easy to check that $\sum_{k=1}^{\infty} \alpha_k x_k$ is absolutely convergent for each $x \in Ces(p,s)$. Moreover, for each $x \in Ces(p,s)$, there exists an integer $m_0 \geq 1$, such that

$$g_{m_0}(x) = \sum_{r=m_0}^{\infty}(2^r)^{-s}(\frac{1}{2^r}\sum_r |x_k|)^{p_r} < 1.$$

If $g_{m_0}(x) \neq 0$ the by the proof of Theorem 2 and by inequality (1) we have

(9)
$$\sum_{m_0}^{\infty} |a_{n,k} - \alpha_k|\, |x_k| \leq E(\sum_{r=m_0}^{\infty} (2^r)^{s(q_r-1)}$$

$$k = 2$$

$$(2^r B_r(n))^{q_r} E^{-q_r} + 1) g_{m_0}(x)^{1/M},$$

where $B_r(n) = \max_r |a_{n,k} - \alpha_k|$.

Cleraly (9) holds if $g_{m_0}(x) = 0$. Since

$$\sum_{r=m_0}^{\infty} (2^r)^{s(q_r-1)} (2^r B_r(n))^{q_r} E^{-q_r} \leq 2\, U(E) < \infty,$$

from (9), it follows immediately that $\lim_{n\to\infty} \sum a_{n,k} x_k = \sum \alpha_k x_k$. This shows that $A \in (Ces(p,s), c)$ which proves the Theorem.

Remark. To be able to get the necessary and sufficient confition for $A \in (Ces(p,s), c_0)$, where c_0 is the space of null sequences, it would be enough to take $\alpha_k = 0$ in the above theorem.

Finally we are grateful to the referee for his suggestions and some useful remarks.

References

[1] Maddox, I.J.: Continuous and Köthe-Toeplitz dual of certain sequence spaces. Proc. Camb. Phil. Soc. 65 (1969), 431-435.

[2] Maddox, I.J.: Elements of Functional analysis. University Press, Cambridge, 1970.

[3] Lim, K.P.: Matrix transformation in the Cesaro sequence spaces. Kyungpook Math. J. 14 (1974), 221-227.

[4] Lim, K.P.: Matrix transformation on certain sequence spaces. Tamkang J. of Mathematics 8, No.2 (1977), 213-220.

REZIME

NIZOVNI PROSTOR $Ces(p,s)$ I ODNOSNE MATRIČNE TRANSFORMACIJE

Glavna svrha rada je da se definiše i ispita nizovni prostor $Ces(p,s)$, te da se odrede matrice klasa $(Ces(p,s),\ell_\infty)$ i $(Ces(p,s),c)$ gde su ℓ_∞ i c prostori ograničenih, odnosno konvergentnih kompleksnih nizova, a za $p=(p_2)$ sa $\inf p_2>0$, prostor $Ces(p,s)$ je definisan za $s\geq 0$ sa:

$$Ces(p,s)=\{x=(x_k):\sum_{r=0}^{\infty}(2^r)^{-s}(\frac{1}{2^r}\sum_r |x_k|)^{p_r}<\infty\},$$

gde $\sum_r$ označava sumu po svim k:

$$z^r\leq k\leq 2^{r+1}.$$

Prostori $Ces(p,s)$ se mogu posmatrati kao $Ces(p)$ prostori sa težinama, koji uopštavaju $Ces(p)$ prostore.

Recived by the editors May 5, 1989.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 105-115

Review of Research
Faculty of Science
Mathematics Series

UNIFORM CONVEXITY AND THE FIXED POINT PROPERTY[1]

Shih-sen Chang
Department of Mathematics, Sichuan University, Chengdu

Abstract

In this paper the concept of (F^k) uniformly convex spaces and some fixed point theorems for generalized nonexpansive mappings are introduced and presented. The results obtained in this paper extend and improve some recent results of Dulst [3], Jaggi [5] and Browder [2], Kirk [6].

AMS Mathematics Subject Classification (1980): 47H10
Key words and phrases: Uniform convexity, fixed point property.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space. A mapping $T : K \to K$, $K \subset X$ is said to be nonexpansive, if

$$(1.1) \qquad \|Tx - Ty\| \le \|x - y\|, \qquad \forall x, y \in K.$$

$T : K \to K$ is said to be generalized nonexpansive, if for every subset E of K with at least two points and $T(E) \subset E$

$$(1.2) \qquad \sup_{y \in E} \|Tx - Ty\| \le \sup_{y \in E} \|x - y\|, \qquad \forall x \in E.$$

---

[1] Research supported by the Science Fund. of the Chinese Academy of Sciences.

It is easy to see that the nonexpansive mapping and the mapping $T: K \to K$ satisfying one of the following conditions:

$$(1.3) \qquad \|Tx - Ty\| \leq \sup\{\|x - z\|,\ z \in \{T^n y\}_{n \geq 0}\}, \qquad x, y \in K$$

$$(1.4) \quad \|Tx - Ty\| \leq \sup\{\|x - z\|,\ z \in (\{T^n x\}_{n \geq 0} \cup \{T^n y\}_{n \geq 0})\},\ x, y \in K$$

are all the special cases of generalized nonexpansive mappings.

Let $(X, \|\cdot\|)$ be a Banach space, and $K \subset X$ a nonempty bounded closed convex subset. K is said to have normal structure, if for each closed convex subset $C \subset K$ consisting of more than one point there exists a point $x \in C$ such that

$$\sup\{\|x - y\|,\ y \in C\} < \operatorname{diam}(C) \qquad \text{(the diameter of } C\text{)}.$$

A Banach space X is said to have normal structure, if for each bounded closed nonempty convex set $K \subset X$ it has the normal structure.

A Banach space X is said to have the fixed point property (F.P.P.) for nonexpansive mappings (or for generalized nonexpansive mappings), if for every weakly compact convex subset K of X and for every nonexpansive mapping (or generalized nonexpansive mapping) $T : K \to K$, T has a fixed point in K.

The following two questions have long been open.

1. Which Banach spaces have the F.P.P. for generalized nonexpansive mappings?
2. Which Banach spaces have the F.P.P. for nonexpansive mapping?

It is obvious that question 2 is a special case of question 1.

Concerning question 2, recently Alspach [1] showed that the answer is negative for $X = L^1[0,1]$. If X is a Hilbert, or more general, if X is a reflexive Banach space having normal structure then the answer to question 2 is also positive (cf. Browder [2] and Kirk [6]).

Concerning question 1, recently Jaggi [5] showed that, if X is a reflexive Banach space having normal structure then the answer is also affirmative.

The purpose of this paper is first to introduce the concept of (F^k) uniformly convex space and then use this concept to prove that every Banach space with the separated property (S.P.) can be reendowed with an equivalent norm $\|\cdot\|_1$ so that $(X, \|\cdot\|_1)$ has the F.P.P. for generalized nonexpansive mapping (hence for every kind of mappings in (1.1), (1.3) or (1.4)). The results presented in this paper extend and improve some recent results of [2,3,5,6].

2. Definition and preliminaries

Let X be a Banach space, and X^* its dual. Throughout this paper we always denote by

$$B(X) = \{x \in X \ : \ \|x\| \leq 1\} ;$$

$$B(X^*) = \{f \in X^* \ : \ \|f\| \leq 1\} ;$$

$$S(X^*) = \{f \in X^* \ : \ \|f\| = 1\} .$$

Definition 1. *A Banach space $(X, \|\cdot\|)$ is said to have the separated property (S.P.), if there exists a countable set $F \subset B(X^*)$ such that if an $x \in X$ satisfies $f(x) = 0$ for every $f \in F$, then $x = 0$.*

In what folows we denote such kind of spaces by $(X, \|\cdot\|, F)$.

Let $(X, \|\cdot\|, F)$ be a Banach space with the S.P. . By virtue of F we can define $\|\cdot\|_1$ as follows:

$$\|x\|_1 = \left(\|x\|^2 + \sum_{i=1}^{\infty} \frac{|f_i(x)|^2}{2^i} \right)^{\frac{1}{2}}, \qquad f_i \in F. \tag{2.1}$$

It is easy to prove that $\|\cdot\|_1$ is a norm on X equivalent to the given norm $\|\cdot\|$, and

$$\|x\| \leq \|x\|_1 \leq \sqrt{2}\|x\|. \tag{2.2}$$

Example Let $(X, \|\cdot\|)$ be a separable Banach space. Then $(X, \|\cdot\|)$ is an example of Banach space with the S.P. . In fact, let $\{x_n\}$ be a countable dense subset of X. By Hahn-Banach Theorem, for each x_n, $n = 1, 2, \ldots$ there exists an $f_n \in S(X^*) \subset B(X^*)$ such that $f_n(x_n) = \|x_n\|$. Letting

$$F = \{f_n\}_{n=1}^{\infty} \tag{2.3}$$

we know that if $x \in X$ satisfies $f(x) = 0$, for every $f \in F$, then $x = 0$. In fact, let $\{x_{n_i}\} \subset \{x_n\}$ and $x_{n_i} \to x$ hence we have

$$\|x_{n_i}\| = f_{n_i}(x_{n_i}) - f_{n_i}(x) \leq \|f_{n_i}\| \cdot \|x_{n_i} - x\| \to 0 \ \ (n_i \to \infty).$$

This implies that $x_{n_i} \to 0$, i.e. $x = 0$.

Definition 2. *Let $(X, \|\cdot\|, F)$ be a Banach space with the S.P. . X is said to be (F^k) uniformly convex for some positive integer k, if for each $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for any $k+1$ points $x_1, x_2, \ldots, x_{k+1} \in B(X)$, if*

$$\|x_1 + x_2 + \cdots + x_{k+1}\| > (k+1) - \delta,$$

then

$$|A(x_1, x_2, \ldots, x_{k+1}; f_1, \ldots, f_k)| < \varepsilon, \qquad \forall \{f_1, f_2, \ldots, f_k\} \subset F,$$

where

$$A(x_1, x_2, \ldots, x_{k+1}; f_1, \ldots, f_k) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ f_1(x_1) & f_1(x_2) & \cdots & f_1(x_{k+1}) \\ \vdots & \vdots & & \vdots \\ f_k(x_1) & f_k(x_2) & \cdots & f_k(x_{k+1}) \end{vmatrix}.$$

Proposition 1. *If $(X, \|\cdot\|, F)$ is (F^k) uniformly convex for some positive integer k, then $(X, \|\cdot\|, F)$ is (F^{k+1}) uniformly convex.*

Proof. Since X is (F^k) uniformly convex, hence for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $x_1, x_2, \ldots, x_{k+1}$ are any $k+1$ points of $B(X)$ with $\|x_1 + x_2 + \cdots + x_{k+1}\| > (k+1) - \delta$, then

$$(2.4) \quad |A(x_1, x_2, \ldots, x_{k+1}; f_1, \ldots, f_k)| < \frac{\varepsilon}{k+1}, \qquad \forall \{f_1, \ldots, f_k\} \subset F.$$

Now suppose that $z_1, z_2, \ldots, z_{k+2}$ are any $k+2$ points in $B(X)$ with

$$\|z_1 + z_2 + \cdots + z_{k+2}\| > (k+2) - \delta.$$

Then from the triangle inequality for each i we have

$$(2.5) \qquad \|z_1 + z_2 + \cdots + z_{i-1} + z_{i+1} + \cdots + z_{k+2}\| \geq$$

$$\geq \|z_1 + z_2 + \cdots + z_{k+2}\| - \|z_i\| > (k+1) - \delta.$$

Now let $f_1, f_2, \ldots, f_{k+1}$ be any $k+1$ functionals of F and consider the determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ f_1(z_1) & f_1(z_2) & \cdots & f_1(z_{k+2}) \\ \vdots & \vdots & & \vdots \\ f_{k+1}(z_1) & f_{k+1}(z_2) & \cdots & f_{k+1}(z_{k+2}) \end{vmatrix}$$

Expanding in minors along the second row and using (2.4) and (2.5) we have

$$|A(z_1, z_2, \ldots, z_{k+2}; f_1, \ldots, f_{k+1})| < \varepsilon, \qquad \forall \{f_1, \ldots, f_{k+1}\} \subset F.$$

This completes the proof. □

Proposition 2. *Let $(X, \|\cdot\|, F)$ be a Banach space with the S.P. and C a subset of X. Suppose that there exists some k such that for any $k+1$ points $x_1, x_2, \ldots, x_{k+1} \in C$ the following holds:*

$$A(x_1, x_2, \ldots, x_{k+1}; f_1, \ldots, f_k) = 0, \qquad \forall \{f_1, \ldots, f_k\} \subset F.$$

Then the dimension of span(C) dim(span$(C)) < \infty$.

The proof of this proposition is straightforward we omit the statement here.

Proposition 3. [7] *If E is the real or complex field, then for each $\varepsilon > 0$, there exists a $\delta(\varepsilon) \in (0,1)$ such that for any $|x| \leq 1$, $|y| \leq 1$, and $|x - y| \geq \varepsilon$, the following holds*

$$\left|\frac{x+y}{2}\right|^2 \leq (1-\delta)\frac{|x|^2 + |y|^2}{2}.$$

Theorem 1. *If $(X, \|\cdot\|, F)$ is a (F^k) uniformly convex Banach space for some positive integer k, then X has normal structure.*

Proof. Let $K \subset X$ be any nonempty bounded closed convex subset with $\operatorname{diam}(K) = d > 0$. If the dimension of K is finite, then K is a compact convex subset of X. By a wellknown result we know that K has normal structure. Therefore without loss of generality, we can suppose that

$\dim(K) = \infty$. By virtue of Proposition 2 there exist some $x_1^*, x_2^*, \dots, x_{k+1}^* \in K$, $f_1^*, f_2^*, \dots, f_k^* \in F$ and $\varepsilon_0 > 0$ such that

$$\frac{1}{d^k}|A(x_1^*, x_2^*, \dots, x_{k+1}^*; f_1^*, \dots, f_k^*)| \geq \varepsilon_0. \tag{2.6}$$

Since X is (F^k) uniformly convex, therefore for ε_0 there exists a $\delta(\varepsilon_0) > 0$ such that for any $y_1, y_2, \dots, y_{k+1} \in B(X)$ if

$$|A(y_1, y_2, \dots, y_{k+1}; f_1^*, \dots, f_k^*)| \geq \varepsilon_0, \tag{2.7}$$

then

$$\|y_1 + \cdots + y_{k+1}\| \leq (k+1) - \delta(\varepsilon_0). \tag{2.8}$$

Next for any $z \in K$ it is easy to see that $\frac{1}{d}(x_i^* - z) \in B(X)$, $i = 1,2, \dots, k+1$ and from (2.6) we have

$$|A(\frac{x_1^* - z}{d}, \dots, \frac{x_{k+1}^* - z}{d}; f_1^*, \dots, f_k^*)| = \tag{2.9}$$

$$= \frac{1}{d^k}|A(x_1^*, \dots, x_{k+1}^*; f_1^*, \dots, f_k^*)| \geq \varepsilon_0.$$

In view of (2.7) and (2.8) we have

$$\left\|\frac{x_1^* - z}{d} + \cdots + \frac{x_{k+1}^* - z}{d}\right\| \leq (k+1) - \delta(\varepsilon_0),$$

$$\text{i.e.} \quad \left\|z - \frac{x_1^* + x_2^* + \cdots + x_{k+1}^*}{k+1}\right\| \leq d - \frac{d}{k+1}\delta(\varepsilon_0) < \operatorname{diam}(K).$$

This implies that $X, \|\cdot\|)$ has normal structure.

This completes the proof. □

Theorem 2. *Let $(X, \|\cdot\|, F)$ be a Banach space with the S.P. . If $\|\cdot\|_1$ is the equivalent norm on X defined by (2.1) then $(X, \|\cdot\|_1, F)$ is a (F^k) uniformly convex space for any positive integer k.*

Proof. Firstly, we show that $(X, \|\cdot\|_1, F)$ has the S.P. . For the purpose, it suffices to prove

$$F \subset B_1(X^*) \triangleq \{f \in X^* : \|f\|_1 \leq 1\},$$

where $B_1(X) \overset{\triangle}{=} \{x \in X : \|x\|_1 \leq 1\}$, and

$$\|f\|_1 = \sup_{x \in B_1(X)} \{|f(x)|\}.$$

In fact, if $f \in F$, it follows from (2.2) that

$$\|f\|_1 = \sup_{\|x\|_1 \leq 1} |f(x)| \leq \sup_{\|x\| \leq 1} |f(x)| \leq 1.$$

This shows that $F \subset B_1(X^*)$.

Now we prove that $(X, \|\cdot\|_1, F)$ is (F^k) uniformly convex for any positive integer k. By Proposition 1 it suffices to prove that $(X, \|\cdot\|_1, F)$ is (F^1) uniformly convex. For the purpose it suffices to prove that for any sequences $\{x_n\}$, and $\{y_n\}$ of $B_1(X)$ with $\|x_n + y_n\|_1 \to 2$, then we have

$$\lim_{n\to\infty} |A(x_n, y_n; f)| = \lim_{n\to\infty} |f(x_n - y_n)| = 0, \quad \forall f \in F.$$

Otherwise, there exist some $\varepsilon_0 > 0$, $f_{k_0} \in F$ and some subsequences $\{x_{n_m}\} \subset \{x_n\}$, and $\{y_{n_m}\} \subset \{y_n\}$ such that

$$\text{(2.10)} \qquad |f_{k_0}(x_{n_m} - y_{n_m})| \geq \varepsilon_0, \quad m = 1, 2, \dots .$$

It follows from $|f_{k_0}(x_{n_m})| \leq 1$, $|f_{k_0}(y_{n_m})| \leq 1$, (2.10), Proposition 3 that there exists $\delta \in (0,1)$ such that

$$\text{(2.11)} \qquad \left|\frac{f_{k_0}(x_{n_m}) + f_{k_0}(y_{n_m})}{2}\right|^2 \leq (1-\delta)\frac{|f_{k_0}(x_{n_m})|^2 + |f_{k_0}(y_{n_m})|^2}{2}.$$

Therefore we have

$$\frac{\delta}{2^{k_0}}\left|\frac{f_{k_0}(x_{n_m} - y_{n_m})}{2}\right|^2 \leq \frac{\delta}{2^{k_0}}\left(\frac{|f_{k_0}(x_{n_m})| + |f_{k_0}(y_{n_m})|}{2}\right)^2 \leq$$

$$\leq \frac{\delta}{2^{k_0}}\frac{|f_{k_0}(x_{n_m})|^2 + |f_{k_0}(y_{n_m})|^2}{2} \leq$$

$$\leq \frac{1}{2^{k_0}}\left(\frac{|f_{k_0}(x_{n_m})|^2 + |f_{k_0}(y_{n_m})|^2}{2} - \left|\frac{f_{k_0}(x_{n_m}) + f_{k_0}(y_{n_m})}{2}\right|^2\right) \leq$$

$$\leq \frac{\|x_{n_m}\|^2 + \|y_{n_m}\|^2}{2} - \left\|\frac{x_{n_m} + y_{n_m}}{2}\right\|^2 +$$

$$+\sum_{i=1}^{\infty}\frac{1}{2^i}\left(\frac{|f_i(x_{n_m})|^2+|f_i(y_{n_m})|^2}{2}-\left|\frac{f_i(x_{n_m}+y_{n_m})}{2}\right|^2\right)=$$

$$=\frac{1}{2}(\|x_{n_m}\|_1^2+\|y_{n_m}\|_1^2)-\left\|\frac{x_{n_m}+y_{n_m}}{2}\right\|_1^2\leq$$

$$\leq 1-\left\|\frac{x_{n_m}+y_{n_m}}{2}\right\|_1^2\to 0 \qquad (n_m\to\infty).$$

This implies that $|f_{k_0}(x_{n_m}-y_{n_m})|\to 0$ and this leads to a contradiction. Therefore we have

$$\lim_{n\to\infty}|f(x_n-y_n)|=0, \qquad \forall f\in F.$$

This completes the proof. □

3. The F.P.P. for generalized nonexpansive mappings

We are now in a position to prove the main results of this paper.

Theorem 3. *Every Banach space having normal structure has the F.P.P. for generalized nonexpansive mapping.*

Proof. Let K be any nonempty weakly compact convex subset of a given Banach space X, and $T: K\to K$ be any generalized nonexpansive mapping. We define Γ as a collection of all nonempty closed convex subsets E of K with $T(E)\subset E$. It is obvious that $\Gamma\neq\emptyset$ as $K\in\Gamma$. Now we define a partial ordering on Γ by the set inclusion relation. It follows from Smulian Theorem [4, p. 433] and Zorn Lemma that there exists a minimal element $S\in\Gamma$. If S contains only one point, say x, then x is a fixed point of T, and the proof is complete. If S consists of more than one point, then it follows from the normal structure of K, there exists an $x_*\in S$ such that

(3.1) $$\alpha=\sup_{y\in S}\|x_*-y\|<\operatorname{diam}(S).$$

Therefore we have

$$\sup_{y\in S}\|Tx_*-Ty\|\leq\sup_{y\in S}\|x_*-y\|=\alpha.$$

This implies that

$$T(S) \subset B(Tx_*, \alpha) \overset{\triangle}{=} \{x \in X : \|Tx_* - x\| \leq \alpha\} . \tag{3.2}$$

Letting $G = S \cap B(Tx_*, \alpha)$ we know that G is a nonempty closed convex set with $T(G) \subset G$. By the minimality of S it gets $G = S$. This implies that $S \subset B(Tx_*, \alpha)$. Therefore we have

$$\sup_{y \in S} \|Tx_* - y\| \leq \alpha. \tag{3.3}$$

Now we consider the set S' defined by

$$S' = \{z \in S : \sup_{y \in S} \|z - y\| \leq \alpha\}. \tag{3.4}$$

It is obvious that S' is a nonempty closed convex subset. We claim that $T(S') \subset S'$. In fact, if $z \in S'$ then $z \in S$ and $Tz \in S$.

Next, it is easy to see that $\overline{\text{co}}T(S) \subset S$, and consequently

$$T(\overline{\text{co}}T(S)) \subset T(S) \subset \overline{\text{co}}T(S) \subset S.$$

This means that $\overline{\text{co}}T(S) \in \Gamma$. In view of the minimality of S we get $S = \overline{\text{co}}T(S)$. Hence $\text{co}T(S)$ is a dense subset of S, and consequently for each $y \in S$ and each $\varepsilon > 0$ there exist $x_i \in S$, $i = 1, 2, \ldots, n$ such that

$$\|y - \sum_{i=1}^{n} \beta_i T x_i\| \leq \varepsilon, \qquad \beta_i \geq 0, \qquad \sum_{i=1}^{n} \beta_i = 1.$$

Hence for each $z \in S'$ we have

$$\|Tz - y\| \leq \|Tz - \sum_{i=1}^{n} \beta_i T x_i\| + \|\sum_{i=1}^{n} \beta_i T x_i - y\| \leq$$

$$\leq \varepsilon + \sum_{i=1}^{n} \beta_i \|Tz - Tx_i\| \leq \varepsilon + \sup_{y \in S} \|z - y\| \leq \varepsilon + \alpha.$$

By the arbitrariness of $y \in S$ and $\varepsilon > 0$ we have $\sup_{y \in S} \|Tz - y\| \leq \alpha$. This means that $Tz \in S'$ i.e. $T(S') \subset S$. Hence $S' = S$. But by the definition of S' it leads

$$\text{diam}(S') = \sup_{x, y \in S'} \|x - y\| \leq \sup_{x \in S', y \in S} \|x - y\| \leq \alpha < \text{diam}(S).$$

Thus we get a contradiction. Therefore S is a singleton.

This completes the proof. □

Remark. The main result of Jaggi [5] is a special case of Theorem 3. In particular, it extends the main results of Browder [2] and Kirk [6].

From Theorem 1, 2 and Theorem 3 we have the following result.

Theorem 4. *Let* $(X, \|\cdot\|, F)$ *be a Banach space with the S.P. . Then* X *admits an equivalent norm* $\|\cdot\|_1$ *defined by (2.1) so that* $(X, \|\cdot\|_1)$ *has the F.P.P. for generalized nonexpansive mappings.*

As a consequence of Theorem 4 we have the following

Corollary 1. *Let* $(X, \|\cdot\|)$ *be a separable Banach space. Then* X *admits an equivalent norm* $\|\cdot\|_1$ *defined by*

$$\|x\|_1 = (\|x\|^2 + \sum_{i=1}^{\infty} \frac{1}{2^i} |f_i(x)|^2)^{\frac{1}{2}}, \qquad f_i \in F,$$

where F *is a set defined by (2.3), so that* $(X, \|\cdot\|_1)$ *has the F.P.P. for generalized nonexpansive mapping (hence for every kind of mappings which satisfies the condition (1.1), (1.3) or (1.4)).*

Remark. This Corollary extends and improves the main results of Dulst [3]. Of course, it also extends the main results of Browder [2] and Kirk [6].

References

[1] D. Z. Alspach, A fixed point free nonexpansive map, Oklahoma, State University, 1980.

[2] F. E. Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Nat. Acad. Sci. USA, 54 (1965), 1041-1044.

[3] D. Van Dulst, Equivalent norms and the fixed point property for nonexpansive mapping, J. London Math. Soc., (2), 25 (1982), 139-144.

[4] N. Dunford, J. T. Schwartz, Linear Operators, Part I, Interscience Publishers, Inc., New York, 1985.

[5] D. S. Jaggi, On fixed points of nonexpansive mappings, Contemporary Math. V. 21 (1983), 147-149.

[6] W. A. Kirk, A fixed point theorem for mappings which do not increase distance, Amer. Math. Monthly, 72 (1965), 1004-1006.

[7] Nan Chao-xin, Geometrical Theory of Banach Spaces, 1983 (Chinese).

REZIME

UNIFORMNA KONVEKSNOST I OSOBINA NEPOKRETNE TAČKE

U ovom radu uveden je pojam (F^k) uniformno konveksnih prostora i dokazane su teoreme o nepokretnoj tački. Rezultati koji su dobijeni proširuju neke rezultate Dulsta[3], Jaggia[5], Browdera[2] i Kirka[6].

Received by the editors May 5, 1988.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 117-125

Review of Research
Faculty of Science
Mathematics Series

ON SEQUENTIAL MEASURES OF NONCOMPACTNESS IN TOPOLOGICAL VECTOR SPACES

Lothar Kaniok
Pädagogische Hochschule "Karl Friedrich Wilhelm Wander" Dresden
Sektion Mathematik
Wigardstraße 17, DDR, Dresden, 8060

Abstract

The notions of the sequential φ–measure of noncompactness and of the sequentially (φ, γ)–condensing mapping in topological vector spaces are introduced. Some examples of sequential φ–measures of noncompactness are given.

AMS Mathematics Subject Classification (1980): 47H10
Key words and phrases: measure of noncompactness, condensing mapping, topological vector space.

1. Introduction

If a measure of noncompactness is not defined on all, but only on the countable subsets of a locally convex space, then Sadovskij call it a sequential measure of noncompactness ([6]). Proceeding on [6; 1.4] and [3; Definition 4], we shall introduce the notion of the sequential φ–measure of noncompactness in topological vector spaces. Referring to this, we shall consider some examples of φ–measures of noncompactness. At this, the φ–measure auf noncompactness $J_{\mathcal{U}}$ introduced in [5] is of exceptional importance, just as the well-known Istrătescu's measure of noncompactness (cp. [1]).

Finally, we shall define the notion of the sequentially (φ,γ)–condensing mapping in topological vector spaces. At this, the condition

$$\varphi(\gamma(T(M))) \geq \gamma(M) \Rightarrow \overline{T(M)} \text{ is compact } (M \subseteq D(T)),$$

which is fundamental for a (φ,γ)–condensing mapping T (cp. [3], [5]), is relaxed by assuming that it holds only for at most countable sets $M \subseteq D(T)$. We shall show, that every sequentially $(\varphi, J_{\mathcal{U}})$–condensing mapping is $(\varphi, J_{\mathcal{U}})$–condensing, too.

2. Notions and definitions

In this paper every topological vector space will be assumed to be separated and real. Let E be a topological vector space and $K \subseteq E$. By $\mathcal{U}(E)$ we shall denote a fundamental system of circled, closed neighbourhoods of zero in E and by $\mathcal{F}_{\mathcal{U}}$ the set of all nonnegative functions on $\mathcal{U}(E)$ with the natural order. Moreover, by $\overline{K}$, co$K$, $\overline{\text{co}}K$ and ∂K we shall denote the closed hull, the convex hull, the closed convex hull and the boundary of K, respectively. We define $2^K := \{M \subseteq K : M \neq \emptyset\}$, $b(K) := \{M \in 2^K : M$ is bounded$\}$, $\text{cc}(K) := \{M \in 2^K : M$ is closed in K, M is convex$\}$ and $\text{fucc}(E) := \{K \subseteq E : K = \bigcup_{i\in I} K_i$, I is finite, $K_i \in \text{cc}(E)$ for all $i \in I\}$.

We say that $K \in 2^E$ is of *Zima's type* ([3]), iff for every $U \in \mathcal{U}(E)$ there exists $V \in \mathcal{U}(E)$ so that $\text{co}(V \cap (K - K)) \subseteq U$. Special examples of sets of Zima's type are given in [2] and [3]. K is said to be *starshaped, relative to some* $u \in K$, iff $tx + (1-t)u \in K$ for all $x \in K$ and all $t \in [0,1]$.

Let $K \in 2^E$, $M \in b(\overline{\text{co}}K)$ and $U \in \mathcal{U}(E)$. As in [5] we define

$$\begin{aligned} J(M,U) \;&:=\; \sup\{a \geq 0 : M \text{ contains a countable set } \{x_n : n \in \mathbb{N}\} \\ &\qquad \text{such that } x_i - x_k \notin aU \text{ for } i \neq k\} \end{aligned}$$

($\sup\emptyset = 0$, by definition). By $[J_{\mathcal{U}}(M)](U) := J(M,U)$ a mapping $J_{\mathcal{U}} : b(\overline{\text{co}}K) \to \mathcal{F}_{\mathcal{U}}$ is defined.

Now, let (E,d) be a metric space, $K \in 2^E$ and $M \in b(\overline{\text{co}}K)$. We define ([1]):

$$\begin{aligned} \alpha(M) \;&:=\; \inf\{a > 0 : M \text{ has a finite } a-\text{net in } E\}, \\ \beta(M) \;&:=\; \inf\{a > 0 : M \text{ has a finite } a-\text{net in } M\}, \end{aligned}$$

$$\begin{aligned} J(M) &:= \sup\{a > 0 : M \text{ contains an infinite } a-\text{discrete set}\} \\ &\quad\ (\sup\emptyset = 0) \end{aligned}$$

and

$$\chi(M) := \inf\{a > 0 : M \text{ has a finite } a-\text{cover}\}.$$

The mappings α, β, J, χ are the well-known Hausdorff's, inner Hausdorff's, Istrătescu's and Kuratowski's function, respectively. For fundamental properties of these functions see [1], [2], [3], [6], [7], respectively.

In [1] Daneš asked the question ([1; Problem 1]): Is J algebraically subadditive? The following result is the answer to this problem.

Proposition 1. *Let (E, d) be a metric vector space and $K \in 2^E$. Then, the inequality*

$$J(M+N) \le J(M) + J(N) \qquad (M \in b(\overline{co}K),\ N \in b(\overline{co}K))$$

holds.

Proof. Without loss of generality we may assume that $J(M+N) > \max\{J(M),\ J(N)\}$. Let $\varepsilon > 0$ such that $J(M) + \varepsilon =: a < b := J(M+N) - \varepsilon$. Then there is an infinite set $\{x_n + y_n : n \in \mathbf{N}\} \subseteq M + N$ such that $d(x_i + y_i,\ x_k + y_k) > b$ for $i \ne k$. Since $J(M) > a$, there is an infinite set

$$\{x_{n_j} + y_{n_j} : j \in \mathbf{N}\} \subseteq \{x_n + y_n : n \in \mathbf{N}\}$$

with

$$b < d(x_{n_i} + y_{n_i}, x_{n_k} + y_{n_k}) \le d(x_{n_i}, x_{n_k}) + d(y_{n_i}, y_{n_k}) \le a + d(y_{n_i}, y_{n_k})$$

for $i \ne k$.

Then there must be $d(y_{n_i}, y_{n_k}) > b - a$ for $i \ne k$. From this the assertion follows.

3. Sequential φ–measures of noncompactness in topological vector spaces

In the following, let E be a topological vector space, $K \in 2^E$, A a partially ordered set with the partial ordering $\le$, $\varphi : A \to A$ and $\mathcal{M}$ a system of subsets of $\overline{\text{co}}K$ such that:

$$M \in \mathcal{M} \Rightarrow (\overline{M} \in \mathcal{M}, \text{co}M \in \mathcal{M}, M \cup \{u\} \in \mathcal{M}(u \in K), N \in \mathcal{M}(N \subseteq M)),$$

if we don't state additional claims.

Moreover we denote by $\mathcal{M}_s$ the system of all at most countable sets of $\mathcal{M}$.

Definition 1. ([3]) *The mapping* $\gamma : \mathcal{M} \to A$ *is said to be a* $\varphi-$measure of noncompactness on K, *iff the following conditions are satisfied:*

$$(1) \quad \gamma(\overline{M}) = \gamma(M \cup \{u\}) = \gamma(M) \geq \gamma(N) \qquad (M \in \mathcal{M},\ N \subseteq M,\ u \in K),$$

$$(2) \qquad \gamma(\text{co}M) \leq \varphi(\gamma(M)) \qquad (M \in \mathcal{M}).$$

If $\varphi(t) = t \quad (t \in A)$, *then* γ *is called measure of noncompactness on* K.

Examples.

(a) Let K be a set of Zima's type with $K \in b(E)$, which is starshaped, relative to some $u \in K$. For every $U \in \mathcal{U}(E)$ we can choose a neighbourhood $W_u \in \mathcal{U}(E)$ (fixed) such that $J(\text{co } M, U) \leq J(M, W_u)$ for every $M \subseteq K$ ([5; Corollary to Lemma 2]). By $v(U) := W_u(U \in \mathcal{U}(E))$ and $\varphi^*(f) := f \circ v(f \in \mathcal{F}_\mathcal{U})$ we define a mapping φ^* of $\mathcal{F}_\mathcal{U}$ into $\mathcal{F}_\mathcal{U}$ and $J_\mathcal{U}$ is a φ^*- measure of noncompactness on K ([5; Proposition 2.]).

(b) Let (E, p) be a paranormed space (s. [2], [3]). E is a metrizable topological vector space. The fundamental system of neighbourhoods of zero in E is given by the family $\mathcal{V} = \{V_r : r > 0\}$, where $V_r := \{x \in E : p(x) < r\}$.

Let $K \in 2^E$ and $\varphi : (0, \infty) \to (0, \infty)$. The set K is said to be of $Z_\varphi-$*type* ([3]), iff, for every $r > 0$, $\text{co}(V_r \cap (K - K) \subseteq V_{\varphi(r)}$. In [3] Hadžić gave an example of a set of $Z_\varphi-$type. Every set, which is of $Z_\varphi-$type, is of Zima's type also, if $\inf\{\varphi(r) : r > 0\} = 0 \quad ([3])$.

Let K be a convex set with $K \in b(E)$. Hadžić proved in [3] that the inner Hausdorff's and the Kuratowski's function satisfy the condition (2) from Definition 1, if K is of $Z_\varphi-$type and $\varphi : [0, \infty) \to [0, \infty)$ is a right continuous and a continuous mapping, respectively. The mapping χ satisfies (1) from Definition 1 also, but β is not monotone, in general (s. [1]).

On K the following inequality holds ([1; Proposition 1]):

$$\alpha \le \beta \le J \le \chi \le 2\alpha.$$

Hence, the Istrătescu's function J and the Kuratowski's function χ are $2\varphi-$measures of noncompactness on K, if K is of $Z_\varphi-$type and $\varphi : [0,\infty) \to [0,\infty)$ is a right continuous, non decreasing mapping. If, in addition, φ is positive homogeneous then the Hausdorff's function α is a $2\varphi-$measure of noncompactness on such a set K, too.

Definition 2. *We say, that the mapping $\gamma : \mathcal{M}_s \to A$ is a sequential $\varphi-$measure of noncompactness on K, iff, for all $M, N \in \mathcal{M}_s$ the following conditions hold:*

$$\text{(1)} \qquad \gamma(N) \le \gamma(M) = \gamma(M \cup \{u\}) \qquad (N \subseteq \overline{M},\ u \in K),$$

$$\text{(2)} \qquad \gamma(N) \le \varphi(\gamma(M)) \qquad (N \subseteq \mathrm{co}M)$$

If $\varphi(t) = t (t \in A)$ then γ will be called a sequential measure of noncompactness on K.

Remark. Such properties as the subadditivity, the algebraic subadditivity and the positive homogenity have the same importance, which is well-known from the theory of the measures of noncompactness, for sequential $\varphi-$measures of noncompactness, too.

It is obvious that every $\varphi-$measure of noncompactness is a sequential $\varphi-$measure of noncompactness also. Sadovskij proved ([6]) that in a metric vector space every sequential measure of noncompactness generates a measure of noncompactness in specified way by the preservation of the basic properties. Analogously to Theorem 1.1. from [6] we can prove the

Theorem 1. *Let E be a metric vector space, $K \in 2^E, \varphi : A \to A$ a non decreasing mapping and γ a sequential $\varphi-$measure of noncompactness on K. Then, by*

$$\text{(i)} \qquad \tilde{\gamma}(M) := \sup\{\gamma(Z) : Z \in \mathcal{M}_s,\ Z \subseteq M\}\ (M \in \mathcal{M})$$

a $\varphi-$measure of noncompactness on K is defined.

If the implication ($M \in \mathcal{M},\ N \in \mathcal{M} \Rightarrow M \cup N \in \mathcal{M},\ M+N \in \mathcal{M},\ cM \in \mathcal{M}\ (c \in \mathbf{R})$) holds and the mapping γ is positively homogeneous, subadditive or algebraically subadditive, then $\tilde{\gamma}$ has the corresponding property.

Proof. (cp. [6; Theorem 1.1.]).

(1) The relations

$$\tilde{\gamma}(N) \leq \tilde{\gamma}(M \cup \{u\}) = \tilde{\gamma}(M) \leq \tilde{\gamma}(\overline{M}) \qquad (M \in \mathcal{M},\ N \subseteq M,\ u \in K)$$

and the positive homogenity, the subadditivity and the algebraic subadditivity follow from the definition of $\tilde{\gamma}$ and from the equivalent properties of γ.

We shall prove the inequality $\tilde{\gamma}(\overline{M}) \leq \tilde{\gamma}(M)$ $(M \in \mathcal{M})$. The assertion is clear for finite sets. Let M be an infinite set of $\mathcal{M}$ and $Z := \{z_n\}$an at most countable subset of $\overline{M}$. Then there are $x_{nm} \in M$ such that $\lim_{m\to\infty} x_{nm} =$ $(z_n \in Z)$. We denote the set of all x_{nm} by Z_0. Z_0 is a countable subset of M with $Z \subseteq \overline{Z}_0$. Therefore $\gamma(Z) \leq \gamma(Z_0)$ and $\tilde{\gamma}(\overline{M}) \leq \tilde{\gamma}(M)$.

(2) Let $M \in \mathcal{M}$ and $Z = \{z_n\}$ an at most countable subset of co M. For every $z_n \in Z$ there exist $x_{n1}, \ldots, x_{nm(n)} \in M$ and $c_{n1}, \ldots, c_{nm(n)} \geq 0$ with $\sum_{k=1}^{m(n)} c_{nk} = 1$ such that $z_n = \sum_{k=1}^{m(n)} c_{nk} x_{nk}$. Now we denote the set of all $x_{nk}(k \in \{1, \ldots, m(n)\})$ by Z_0. Z_0 is an at most countable subset of M with $Z \subseteq \text{co } Z_0$. Hence $\gamma(Z) \leq \varphi(\gamma(Z_0))$. Thus $\tilde{\gamma}(\text{co } M) \leq \varphi(\tilde{\gamma}(M))$.

The question of the unicity of the extension of a sequential φ−measure of noncompactness from $\mathcal{M}_s$ onto $\mathcal{M}$ arises. In particular, Sadovskij asked the question ([6]):
Is $\alpha = \tilde{\alpha}$, if the Hausdorff's measure of noncompactness α is extended to $\tilde{\alpha}$ according to (i)? The equality holds in separable spaces ([6; p. 96]). In arbitrary metric vector spaces the question is unanswered.

Now we deal with this problem and give a partial answer.

Lemma 1. *Let $M \in b(\overline{\text{co}}K)$ and $U \in \mathcal{U}(E)$. It is*

$$J(M,U) = \max\{J(Z,U) : Z \subseteq M,\ Z \textit{ is at most countable}\}.$$

Proof. Since $J_{\mathcal{U}}$ is monotone ([5]), the inequality $\sup\{J(Z,U) : Z \subseteq M$, Z is at most countable $\} \leq J(M,U)$ holds. It is easy to see that the equality holds, if $J(M,U) = 0$. Let $\varepsilon > 0$ and $J(M,U) > 0$. There exists a countable subset $Z = \{z_n : n \in \mathbb{N}\}$ of M such that $z_m - z_n \notin (1+\varepsilon)^{-1} \cdot J(M,U)U$ for $m \neq n$. Hence $J(Z,U) \geq (1+\varepsilon)^{-1} \cdot J(M,U)$. Thus $\sup\{J(Z,U) : Z \subseteq M$, Z is at most countable$\} \geq J(M,U)$, too. Finally, we can write max instead of sup, since always from $\lim_{n\to\infty} J(Z_n,U) = J(M,U)$ follows $J(\bigcup_{n\in\mathbf{N}} Z_n, U) = J(M,U)$.

Corollary 1. *The equality* $J_{\mathcal{U}} = \tilde{J}_{\mathcal{U}}$ *(s.* (i)*) holds on every set* $K \in 2^E$ *of an arbitrary topological vector space* E.

Remark. In the same way as in the proof of the above Lemma it can be proved that for the Istrătescu's measure of noncompactness J the equality $J = \tilde{J}$ (s. (i) with max instead of sup) is true on every set $K \in 2^E$ of a metric space E. Since the functions α and χ are monotone and $\alpha \le \beta \le J \le \chi \le 2\alpha$ ([1]) the following inequalities hold in any metric space:

$$\frac{1}{2}\gamma \le \tilde{\gamma} \le \gamma \quad (\gamma \in \{\alpha,\chi\}) \qquad \text{and} \qquad \frac{1}{2}\beta \le \tilde{\beta} \le 2\beta.$$

The same relations are true for the well-known Hausdorff's and Kuratovski's measure of noncompactness in locally convex spaces. This is a partial answer to the above-mentioned problem from [6; p. 96].

4. Sequentially (φ,γ)–condensing maps

The same agreements from the section 2 let be true for $E, K, A, \varphi, \mathcal{M}$ and $\mathcal{M}_s$. Moreover, let $M \in 2^K$ and $F : M \to \mathrm{cc}(K)$ be an upper semicontinuous mapping.

Definition 3 *(cp. [5]). Let* γ *be a* φ*–measure of noncompactness on* K. *The mapping* F *is said to be a* (φ,γ)*– condensing mapping, iff for every* $N \subseteq M$ *the following implication holds:*

(ii)
$$\gamma(N) \le \varphi(\gamma(F(N))) \Rightarrow \overline{F(N)} \text{ is compact.}$$

We call F *sequentially* (φ,γ)*–condensing, iff* (ii) *holds for every at most countable subset* N *of* M.

In [5] we gave an example of a $(\varphi^*, J_{\mathcal{U}})$–condensing mapping in an arbitrary topological vector space, where φ^* is the mapping defined in the example (a) of section 3.

It is obvious that a (φ,γ)–condensing mapping is sequentially (φ,γ)–condensing as well.

Theorem 2. *In any topological vector space a sequentially* $(\varphi, J_{\mathcal{U}})$*–condensing mapping* $F: M \to \mathrm{cc}(K)$ *is* $(\varphi, J_{\mathcal{U}})$*–condensing.*

Proof. Let $N \subseteq M$ and let $F(N)$ be not relatively compact. From the above Lemma it follows that there is a countable, not relatively compact subset Z of $F(N)$ such that $J_{\mathcal{U}}(Z) = J_{\mathcal{U}}(F(N))$. We choose an at most countable subset Z_0 of N with $F(Z_0) = Z$. Since $\overline{F(Z_0)}$ is not compact and F is sequentially $(\varphi, J_{\mathcal{U}})$–condensing,we obtain

$$J_{\mathcal{U}}(N) \geq J_{\mathcal{U}}(Z_0) > \varphi(J_{\mathcal{U}}(F(Z_0))) = \varphi(J_{\mathcal{U}}(Z)) = \varphi(J_{\mathcal{U}}(F(N))).$$

Hence, the inequality $J_{\mathcal{U}}(N) \leq \varphi(J_{\mathcal{U}}(F(N)))$ $(N \subseteq M)$ implies the compactness of $\overline{F(N)}$. This means that F is $(\varphi, J_{\mathcal{U}})$–condensing.

Remark. Just as in Theorem 2 it can be proved that in any metric vector space a sequentially (φ, J)–condensing mapping is (φ, J)–condensing, since $J = \tilde{J}$.

Now, from [5; Corollary to Theorem 1 and Corollary to Theorem 2] we obtain the following fixed point theorem.

Theorem 3. *Let* K *be a set of Zima's type with* $K \in b(E)$, *which is star-shaped, relative to some* $u \in K$, *and* φ^* *the mapping defined in the example* (a) *of the section 2. Moreover, one of the following conditions holds:*

(1) Let $K \in \mathrm{fucc}(E)$, $U \subseteq K$ *an in* K *closed neighbourhood of* u *and* $F: U \to \mathrm{cc}(K)$ *a sequentially* $(\varphi^*, J_{\mathcal{U}})$*–condensing mapping with* $x \notin tF(x) + (1-t)u \qquad (x \in \partial_K U,\ t \in (0,1))$.

(2) Let $F: K \to \mathrm{cc}(K)$ *be a sequentially* $(\varphi^*, J_{\mathcal{U}})$*–condensing mapping.*

Then F *has a fixed point.*

For continuous single-valued maps the notion of the sequentially (φ,γ)–condensing mapping can be defined as in Definition 3 by proceeding on the assumption that γ is a sequential φ–measure of noncompactness.

Using Theorem 1, where in (i) "sup" can be replaced by "max" also (cp. with the proof of the above Lemma), with the methods of the proof of Theorem 2 it can be proved the

Theorem 4. *Let E be a metric vector space, $K \in 2^E$, $M \in 2^K$, $\varphi : A \to A$ non decreasing and γ a sequentially $\varphi-$measure of noncompactness on K. Then a sequentially $(\varphi, \gamma)-$condensing mapping $G : M \to K$ is $(\varphi, \tilde{\gamma})-$condensing as well, where $\tilde{\gamma}$ is the $\varphi-$measure of noncompactness defined by* (i).

References

[1] Daneš, J.: On the Istrătescu's measure of noncompactness, Bull. Math. Soc. R. S. Roumanie, 16(64), No. 4, 1972.

[2] Hadžić, O.: Fixed point theory in topological vector spaces, Institute of Mathematics, Novi Sad, 1984, 327 pp.

[3] Hadžić, O.: Some properties of measures of noncompactness in paranormed spaces, Proc. Amer. Math. Soc., Vol. 102, No. 4, 1988, 843-849.

[4] Janos, L., Martelli, M.: Sequentially condensing maps, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 16,1(1986), 85-94.

[5] Kaniok, L.: On measures of noncompactness in general topological vector spaces, Comment. Math. Univ. Carolinae, 31,3(1990), 479-487.

[6] Sadovskij, B. N.: On measures of noncompactness and densifying operators, Probl. Mat. Anal. Slozn. Sistem, Vol. 2, Voronez, 1968 (in Russian).

[7] Sadovskij, B. N.: Asymptotically compact and densifying operators, Usp. Mat. Nauk 27, No. 1, 1972 (in Russian).

REZIME

O SEKVENCIJALNOJ MERI NEKOMPAKTNOSTI U VEKTORSKO TOPOLOŠKIM PROSTORIMA

Uvedeni su pojmovi sekvencijalne φ - mere nekompaktnosti i sekvencijalnog (φ, γ) - kondenzujućeg preslikavanja u vektorsko topološkim prostorima. Dati su neki primeri sekvencijalnih φ - mera nekompaktnosti.

Received by the editors February 26, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 127-133

Review of Research
Faculty of Science
Mathematics Series

ZUR APPROXIMATION VON FIXPUNKTEN BEI MENGENWERTIGEN OBERHALBSTETIGEN KOMPAKTEN ABBILDUNGEN

S. Hahn und **S. Juckelandt**
Pädagogische Hochschule "Karl Friedrich Wilhelm Wander" Dresden
Sektion Mathematik
Wigardstraße 17, DDR, Dresden, 8060

Abstract

In der vorliegenden Arbeit wird gezeigt, daß ein bekanntes Verfahren zur Berechnung eines Fixpunktes für den Fixpunktsatz von Kakutani auch auf allgemeinere Fixpunktsätze angewendet werden kann.

AMS Mathematics Subject Classification (1980): 47H10, 54H25, 58C30
Kennwörter: Mengenwertige oberhalbstetige Abbildungen, Simpliciale Approximation von Fixpunkten, Merrill-Algorithmus

1. Einleitung

Sei K eine nichtleere Teilmenge des m–dimensionalen euklidischen Raumes R^m. Mit co$K, \partial K$, int$K$, und $\overline{K}$ bezeichnen wir die konvexe Hülle, den Rand, das Innere bzw. die Abschließung von K. Das Symbol ac(K) bezeichne das System aller nichtleeren, abgeschlossenen und konvexen Teilmengen von K. Sei $M \subseteq R^m$. Die ("mengenwertige") Abbildung $F : M \to \text{ac}(K)$ heißt oberhalbstetig, wenn für jedes $x_0 \in M$ und jede offene Menge $G \subseteq R^m$ mit $F(x_0) \subseteq G$ eine Umgebung U von x_0 existiert, so daß $F(x) \subseteq G$ für jedes $x \in U \cap M$ gilt. F heißt kompakt, wenn F oberhalbstetig und $F(M)$ eine beschränkte Teilmenge des R^m ist. Wenn $F(M)$ beschränkt ist, ist die Oberhalbstetigkeit äquivalent mit der Abgeschlossenheit des Graphen $\mathrm{Gr}F := \{(x,y) : x \in M,\ y \in F(x)\}$ in $M \times K$. Kakutani bewies folgenden fundamentalen Fixpunktsatz

Satz von Kakutani. Es seien C eine nichtleere, abgeschlossene, konvexe und beschränkte Teilmenge von R^m und $F : C \to \mathrm{ac}(C)$ eine oberhalbstetige Abbildung. Dann hat F einen Fixpunkt, das heißt, es existiert ein $x \in C$ mit $x \in F(x)$.

Der klassische Beweis von Kakutani wurde von Lüthi [3] (vgl. auch [2], [6]) ausgebaut, um ein konstruktives Verfahren zur Bestimmung eines solchen Fixpunktes zu erhalten. Dazu werden Triangulationen des R^m verwendet. Zum Begriff der Triangulation und des Simplex verweisen wir auf [2], [3], [6]. Den Grundgedanken von Lüthi verwenden wir, um für den folgenden allgemeinen (auch für lokalkonvexe Räume gültigen) Fixpunktsatz aus [1] ein Konstruktionsverfahren zu entwickeln.

Theorem 1. *Es seien $U \subseteq R^m$ eine abgeschlossene Umgebung eines Punktes $a \in R^m$ und K eine abgeschlossene, konvexe Teilmenge von R^m mit $a \in K$. Weiter sei $F : U \cap K \to \mathrm{ac}(K)$ eine kompakte Abbildung mit folgender Eigenschaft*

$$(*) \qquad \beta x + (1-\beta)\, a \notin F(x) \qquad (x \in \partial U \cap K,\ \beta > 1).$$

Dann hat F einen Fixpunkt.

Fur $K = R^m$ wurde Theorem 1 von Ma [4] mit Abbildungsgrad bewiesen. Für den Spezialfall $U = R^m$ erhalten wir aus Theorem 1 eine modifizierte Fassung des Satzes von Kakutani (im Satz von Kakutani kann die Voraussetzung der Beschränktheit von C ersetzt werden durch die der Kompaktheit von F).

Der erste Teil des Beweises unseres Theorems 2 wird gleichzeitig eine äußerst elementare Herleitung von Theorem 1 aus dem Fixpunktsatz von Kakutani sichtbar machen.

2. Hilfsaussagen

Sei T eine Triangulation von R^m und $F : R^m \to \mathrm{ac}(R^m)$. Wir wählen für jeden Eckpunkt w von T ein $f(w) \in F(w)$. Für beliebiges $x \in R^m$ seien $w_1, \ldots, w_{t+1}$ $(t \in \{0, 1, \ldots, m\})$ die eindeutig bestimmten Eckpunkte des t–dimensionalen Simplexes, das x in seinem Inneren (relativ der Dimension t) enthält. Dann existiert genau ein $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{t+1})$ mit $\lambda_i > 0$ $(i = 1, \ldots, t+1)$ und $\sum_{i=1}^{t+1} \lambda_i = 1$ sowie $x = \sum_{i=1}^{t+1} \lambda_i w_i$. Man definiert eine Funktion

$$f : R^m \to R^m \quad \text{durch} \quad f(x) := \sum_{i=1}^{t+1} \lambda_i f(w_i), \quad \text{wobei} \quad x = \sum_{i=1}^{t+1} \lambda_i w_i \in R^m$$

ist. f nennt man eine stückweise lineare Approximation von F bezüglich der Triangulation T.

Man sieht leicht, daß f stetig und $f(R^m) \subseteq \mathrm{co}F(R^m)$ ist. Falls F kompakt ist, muß auch f kompakt sein und aus dem Fixpunktsatz von Brouwer folgt dann die Existenz eines Fixpunktes für f. Unter gewissen Voraussetzungen kann mit dem Algorithmus von Merrill [5], der in [3] ausführlich beschrieben wird und verschiedene Vorteile gegenüber anderen bekannten Algorithmen hat (s. [2], [6]), ein Fixpunkt einer stückweise linearen Approximation berechnet werden. In [3, s. 113-115] wird dazu folgende Aussage bewiesen.

Lemma 1. (Merrill) *Der Algorithmus von Merrill berechnet einen Fixpunkt einer stückweise linearen Approximation f einer oberhalbstetigen Abbildung $F : R^m \to \mathrm{ac}(R^m)$ in endlich vielen Schritten, wenn gilt:*

(1) *Es existieren positive Zahlen M und δ und ein $x \in R^m$ derart, daß für alle $z \in R^m$ mit $\|z - x\| > M$, $u(z) \in F(z)$ und $y \in R^m$ mit $\|y - z\| \leq \delta$ sowie $\|y - x\| > M$ das Skalarprodukt von $u(z) - z$ mit $y - x$ negativ ist, also $(u(z) - z)(y - x) < 0$ gilt.*

(2) *Die zu f gehörige Triangulation hat einen Durchmesser, der kleiner als δ ist.*

Lemma 2. *Es sei $F : R^m \to \mathrm{ac}(R^m)$ eine kompakte Abbildung. Dann gilt*

(1) *Es existiert eine Folge* (x_k) *aus* R^m, *die gegen einen Fixpunkt von* F *konvergiert. Debei ist jedes* x_k $(k = 1, 2, \ldots)$ *Fixpunkt einer stückweise linearen Approximation von* F *bezüglich einer Triangulation* T_k *von* R^m.

(2) *Jedes* x_k $(k = 1, 2, \ldots)$ *aus* (1) *läßt sich in endlich vielen Schritten durch den Algorithmus von Merrill berechnen.*

Beweis. **(1)** Sei (T_n) eine Folge von Triangulationen des R^m mit $d(T_n) \to 0$ ($d(T_n)$ bezeichne den Durchmesser von T_n). Sei $f_n : R^m \to R^m$ eine stückweise lineare Approximation von F bezüglich T_n und x_n ein Fixpunkt von f_n $(n = 1, 2, \ldots)$. Für jedes n existieren Eckpunkte $w_1(n), w_2(n), \ldots, w_{m+1}(n)$ von T_n und eindeutig bestimmte Zahlen $\lambda_i(n) \in [0, 1]$ $(i = 1, \ldots, m+1)$ mit $\sum_{i=1}^{m+1} \lambda_i(n) = 1$ und $x_n = \sum_{i=1}^{m+1} \lambda_i(n) w_i(n)$. Nach Definition der stückweise linearen Approximation f_n von F gilt

$$(*) \qquad x_n = f_n(x_n) = \sum_{i=1}^{m+1} \lambda_i(n) f_n(w_i(n)) \qquad \text{für jedes } n.$$

Weil [0,1] und $F(R^m)$ beschränkt sind, existieren konvergente Teilfolgen (x_k) von (x_n) und $(f_k(w_i(k)))$ von $(f_n(w_i(n)))$ mit $\lim_{k\to\infty} x_k = a \in R^m$ und $\lim_{k\to\infty} f_k(w_i(k)) = u_i$ $(i = 1, 2, \ldots, m+1)$. Aus $d(T_k) \to 0$ folgt $\lim_{k\to\infty} w_i(k) = a$ für alle $i = 1, \ldots, m+1$. Da Gr F abgeschlossen ist, folgt aus $f_k(w_i(k)) \in F(w_i(k))$ und $f_k(w_i(k)) \to u_i$ für jedes $i = 1, \ldots, n+1$ die Beziehung $u_i \in F(a)$. Aus (*) erhalten wir durch Grenzübergang $a = \sum_{i=1}^{m+1} \lambda_i u_i$, woraus wegen der Konvexität von $F(a)$ schließlich $a \in F(a)$ folgt.

(2) Sei T_k eine Triangulation von R^m entsprechend des Beweises von (1) und x_k Fixpunkt einer stückweise linearen Approximation f_k von F. Wir wählen ein $\delta > 0$ mit $\delta > d(T_k)$ und ein $x \in F(R^m)$. Da $f(R^m)$ beschränkt ist, existiert ein $M > 0$ derart, daß $\|u - x\| < M - \delta$ für alle $u \in F(R^m)$ gilt. Sei z und y aus R^m mit $\|z - x\| > M$, $\|y - x\| > M$, $\|y - z\| \leq \delta$ und $u(z) \in F(z)$. Dann erhalten wir unter Verwendung der Schwarz'schen Ungleichung die Abschätzung

$$\begin{aligned}(u(z) - z)(y - x) &= (u(z) - x)(y - x) + (y - z)(y - x) - \|y - x\|^2 \\ &\leq \|y - x\|(M - \|y - x\|) < 0.\end{aligned}$$

Damit sind die Voraussatzungen von Lemma 1 erfüllt, woraus die Behauptung folgt.

3. Hauptergebnis

Theorem 2. *Sei $U \subseteq R^m$ eine abgeschlossene Umgebung eines Punktes $a \in R^m$ und K eine abgeschlossene, konvexe Teilmenge mit $a \in K$. Weiter sei $F : U \cap K \to \mathrm{ac}(K)$ eine kompakte Abbildung mit folgenden Eigenschaften:*

(LS) $$\beta x + (1-\beta)a \notin F(x) \qquad (x \in \partial U \cap K, \beta > 1).$$

Dann gilt

(1) *F hat einen Fixpunkt*

(2) *Es existiert eine Folge (x_k) aus R^m, die gegen einen Fixpunkt b von F konvergiert, und alle $x_k (k = 1, 2, \ldots)$ können mit dem Algorithmus von Merrill in endlich vielen Schritten berechnet werden.*

Beweis. **(1)** Sei $F_1(x) = \begin{cases} F(x) & \text{für } x \in (\mathrm{int}U) \cap K \\ \mathrm{co}(F(x) \cup \{a\}) & \text{für } x \in \partial U \cap K \\ \{a\} & \text{für } x \in K \backslash (U \cap K). \end{cases}$
Wie in [6, S. 58] zeigt man leicht, daß $F_1 : K \to \mathrm{ac}(K)$ oberhalbstetig ist. Da F kompakt und $F_1(K) \subseteq \mathrm{co}(F(U \cap K) \cup \{a\})$ ist, muß F_1 sogar eine kompakte Abbildung sein. Aus dem Satz von Kakutani folgt die Existenz eines Fixpunktes $b \in K$ von F_1. Wir zeigen nun, daß b auch Fixpunkt von F ist.

Dazu sei $b \in \partial U \cap K$ angenommen, denn der Fall $b \in (\mathrm{int}U) \cap K$ liefert bereits die Behauptung und der Fall $b \notin U \cap K$ ist wegen $a \in U \cap K$ unmöglich. Wegen $a \in \mathrm{int}U$ gilt $b \neq a$. Aus $b \in \mathrm{co}(F(b) \cup \{a\})$ folgt die Existenz von Zahlen $t_i \in [0,1]$ $(i = 1, \ldots, n)$ mit $\sum_{i=1}^{n} t_i = 1$ und von Elementen $x_i \in F(b)$ $(i = 1, \ldots, n-1)$ mit $b = \sum_{i=1}^{n-1} t_i x_i + t_n a = (1 - t_n) \sum_{i=1}^{n-1} \frac{t_i}{1-t_n} x_i + t_n a$. Da $\sum_{i=1}^{n-1} \frac{t_i}{1-t_n} = 1$ und $x_i \in F(b)$ gilt, folgt aus der

Konvexität von $F(b)$, daß $u := \sum_{i=1}^{n-1} \frac{t_i}{1-t_n} x_i \in F(b)$ ist. Angenommen, es gilt $b \notin F(b)$. Dann folgt aus $b = (1-t_n)u + t_n a$ die Beziehung $t_n \neq 0$ und wegen $b \neq a$ auch $t_n \neq 1$. Daraus ergibt sich $\beta b + (1-\beta)a = u \in F(b)$ mit $\beta = \frac{1}{1-t_n} > 1$ und $b \in \partial U \cap K$ im Widerspruch zur Voraussetzung über F.

(2) Nach einem bekannten Ergebnis von Dugundji ist K ein Retrakt von R^m. Daher existiert eine stetige Abbildung $r : R^m \to K$ mit $r(x) = x$ für alle $x \in K$.

Wir setzen $F_2(x) = F_1(r(x))$, $(x \in R^m)$. Dann ist $F_2 : R^m \to \mathrm{ac}(R^m)$ eine kompakte Abbildung, auf die wir Lemma 2 anwenden können. Daraus folgt die Behauptung von Theorem 2, denn jeder Fixpunkt b von F_2 ist auch Fixpunkt von F_1 (da aus $F_2(R^m) \subseteq K$ stets $b \in K$ und damit $F_1(b) = F_2(b)$ folgt) und damit nach (1) auch Fixpunkt für F.

Bemerkungen.

1.) Speziell kann in Theorem 2 für K ein Kegel (also eine abgeschlossene, nichtleere, konvexe Teilmenge mit $K \notin \{0\}$, für die mit $x \in K \backslash \{0\}$ stets $tx \in K$ für alle $t \geq 0$, nicht aber $-x \in K$ gilt) verwendet werden, wodurch wir Anschluß an die Theorie der positiven Operatoren erhalten.

2.) Wie man leicht zeigen kann, ist bekanntlich die Randbedingung (LS) aus Theorem 2 speziell erfüllt, wenn U konvex ist und $F(\partial U \cap K) \subseteq U$ gilt.

Wir geben abschließend ein Beispiel einer kompakten Abbildung an, die die Voraussetzungen von Theorem 2, nicht aber die des Satzes von Kakutani erfüllt.

Beispiel. Sei $U = \{x \in R^m : \|x\| \leq 1\}$ und $F : U \to \mathrm{ac}(R^m)$ definiert durch $F(x) = \{y \in R^m : \|y - (3u_0 - 6x)\| \leq 1\}$ mit $u_0 \in \partial U$ fest. Es gilt $F(U) = \bigcup_{x \in U} F(x) = \{y \in R^m : \|y - 3u_0\| \leq 7\}$. Man sieht leicht, daß F kompakt ist. Wir zeigen, daß F die Bedingung (LS) aus Theorem 2 erfüllt (mit $a = 0$). Angenommen, es gilt $\beta x \in F(x)$ für ein $x \in \partial U$ und ein $\beta > 1$. Wegen $\|x\| = \|u_0\| = 1$ folgt der Widerspruch $1 \geq \|\beta x - (3u_0 - 6x)\| \geq |\,\|\beta x + 6x\| - \|3u_0\|\,| = |\,|\beta + 6| - 3| > 4$. Somit sind alle Voraussetzungen von Theorem 2 erfüllt (mit $K = R^m$) Jedoch gilt nicht $F(U) \subseteq U$, sogar $F(\partial U) \subseteq U$ ist nicht erfüllt, denn es gilt z. B. $F(u_0) \cap U = \emptyset$.

References

[1] Hahn, S.: Fixpunktsätze für mengenwertige Abbildungen in lokalkonvexen Räumen. Math. Nachr. 73 (1976), 269-293.

[2] Laan, G. van der: Simplicial fixed point algorithms Ph. D. Dissertation, University of Amsterdam 1980.

[3] Lüthi, H. J.: Komplementaritäts - und Fixpunktalgorithmen in der mathematischen Programmierung, Spieltheorie und Ökonomie. Springer – Verlag Berlin, Heidelberg, New York 1976.

[4] Ma, T. W.: Topological degrees of set – valued compact fields in locally convex spaces. Diss. Math. 92 (1972).

[5] Merrill, O. H.: Applications and extensions of an algorithm that computs fixed points of certain non-empty upper semi-continuous point to set mappings. Ph. D. dissertation University of Michigan (1972).

[6] Todd, M. J.: The computation of fixed points and applications. Springer Verlag Berlin, Heidelberg, New York 1976.

REZIME

APROKSIMACIJA NEPOKRETNE TAČKE VIŠEZNAČNIH OD GORE POLUNEPREKIDNIH KOMPAKTNIH PRESLIKAVANJA

U ovom radu je pokazano da jedan poznati postupak za izračunavanje nepokretne tačke za teoremu o nepokretnoj tački Kakutanija može da se primeni takodje i na opšte teoreme o nepokretnoj tački.

Received by the editors August 31, 1989.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 135-146

Review of Research
Faculty of Science
Mathematics Series

A NON-COMMUTATIVE NEUTRIX PRODUCT OF DISTRIBUTIONS ON R^m

Brian Fisher
Department of Mathematics, The University,
Leicester,LE1 7RH, England.

Li Chen Kuan
Department of Mathematics and Statistics,
University of Regina, S4S OA2, Canada.

Abstract

We let $\rho(r)$ be a fixed infinitely differentiable function of $r = (x_1^2 + \ldots + x_m^2)^{1/2}$ satisfying the properties (i) $\rho \geq 0$, (ii) $\rho(r) = 0$, $r \geq 1$, (iii) $\int_{R^m} \rho(r)\, d\mathbf{x} = 1$. The function $\delta_n(\mathbf{x})$, with $\mathbf{x}$ in R^m, is then defined by $\delta_n(\mathbf{x}) = n^m \rho(nr)$ for $n = 1, 2, \ldots$. The product $f \circ g$ of two distributions f and g is then defined to be the neutrix limit of the sequence $\{fg_n\}$, where $g_n = g * \delta_n$. Some results are given.

AMS Mathematics Subject Classification (1980): 46F10.
Key Words and Phrases: Neutrix, neutrix limit, Dirac delta-function, distribution.

1. Introduction

The product $g\phi$ of an infinitely differentiable function g and a test function ϕ in the space $\mathcal{D}$ of infinitely differentiable functions with compact support is itself in $\mathcal{D}$. This leads to the following definition $fg = gf$ of a distribution f in $\mathcal{D}'$ and an infinitely differentiable function g.

Definition 1. *Let f be a distribution in $\mathcal{D}'$ and let g be an infinitely differentiable function. The product $fg = gf$ is defined by*

$$\langle fg, \phi \rangle = \langle gf, \phi \rangle = \langle f, g\phi \rangle$$

for all ϕ in $\mathcal{D}$.

It follows easily by induction that we then have

$$f^{(s)}g = \sum_{i=0}^{s} \binom{s}{i} (-1)^i \left[fg^{(i)} \right]^{(s-i)}$$

for $s = 1, 2, \ldots$, where

$$\binom{s}{i} = \frac{s!}{i!(s-i)!}.$$

This suggests the following definition for the product of two distributions, see [2].

Definition 2. *Let f and g be distributions in $\mathcal{D}'$ for which on the interval (a,b) f is the s-th derivative of a locally summable function F in $L^p(a,b)$ and $g^{(s)}$ is a locally summable function in $L^q(a,b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ is defined by*

$$fg = gf = \sum_{i=0}^{s} \binom{s}{i} (-1)^i \left[Fg^{(i)} \right]^{(s-i)}.$$

Now let ρ be a fixed infinitely differentiable function with the properties:

(a) $\rho(x) = 0, \ |x| \geq 1$,

(b) $\rho(x) \geq 0$,

(c) $\rho(x) = \rho(-x)$,

(d) $\int_{-1}^{1} \rho(x)\, dx = 1$.

The function δ_n is defined by $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$. It is obvious that $\{\delta_n\}$ is a sequence of infinitely differentiable functions converging to the Dirac delta-function δ.

For an arbitrary distribution g in $\mathcal{D}'$ the function g_n is defined by

$$g_n(x) = (g * \delta_n)(x) = \langle g(x-t), \delta_n(t) \rangle.$$

It follows that $\{g_n\}$ is a sequence of infinitely differentiable functions converging to g.

The next definition for the neutrix product $f \circ g$ of two distributions f and g in $\mathcal{D}'$ generalizes Definition 2 and was given in [4].

Definition 3. *Let f and g be distributions in $\mathcal{D}'$ and let $g_n = g * \delta_n$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to h on the interval (a, b) if*

$$\operatorname*{N-lim}_{n\to\infty} \langle f g_n, \phi \rangle = \langle h, \phi \rangle$$

for all test functions ϕ in $\mathcal{D}$ with support contained in the interval (a, b), where N is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots n, \ldots\}$ and range N'' the real numbers with negligible functions finite linear sums of the functions

$$n^{\lambda} \ln^{r-1} n, \quad \ln^r n$$

for $\lambda > 0$ and $r = 1, 2, \ldots$ and all functions which converge to zero in the normal sense as n tends to infinity.

Note that if

$$\lim_{n\to\infty} \langle f g_n, \phi \rangle = \langle h, \phi \rangle$$

for all ϕ in $\mathcal{D}$, we simply say that the product $f \circ g$ exists and equals h, see [3].

The following two theorems hold, see [3] and [4].

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$. If the product fg exists on the interval (a, b) then the products (and so the neutrix products) $f \circ g$ and $g \circ f$ exist and*

$$f \circ g = g \circ f = fg$$

on this interval.

Theorem 2. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix products $f \circ g$ and $f \circ g'$ (or $f' \circ g$) exist on the interval (a, b). Then the neutrix product $f' \circ g$ (or $f \circ g'$) exists and*

$$(f \circ g)' = f' \circ g + f \circ g'$$

on this interval.

In the following, we now give a definition for the neutrix product $f \circ g$ of two distributions f and g in $\mathcal{D}'_m$ the space of distributions defined on the space $\mathcal{D}_m$ of infinitely differentiable functions of $\mathbf{x} = (x_1, \ldots, x_m)$ with compact support.

2. The function $\delta_n(\mathbf{x})$ in $\mathcal{D}_m$

We first of all generalize the infinitely differentiable function ρ defined above. From now on we let $\rho(r)$, where

$$r = (x_1^2 + \ldots + x_m^2)^{1/2},$$

be a fixed function which is infinitely differentiable with respect to r and with respect to $x_1, \ldots, x_m$ having the properties:

(i) $\rho(r) \geq 0$,

(ii) $\rho(r) = 0,\ r \geq 1$,

(iii) $\int_{R^m} \rho(r)\, d\mathbf{x}$.

Property (iii) is represented in spherical polar coordinates by

(iv) $\Omega_m \int_0^1 \rho(r) r^{m-1}\, dr = 1$,

where Ω_m is the surface area of the unit sphere in R^m.

The function

$$\rho(r) = \begin{cases} ke^{1/(r^2-1)}, & r < 1, \\ 0, & r \geq 1, \end{cases}$$

where

$$k^{-1} = \int_{R^m} e^{1/(r^2-1)}\, d\mathbf{x}$$

is an example of a function satisfying the above conditions.

We now define the function $\delta_n(\mathbf{x})$, with $\mathbf{x}$ in R^m, by

$$\delta_n(\mathbf{x}) = n^m \rho(nr)$$

for $n = 1, 2, \ldots$. It is obvious that $\{\delta_n\}$ is a sequence of infinitely differentiable functions converging to δ in the sense that

$$\lim_{n \to \infty} \langle \delta_n(\mathbf{x}), \phi(\mathbf{x}) \rangle = \langle \delta(\mathbf{x}), \phi(\mathbf{x}) \rangle = \phi(0)$$

for all functions ϕ in $\mathcal{D}_m$.

Note that we can not only consider $\delta_n(\mathbf{x})$ as an infinitely differentiable function of the variables $x_1, \ldots, x_m$ but also as an infinitely differentiable function of the variable r, which is very useful in calculating the product of distributions in $\mathcal{D}'_m$.

For an arbitrary distribution g in $\mathcal{D}'_m$ the function g_n is defined by

$$g_n(\mathbf{x}) = (g * \delta_n)(\mathbf{x}) = \langle g(\mathbf{x} - \mathbf{t}), \delta_n(\mathbf{t}) \rangle.$$

It again follows that $\{g_n\}$ is a sequence of infinitely differentiable functions converging to g.

3. The definition of the product in $\mathcal{D}'_m$

The following definition was introduced by Gel'fand and Shilov in [5] as a natural extension of Definition 1.

Definition 4. *Let* f *be a distribution in* $\mathcal{D}'_m$ *and let* g *be an infinitely differentiable function of* $\mathbf{x}$. *The product* $fg = gf$ *is defined by*

$$\langle fg, \phi \rangle = \langle gf, \phi \rangle = \langle f, g\phi \rangle$$

for all ϕ *in* $\mathcal{D}_m$.

We denote the partial derivative with respect to the variable x_i by D_i. D_i is defined in the obvious way on $\mathcal{D}'_m$ by

$$\langle D_i f, \phi \rangle = -\langle f, D_i \phi \rangle$$

for all ϕ in $\mathcal{D}_m$. It then follows by induction that if

$$\mathbf{s} = (s_1, \ldots, s_m), \quad \mathbf{i} = (i_1, \ldots, i_m),$$

$$(-1)^{\mathbf{i}} = (-1)^{i_1 + \ldots + i_m}, \quad \mathbf{D}^{\mathbf{s}} = D_1^{s_1} \ldots D_m^{s_m},$$

$$\binom{\mathbf{s}}{\mathbf{i}} = \binom{s_1}{i_1} \cdots \binom{s_m}{i_m}, \quad \sum_{\mathbf{i}=0}^{\mathbf{s}} = \sum_{i_1=0}^{s_1} \cdots \sum_{i_m=0}^{s_m},$$

we have

$$(\mathbf{D}^{\mathbf{s}} f)g = \sum_{\mathbf{i}=0}^{\mathbf{s}} \binom{\mathbf{s}}{\mathbf{i}} (-1)^{\mathbf{i}} \mathbf{D}^{\mathbf{s}-\mathbf{i}}(f\mathbf{D}^{\mathbf{i}} g)$$

for any differentiable function g and $s_1, \ldots, s_m = 0, 1, 2, \ldots$.

This suggests the following generalization of Definition 2.

Definition 5. *Let f and g be distributions in $\mathcal{D}'_m$ for which on the interval $(\mathbf{a}, \mathbf{b})$, where $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$, $f = \mathbf{D}^{\mathbf{s}} F$ where F is a locally summable function in $L^p(\mathbf{a}, \mathbf{b})$ and $\mathbf{D}^{\mathbf{s}} g$ is a locally summable function in $L^q(\mathbf{a}, \mathbf{b})$ with $1/p + 1/q = 1$. Then the product $fg = gf$ is defined by*

$$fg = gf = \sum_{\mathbf{i}=0}^{\mathbf{s}} \binom{\mathbf{s}}{\mathbf{i}} (-1)^{\mathbf{i}} \mathbf{D}^{\mathbf{s}-\mathbf{i}}(F(\mathbf{D}^{\mathbf{i}} g).$$

The next definition is a generalization of Definition 3.

Definition 6. *Let f and g be distributions in $\mathcal{D}'_m$ and let $g_n = g * \delta_n$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to h on the interval $(\mathbf{a}, \mathbf{b})$ if*

$$\underset{n\to\infty}{\mathrm{N-lim}} \langle f g_n, \phi \rangle = \langle h, \phi \rangle$$

for all ϕ in $\mathcal{D}_m$ with support contained in the interval $(\mathbf{a}, \mathbf{b})$.

Note that if

$$\lim_{n\to\infty} \langle f g_n, \phi \rangle = \langle h, \phi \rangle$$

for all ϕ, we again simply say that the product $f \circ g$ exists and equals h.

The proofs of Theorems 1 and 2 can be modified to give the following two theorems.

Theorem 3. *Let f and g be distributions in $\mathcal{D}'_m$. If the product fg exists on the interval $(\mathbf{a}, \mathbf{b})$ then the products (and so the neutrix products) $f \circ g$ and $g \circ f$ exist and*

$$f \circ g = g \circ f = fg$$

on this interval.

Theorem 4. *Let f and g be distributions in $\mathcal{D}'_m$ and suppose that the neutrix products $f \circ g$ and $f \circ D_i g$ (or $D_i f \circ g$) exist on the interval $(\mathbf{a}, \mathbf{b})$. Then the neutrix product $D_i f \circ g$ (or $f \circ D_i g$) exists and*

$$D_i(f \circ g) = D_i f \circ g + f \circ D_i g$$

on this interval.

4. Some results

Theorem 5. *The neutrix products $\delta(\mathbf{x}) \circ r^{-k}$ and $r^{-k} \circ \delta(\mathbf{x})$ exist for $k = 1, 2, \ldots, m-1$ and*

$$\delta(\mathbf{x}) \circ r^{-k} = 0, \tag{1}$$

for $k = 1, 2, \ldots, m-1$,

$$r^{1-2k} \circ \delta(\mathbf{x}) = 0, \tag{2}$$

for $k = 1, 2, \ldots, [\frac{1}{2}m]$ and

$$r^{-2k} \circ \delta(\mathbf{x}) = \frac{\Delta^k \delta(\mathbf{x})}{2^k k! m(m+2)\ldots(m+2k-2)} \tag{3}$$

for $k = 1, 2, \ldots, [\frac{1}{2}m - 1]$, where Δ denotes the Laplace operator.

Proof. We note that r^{-k} is a locally summable function on R^m for $k = 1, 2, \ldots, m-1$ and so on putting

$$(r^{-k})_n = r^{-k} * \delta_n$$

we have

$$\begin{aligned} \langle \delta(\mathbf{x})(r^{-k})_n, \phi(\mathbf{x}) \rangle &= \phi(0) \int_{R^m} r^{-k} \delta_n(\mathbf{x})\, d\mathbf{x} \\ &= n^m \phi(0) \int_{R^m} r^{-k} \rho(nr)\, d\mathbf{x} \\ &= n^m \Omega_m \phi(0) \int_0^{1/n} r^{m-k-1} \rho(nr)\, dr \\ &= n^k \Omega_m \phi(0) \int_0^1 t^{m-k-1} \rho(t)\, dt \end{aligned}$$

on changing to spherical polar coordinates, see Gel'fand and Shilov [5], and then putting $nr = t$. It follows that

$$\underset{n\to\infty}{\mathrm{N-lim}}\, \langle \delta(\mathbf{x})(r^{-k})_n, \phi(\mathbf{x})\rangle = 0 = \langle 0, \phi(\mathbf{x})\rangle$$

for arbitrary ϕ in $\mathcal{D}_m$ and equation (1) follows.

Next we have

$$\begin{aligned} \langle r^{-k}\delta_n(\mathbf{x}), \phi(\mathbf{x})\rangle &= n^m \int_{R^m} r^{-k}\rho(nr)\phi(\mathbf{x})\, d\mathbf{x} \\ &= n^m \Omega_m \int_0^{1/n} r^{m-k-1}\rho(nr) S_\phi(r)\, dr \end{aligned}$$

on changing to spherical polar coordinates, where $S_\phi(r)$ is the mean value of ϕ on the sphere $x_1^2 + \ldots + x_m^2 = r^2$, see [5].

By Taylor's theorem we have

$$S_\phi(r) = \sum_{i=0}^{k-1} \frac{S_\phi^{(i)}(0)}{i!} r^i + \frac{S_\phi^{(k)}(0)}{k!} + \frac{S_\phi^{(k+1)}(\xi r)}{(k+1)!} r^{k+1}, \tag{4}$$

where $0 < \xi < 1$. Hence

$$\begin{aligned} \langle r^{-k}\delta_n(\mathbf{x}), \phi(\mathbf{x})\rangle &= n^m \Omega_m \sum_{i=0}^{k-1} \frac{S_\phi^{(i)}(0)}{i!} \int_0^{1/n} r^{m+i-k-1}\rho(nr)\, dr + \\ &+ \frac{n^m \Omega_m S_\phi^{(k)}(0)}{k!} \int_0^{1/n} r^{m-1}\rho(nr)\, dr + \frac{n^m \Omega}{(k+1)!} \int_0^{1/n} r^m \rho(nr) S_\phi^{(k+1)}(\xi r)\, dr \\ &= I_1 + I_2 + I_3. \end{aligned}$$

On making the substitution $nr = t$ we have

$$I_1 = \Omega_m \sum_{i=0}^{k-1} \frac{n^{k-i} S_\phi^{(i)}(0)}{i!} \int_0^1 t^{m+i-k-1}\rho(t)\, dt$$

and so

$$\underset{n\to\infty}{\mathrm{N-lim}}\, I_1 = 0.$$

Similarly

$$I_2 = \frac{\Omega_m S_\phi^{(k)}(0)}{k!} \int_0^1 t^{m-1}\rho(t)\, dt = \frac{S_\phi^{(k)}(0)}{k!}$$

on using (iv) and so

$$\underset{n\to\infty}{\mathrm{N-lim}}\, I_2 = \frac{S_\phi^{(k)}(0)}{k!}.$$

Finally, on putting

$$M_\phi = \max\left\{\left|S_\phi^{(k+1)}(r)\right| : r \geq 0\right\}$$

we have on again putting $nr = t$

$$|I_3| \leq \frac{M_\phi \Omega_m}{n(k+1)!}\int_0^1 t^m \rho(t)\,dt$$

and so

$$\underset{n\to\infty}{\mathrm{N-lim}}\, I_3 = 0.$$

It follows that

$$\underset{n\to\infty}{\mathrm{N-lim}}\langle r^{-k}\delta_n(\mathbf{x}), \phi(\mathbf{x})\rangle = \frac{S_\phi^{(k)}(0)}{k!}$$

for arbitrary ϕ in $\mathcal{D}_m$.

On using the following equations given in [5]

$$S_\phi^{(2k-1)}(0) = 0, \quad S_\phi^{(2k)}(0) = \frac{(2k)!\langle \Delta^k \delta(\mathbf{x}), \phi(\mathbf{x})\rangle}{2^k k! m(m+2)\ldots(m+2k-2)}$$

for $k = 1, 2, \ldots$, equations (2) and (3) follow. This completes the proof of the theorem. □

Theorem 6. *The neutrix product $r^{-k} \circ \delta(\mathbf{x})$ exists and*

$$\langle r^{-k} \circ \delta(\mathbf{x}), \phi(\mathbf{x})\rangle = \frac{S_\phi^{(k)}(0)}{k!}$$

for $k = m, m+1, \ldots$ and arbitrary ϕ in $\mathcal{D}_m$.

Proof. Since r^{-k} is not locally summable on R^m for $k = m, m+1, \ldots$, it was normalized by Gel'fand and Shilov [5] using the following equation

$$\langle r^{-k}, \phi(\mathbf{x})\rangle = \Omega_m \int_0^\infty r^{m-k-1}\left[S_\phi(r) - \sum_{i=0}^{k-m-1} \frac{S_\phi^{(i)}(0)}{i!} r^i - \frac{S_\phi^{(k-m)}(0)}{(k-m)!} H(1-r) r^{k-m}\right] dr,$$

where H denotes Heaviside's function. Using Definition 4 we therefore have

$$\langle r^{-k}\delta_n(\mathbf{x}), \phi(\mathbf{x}) = \langle r^{-k}, \delta_n(\mathbf{x})\phi(\mathbf{x})\rangle =$$

$$= \Omega_m \int_0^\infty r^{m-k-1} \left[S_{\psi_n}(r) - \sum_{i=0}^{k-m-1} \frac{S_{\psi_n}^{(i)}(0)}{i!} r^i - \frac{S_{\psi_n}^{(k-m)}(0)}{(k-m)!} H(1-\right.$$

$$\left. -r)\, r^{k-m} \right] dr = \Omega_m \int_0^{1/n} r^{m-k-1} \left[S_{\psi_n}(r) - \sum_{i=0}^{k-m} \frac{S_{\psi_n}^{(i)}(0)}{i!} r^i \right] dr$$

$$-\Omega_m \sum_{i=0}^{k-m-1} \int_{1/n}^\infty \frac{S_{\psi_n}^{(i)}(0)}{i!} r^{m+i-k-1}\, dr - \Omega_m \int_{1/n}^1 \frac{S_{\psi_n}^{(k-m)}(0)}{(k-m)!} r^{-1}\, dr =$$

$$= I_4 - I_5 - I_6,$$

where

$$\psi_n(\mathbf{x}) = \delta_n(\mathbf{x})\phi(\mathbf{x}) = n^m \rho(nr)\phi(\mathbf{x})$$

and so

$$S_{\psi_n}(r) = n^m \rho(nr) S_\phi(r),$$

$$S_{\psi_n}^{(i)}(0) = n^m \sum_{j=0}^{i} \binom{i}{j} n^j \rho^{(j)}(0) S_\phi^{(i-j)}(0) \tag{5}$$

for $i = 0, 1, 2 \ldots$.

Using equation (4) again we have

$$I_4 = \Omega_m \int_0^{1/n} r^{m-k-1} \left\{ n^m \rho(nr) \left[\sum_{i=0}^{k-1} \frac{S_\phi^{(i)}(0)}{i!} r^i + \frac{S_\phi^{(k)}(0)}{k!} r^k + \right.\right.$$

$$\left.\left. + \frac{S_\phi^{(k+1)}(\xi r)}{(k+1)!} r^{k+1} \right] - \sum_{i=0}^{k-m} \frac{S_{\psi_n}^{(i)}(0)}{i!} r^i \right\} dr =$$

$$= \Omega_m \int_0^{1/n} \left[n^m \rho(nr) \sum_{i=0}^{k-1} \frac{S_\phi^{(i)}(0)}{i!} r^{m+i-k-1} - \sum_{i=0}^{k-m} \frac{S_{\psi_n}^{(i)}(0)}{i!} r^{m+i-k-1} \right] dr +$$

$$+ I_2 + I_3 =$$

$$= \Omega_m \int_0^1 \left[n^{m-1} \rho(t) \sum_{i=0}^{k-1} \frac{S_\phi^{(i)}(0)}{i!} (t/n)^{m+i-k-1} \right.$$

$$- \sum_{i=0}^{k-m} \frac{S_{\psi_n}^{(i)}(0)}{i!}(t/n)^{m+i-k-1}\Bigg] \, dt + I_2 + I_3 =$$

$$= I_7 + I_2 + I_3$$

on making the substitution $nr = t$. On using equation (5) it follows that

$$\operatorname*{N-lim}_{n\to\infty} I_7 = 0$$

and as above

$$\operatorname*{N-lim}_{n\to\infty} I_2 = \frac{S_\phi^{(k)}(0)}{k!}, \quad \operatorname*{N-lim}_{n\to\infty} I_3 = 0.$$

Thus

$$\operatorname*{N-lim}_{n\to\infty} I_4 = \frac{S_\phi^{(k)}(0)}{k!}.$$

Next we have

$$I_5 = -\Omega_m \sum_{i=0}^{k-m-1} \frac{S_{\psi_n}^{(i)}(0)}{i!(m+i-k)} n^{k-m-i}$$

and on using equation (5) it follows that

$$\operatorname*{N-lim}_{n\to\infty} I_5 = 0.$$

Finally we have

$$I_6 = \frac{\Omega_m S_{\psi_n}^{(k-m)}(0)}{(k-m)!} \ln n$$

and again on using equation (5) it follows that

$$\operatorname*{N-lim}_{n\to\infty} I_6 = 0.$$

We have therefore proved that

$$\operatorname*{N-lim}_{n\to\infty} \langle r^{-k}\delta_n(\mathbf{x}), \phi(\mathbf{x})\rangle = \frac{S_\phi^{(k)}(0)}{k!}$$

and the result of the theorem follows.

References

[1] van der Corput, J.G.: Introduction to the neutrix calculus, J. Analyse Math., 7(1959-60), 291-398.

[2] Fisher, B.: The product of distributions, Quart. J. Math. (2) 22(1971), 291-298.

[3] Fisher, B.: On defining the product of distributions, Math. Nachr., 99(1980), 239-249.

[4] Fisher, B.: A non-commutative neutrix product of distributions, Math. Nachr., 108(1982), 117-127.

[5] Gel'fand, I.M., Shilov, G.E.: Generalized functions, Vol. I, Academic Press, 1964.

REZIME

O NEKOMUTATIVNOM NEUTRIKS PROIZVODU DISTRIBUCIJA NA R^m

U radu se definiše nekomutativni proizvod distribucija u R^m pomoću fiksirane beskonačno diferencijabilne radijalne funkcije $\rho : R^m \to R$, $\rho = \rho(r) = \rho((x_1^2 + x_2^2 + ... + x_m^2)^{\frac{1}{2}})$ i pojma neutriksa iz [1]. Pored osnovnih osobina, eksplicitno se izračunava proizvod $r^{-k} \circ \delta(x)$.

Received by the editors December 6, 1988.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 147-156

Review of Research
Faculty of Science
Mathematics Series

COMMON FIXED POINTS OF COMMUTING SET-VALUED MAPPINGS

Brian Fisher
Department of Mathematics,
The University, Leicester, LE1 7RH, U. K.

Abstract

Let F and G be continuous, commuting mappings of a complete metric space (X,d) into $B(X)$ satisfying the inequality

$$\delta(F^p x, G^p y) \leq \max\{c\delta(F^r x, G^s y), \frac{1}{2}\delta(F^r x, F^{r'} x), \frac{1}{2}\delta(G^s y, G^{s'} y) : 0 \leq r,\ s \leq p;\ 0 \leq r',\ s' < p\}$$

for all x, y in X, where $0 \leq c < 1$ and p is a fixed positive integer. It is proved that if F and G also map $B(X)$ into itself, then F and G have a unique common fixed point z.

AMS Mathematics Subject Classification (1980): 47H10
Key words and phrases: Common fixed points, metric spaces.

In the following, as in [1], we let (X,d) be a complete metric space and let $B(X)$ be the set of all nonempty, bounded subsets of X. The function $\delta(A,B)$ with A and B in $B(X)$ is defined by

$$\delta(A,B) = \sup\{d(a,b) :\ a \in A,\ b \in B\}.$$

If A consists of a single point a we write

$$\delta(A,B) = \delta(a,B).$$

If B also consist of a single point b we write

$$\delta(A,B)=\delta(a,b)=d(a,b).$$

It follows immediately that

$$\begin{aligned}\delta(A,B) &= \delta(B,A)\geq 0,\\ \delta(A,B) &\leq \delta(A,C)+\delta(C,B)\end{aligned}$$

for all A, B and C in $B(X)$.

If now $\{A_n : n=1,2,\ldots\}$ is a sequence of sets in $B(X)$, we say that it converges to the subset A of X if

(i) each point a in A is the limit of some convergent sequence $\{a_n \in A_n : n=1,2,\ldots\}$,

(ii) for arbitrary $\varepsilon > 0$, there exists an integer N such that $A_n \subset A_\varepsilon$ for $n > N$, where A_ε is the union of all open spheres with centers in A and radius ε.

The set A is then said to be the limit of the sequence $\{A_n\}$.

The following lemma was proved in [1].

Lemma 1. *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space (X,d) which converge to the bounded subsets A and B respectively, then the sequence $\{\delta(A_n,B_n)\}$ converges to $\delta(A,B)$.*

Now let F be a mapping of a complete metric space (X,d) into $B(X)$. We say that the mapping F is continuous at a point x in X if whenever $\{x_n\}$ is a sequence of points in X converging to x, the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx in $B(X)$. We say that F is a continuous mapping of X into $B(X)$ if F is continuous at each point x in X. We say that a point z in X is a fixed point of F if z is in Fz. If A is any nonempty subsets of X we define the set FA by

$$FA=\bigcup_{a\in A} Fa.$$

If G is a second mappings of X into $B(X)$ we say that F and G commute if $FGx = GFx$ for all x in X. It then follows that $FGA = GFA$ for all nonempty subsets A of X.

We now prove the following theorem.

Theorem 1. *Let F and G be continuous, commuting mappings of a complete metric space (X,d) into $B(X)$ satisfying the inequality*

$$(1)\quad \delta(F^p x, G^p y) \;\le\; \max\{c\delta(F^r x, G^s y),\ \tfrac{1}{2}\delta(F^r x, F^{r'} x),\ \tfrac{1}{2}\delta(G^s y, G^{s'} y) : 0 \le r,\ s \le p;\ 0 \le r',\ s' < p\}$$

for all x, y in X where $0 \le c < 1$ and p is a fixed positive integer. If F and G also map $B(X)$ into itself, then F and G have a unique common fixed point z. Further $Fz = Gz = \{z\}$.

Proof. We will first of all assume, without loss of generality, that $c > \frac{1}{2}$. This will mean that $(1-c)/c < 1$.

Since we are supposing that F and G map $B(X)$ into itself we note that both sides of inequality (1) are finite. Further, if A and B are any sets in $B(X)$ then it follows that

$$(2)\quad \delta(F^p A, G^p B) \;\le\; \max\{c\delta(F^r A, G^s B),\ \tfrac{1}{2}\delta(F^r A, F^{r'} A),\ \tfrac{1}{2}\delta(G^s B, G^{s'} B) : 0 \le r, s \le p;\ 0 \le r',\ s' < p\},$$

both sides of the inequality again being finite.

Now let x be an arbitrary point in X and put $X_{mn} = F^m G^n x$ for $m, n = 0, 1, 2, \ldots$, where $X_{00} = x$. Let us suppose that the set of real numbers $\{K_n : n = 0, 1, 2, \ldots\}$ is unbounded, where

$$K_n = \max\{\delta(X_{n-i,i}, X_{pp}) : 0 \le i \le n\}.$$

Then there exists an integer $n \ge 2p$ such that

$$(3)\quad (1-c)K_n > c.\max\{K_r, \delta(X_{ps}, X_{ps'}),\ \delta(X_{sp}, X_{s'p}) : 0 \le r, s, s' \le p\}$$

and

$$(4)\quad K_n > \max\{K_r\ :\ 0 \le r < n\}.$$

Inequality (3) implies that

$$(5)\quad K_n > \max\{\delta(X_{ps}, X_{ps'}),\ \delta(X_{sp}, X_{s'p})\ :\ 0 \le s, s' \le p\}$$

since $(1-c)/c < 1$.

Inequalities (3) and (4) imply that

$$\begin{aligned} c\delta(X_{r-i,i}, X_{ps}) &\le c\delta(X_{r-i,i}, X_{pp}) + c\delta(X_{pp}, X_{ps}) \\ &< cK_n + (1-c)K_n \\ &= K_n \end{aligned}$$

for $0 \le i \le r \le n$ and $0 \le s \le p$. Similarly

$$c\delta(X_{r-i,i}, X_{sp}) < K_n$$

for $0 \le i \le r \le n$ and $0 \le s \le p$ and so

(6) $K_n > c.\max\{\delta(X_{r-i,i}, X_{ps}), \delta(X_{r-i,i}, X_{sp}) : 0 \le i \le r \le n; 0 \le s \le p\}$.

In the case when $0 \le i \le p$, inequality (4) implies that

$$\frac{1}{2}\delta(X_{r-i,i}, X_{r'-i,i}) \le \frac{1}{2}\delta(X_{r-i,i}, X_{pp}) + \frac{1}{2}\delta(X_{pp}, X_{r'-i,i}) < K_n$$

for $n-p \le r \le n$ and $n-p \le r' < n$ and so

$$\begin{aligned} \text{(7)} \qquad K_n > \frac{1}{2}.\max\{\delta(X_{r-i,i}, X_{r'-i,i}) &: 0 \le i \le p; \\ &n-p \le r \le n;\ n-p \le r' \le n\}. \end{aligned}$$

Similarly, when $p < i \le n$, inequality (4) implies that

$$\begin{aligned} \text{(8)} \qquad K_n > \frac{1}{2}.\max\{\delta(X_{n-i,r}, X_{n-i,r'}) &: p < i \le n; \\ &i-p \le r \le i;\ i-p \le r' < i\}. \end{aligned}$$

On using inequality (2) it follows that

$$\begin{aligned} \delta(X_{n-i,i}, X_{pp}) \le \max\{&c\delta(X_{r-i,i}, X_{ps}), \frac{1}{2}\delta(X_{r-i,i}, X_{r'-i,i}), \frac{1}{2}\delta(X_{ps}, X_{ps'}) : \\ &n-p \le r \le n; n-p \le r' < n; 0 \le s \le p; 0 \le s < p\} \end{aligned}$$

for $0 \le i \le p$. Inequalities (5), (6) and (7) now imply that

$$\text{(9)} \qquad \max\{\delta(X_{n-i,i}, X_{pp}) : 0 \le i \le p\} < K_n.$$

Again on using inequality (2) it follows that

$$\delta(X_{n-i,i}, X_{pp}) = \delta(X_{pp}, X_{n-i,i}) \leq$$

$$\leq \max\{c\delta(X_{sp}, X_{n-i,r}),\ \frac{1}{2}\delta(X_{sp}, X_{s'p}),\ \frac{1}{2}\delta(X_{n-i,r}, X_{n-i,r'})\ :$$
$$i-p \leq r \leq i;\ i-p \leq r' < i;\ 0 \leq s \leq p;\ 0 \leq s' < p\}$$

for $p < i \leq n$. Inequalities (5), (6) and (8) now imply that

$$\text{(10)} \qquad \max\{\delta(X_{n-i,i}, X_{pp})\ :\ i < p \leq n\} < K_n$$

and inequalities (9) and (10) together imply that $K_n < K_n$, a contradiction.

Thus

$$\sup\{K_n :\ n = 0,1,2,\ldots\} = \sup\{\delta(X_{n-i,i}, X_{pp}) : 0 \leq i \leq n; n = 0,1,2,\ldots\} =$$
$$= \sup\{\delta(X_{mn}, X_{pp}) : m,n = 0,1,2,\ldots\} < \infty$$

and so

$$\sup\{\delta(X_{mn}, X_{hk})\ :\ m,n,h,k = 0,1,2,\ldots\} \leq$$
$$\leq \sup\{\delta(X_{mn}, X_{pp}) + \delta(X_{pp}, X_{hk})\ :\ m,n,h,k = 0,1,2,\ldots\} =$$
$$= M < \infty.$$

We now note that since we are assuming that $c > \frac{1}{2}$ the following inequality holds

$$\text{(11)} \qquad \delta(F^p A, G^p B) \ \leq\ c.\max\{\delta(F^r A, G^s B),\ \delta(F^r A, F^{r'} A),$$
$$\delta(G^s B, G^{s'} B) :\ 0 \leq r,\ r',\ s,\ s' \leq p\}$$

for all A, B in $B(X)$. For arbitrary $\varepsilon > 0$, choose an integer N such that $c^N M < \varepsilon$. Then if $m,n,h,k \geq Np$ we have with repeated use of inequality (11)

$$\delta(X_{mn}, X_{kh}) \ \leq\ c.\max\{\delta(X_{rn}, X_{hj}),\ \delta(X_{rn}, X_{r'n}),\ \delta(X_{hj}, X_{hj'})\ :$$
$$m-p \leq r,\ r' \leq m;\ k-p \leq j,\ j' \leq k\}$$
$$\leq\ c.\max\{\delta(X_{rs}, X_{ij}),\ \delta(X_{rs}, X_{r's'}),\ \delta(X_{ij}, X_{i'j'})\ :$$
$$m-p \leq r,\ r' \leq m;\ n-p \leq s,\ s' \leq n;$$
$$h-p \leq i,\ i' \leq h;\ k-p \leq j,\ j' \leq k\}$$

$$
\begin{aligned}
&\leq c^2.\max\{\delta(X_{rs},X_{ij}),\ \delta(X_{rs},X_{r's'}),\ \delta(X_{ij},X_{i'j'}):\\
&\qquad m-2p\leq r,\ r'\leq m;\ n-2p\leq s,\ s'\leq n;\\
&\qquad h-2p\leq i,\ i'\leq h;\ k-2p\leq j,\ j'\leq k\}\\
&\leq c^N.\max\{\delta(X_{rs},X_{ij}),\ \delta(X_{rs},X_{r's'}),\ \delta(X_{ij},\ X_{i'j'}):\\
&\qquad m-Np\leq r,\ r'\leq m;\ n-Np\leq s,\ s'\leq n;\\
&\qquad h-Np\leq i,\ i'\leq h;\ k-Np\leq j,\ j'\leq k\}\\
&\leq c^NM<\varepsilon.
\end{aligned}
$$

Choosing a point x_n in X_{nn} for $n = 1,2,\ldots$ it follows that

$$d(x_m,x_n)\leq\delta(X_{mm},X_{nn})<\varepsilon$$

for $m,n > Np$. The sequence $\{x_n\}$ is therefore a Cauchy sequence in the complete metric space X and so has a limit z in X. Further

$$
\begin{aligned}
\delta(z,Fx_n) &\leq d(z,x_m)+\delta(x_m,Fx_n)\\
&\leq d(z,x_m)+\delta(X_{mm},X_{n+1,n})
\end{aligned}
$$

since x_m is in X_{mm} and Fx_n is in $X_{n+1,n}$. Thus

$$\delta(z,Fx_n)<d(z,x_m)+\varepsilon$$

for $m,n+1>Np$. Letting m tend to infinity it follows that

$$\delta(z,Fx_n)\leq\varepsilon$$

for $n+1>Np$. Using the continuity of F and the lemma, it follows on letting n tend to infinity that

$$\delta(z,Fz)\leq\varepsilon.$$

Since ε is arbitrary, $\delta(z,Fz)=0$ and so we must have $Fz=\{z\}$.

We can prove similarly that there exists a point z' in X such that $Gz'=\{z'\}$. Then

$$
\begin{aligned}
d(z,z') &= \delta(F^pz,G^pz')\\
&\leq c.\max\{\delta(F^rz,G^sz'),\ \delta(F^rz,F^{r'}z),\ \delta(G^sz',G^{s'}z'):\\
&\qquad 0\leq r,r',s,s'\leq p\}\\
&= cd(z,z')
\end{aligned}
$$

and so $z = z'$. Then point z is therefore a common fixed point of F and G.

Now suppose that F and G have a second common fixed point w so that $F^r G^s w$ is contained in $F^p G^p w$ for $r, s = 0, 1, 2, \ldots, p$. Then on using inequlity (11) we have

$$\delta(F^p G^p w, F^p G^p w) = \delta(F^p G^p w, G^p F^p w) \leq$$

$$\leq c.\max\{\delta(F^r G^p w, G^s F^p w),\ \delta(F^r G^p w, F^{r'} G^p w),\ \delta(G^s F^p w, G^{s'} F^p w) :$$

$$0 \leq r, r', s, s' \leq p\} =$$

$$= c\delta(F^p G^p w, F^p G^p w)$$

and so $\delta(F^p G^p w, F^p G^p w) = 0$. It follows that the set $F^p G^p w$ consists of a single point which must be w. This means that $Fw = Gw = \{w\}$. Thus

$$d(z, w) = \delta(F^p z, G^p w) \leq$$

$$\leq c.\max\{\delta(F^r, G^s w),\ \delta(F^r z, F^{r'} z),\ \delta(G^s w, G^{s'} w) : 0 \leq r, r', s, s' \leq p\} =$$

$$= cd(z, w)$$

and it follows that the common fixed point z of F and G is unique. This completes the proof of the theorem.

Corollary 1. *Let S and T be continuous, commuting mappings of a complete metric space (X, d) into itself satisfying the inequality*

$$(12) \qquad d(S^p x, T^p y) \leq \max\{cd(S^r x, T^s y),\ \frac{1}{2}d(S^r x, S^{r'} x),$$

$$\frac{1}{2}d(T^s y, T^{s'} y) : 0 \leq r, r', s, s' \leq p\}$$

for all x, y in X, where $0 \leq c \leq 1$ and p is a fixed positive integer. Then S and T have a unique common fixed point z. Further z is the unique fixed point of S and T.

Proof. Define mappings F and G of X into $B(X)$ by putting

$$Fx = \{Sx\}, \qquad Gx = \{Tx\}$$

for all x in X. The conditions of the theorem are satisfied for F and G since when $r = r' = s = s' = p$

$$d(F^r x, F^{r'} x) = d(G^s y, G^{s'} y) = 0.$$

F and G therefore have a unique common fixed point z and z is then of course the unique common fixed point of S and T.

Now suppose that S has a second fixed point w. Then on using inequality (12)

$$d(w,z) = d(S^p w, T^p z) \leq \max\{cd(w,z),\ \frac{1}{2}d(w,z)\}$$

and the uniqueness of z follows. Similarly z is the unique fixed point of T.

Theorem 2. *Let F and G be commuting mappings of a complete metric space (X,d) into $B(X)$ satisfying the inequality*

$$(13)\quad \delta(F^p x, Gy) \leq \max\{c\delta(F^r x, G^s y),\ \frac{1}{2}\delta(F^r x, F^{r'} x),\ \frac{1}{2}\delta(y, Gy) : 0 \leq r \leq p;\ 0 \leq r' < p;\ s = 0,\ 1\}$$

for all x, y in X, where $0 \leq c < 1$ and p is a fixed positive integer. If F is continuous and if F and G also map $B(X)$ into itself, then F and G have a unique common fixed point z. Further $Fz = Gz = \{z\}$.

Proof. Since F is continuous and F and G obviously satisfy inequality (2) it follows as in the proof of theorem 1 that F has a fixed point z and $Fz = \{z\}$. Further on using inequality (13)

$$\begin{aligned}\delta(z, Gz) &= \delta(F^p z, Gz)\\ &\leq \max\{c\delta(z, Gz),\ \frac{1}{2}\delta(z, Gz)\}\end{aligned}$$

and it follows that $Gz = \{z\}$. The uniqueness of z follows easily. This completes the proof of the theorem.

The corollary follows immediately.

Corollary 2. *Let S and T be commuting mappings of a complete metric space (X,d) into itself satisfying the inequality*

$$d(S^p x, Ty) \leq \max\{cd(S^r x, T^s y),\ \frac{1}{2}d(S^r x, S^{r'} x),\ \frac{1}{2}d(y, Sy) : 0 \leq r, r' \leq p;\ s = 0, 1\}$$

for all x, y in X, where $0 \leq c < 1$ and p is a fixed positive integer. If S is continuous then S and T have a unique common fixed point z. Further z is the unique fixed point of S and T.

The next theorem and its corollary follow easily.

Theorem 3. *Let F and G be commuting mappings of a complete metric space (X,d) into $B(X)$ satisfying the inequality*

$$\delta(Fx,Gy) \leq \max\{cd(x,y),\ c\delta(x,Gy),\ c\delta(y,Fx),\ \frac{1}{2}\delta(x,Fx),\ \frac{1}{2}\delta(y,Gy)\}$$

for all x,y in X, where $0 \leq c < 1$. If F and G also map $B(X)$ into itself, then F and G have a unique common fixed point z. Further $Fz = Gz = \{z\}$.

Corollary 3. *Let S and T be commuting mappings of a complete metric space (X,d) into itself satisfying the inequality*

$$d(Sx,Ty) \leq \max\{cd(x,y),\ cd(x,Ty),\ cd(y,Sx),\ \frac{1}{2}d(x,Sx),\ \frac{1}{2}d(y,Ty)\}$$

for all x,y in X, where $0 \leq c < 1$. Then S and T have a unique common fixed point z. Further z is the unique fixed point of S and T.

The result of the above corollary was given in [2].

We finally show that although the mappings F and G in theorems 1, 2 and 3 necessarily have a unique common fixed point it is possible for either F or G to have a second fixed point. To see this let $X = \{x,y,z\}$ with the metric d defined by

$$d(x,x) = d(y,y) = d(z,z) = 0,$$
$$d(x,y) = d(x,z) = 1,\ \ d(y,z) = 2.$$

Define mappings F and G on X by

$$Fx = Fy = \{x\},\ \ Fz = \{y,z\},$$
$$Gx = Gy = Gz = \{x\}.$$

The conditions of the theorem are satisfied with $c = \frac{1}{2}$ but F has two fixed points x and z.

References

[1] Fisher, B.: Common fixed points of mappings and set-valued mappings, Rostock. Math. Kolloq., 18 (1981), 69-77.

[2] Fisher, B.: A fixed point theorem for commuting mappings, Bull. Malaysian Math. Soc., (2) 5 (1982), 65-67.

REZIME

ZAJEDNIČKE NEPOKRETNE TAČKE KOMUTIRAJUĆIH SKUPOVNIH FUNKCIJA

Neka su F i G neprekidna, komutirajuća preslikavanja kompletnog metričkog prostora (X, d) u $B(X)$ za koje važi nejednakost

$$\begin{aligned}\delta(F^p x, G^p y) \quad \leq \quad & \max\{c\delta(F^r x, G^s y), \frac{1}{2}\delta(F^r x, F^{r'} x), \frac{1}{2}\delta(G^s y, G^{s'} y) : \\ & 0 \leq r,\ s \leq p;\ 0 \leq r',\ s' < p\}\end{aligned}$$

za sve x, y u X, gde je $0 \leq c < 1$ i p fiksiran pozitivan ceo broj. Dokazano je da ako $F, G : B(X) \to B(X)$ tada F i G imaju jedinstvenu nepokretnu tačku z.

Received by the editors November 23, 1987.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 157-164

Review of Research
Faculty of Science
Mathematics Series

CONVERGENCE WITH RESPECT TO SOME σ - IDEALS

Jacek Hejduk
Institute of Mathematics, University of Łódź
S. Banacha 22, 90-238 Łódź, Poland

Abstract

The paper concerns the problem of topologizing the space of measurable real-valued functions by convergence with respect to a totally imperfect σ - ideal of subsets of a perfect Polish space.

AMS Mathematics Subject Classification (1980): 28A20, 54D55
Key words and phrases: σ - ideal, totally imperfect σ - ideal, convergence with respect to σ - ideal.

Suppose that we are given a nonempty set X. In all that follows below, we consider σ - fields and σ - ideals of subsets of X.

To begin with, let us recall two definitions.

Definition 1. *(cf. [5]). Let $\mathcal{I}$ be a σ - ideal. We say that a property holds $\mathcal{I}$ - almost everywhere on X (abrr. $\mathcal{I}$ - a.e.) if the set of all points which do not have this property belongs to $\mathcal{I}$.*

Definition 2. *(cf. [5]). We say that a sequence $\{f_n\}_{n\in N}$ of real functions defined on X converges with respect to a σ - ideal $\mathcal{I}$ to a function f (abbr. $f_n \xrightarrow[n\to\infty]{\mathcal{I}} f$) if each subsequence of $\{f_n\}n \in N$ contains a subsequence converging $\mathcal{I}$ - a.e. on X to f.*

For a σ - field $\mathcal{S}$ and σ - ideal $\mathcal{I}$, denote by $\mathcal{S}\triangle\mathcal{I}$ the smallest σ - field containing both $\mathcal{S}$ and $\mathcal{I}$. It is well known that $\mathcal{S}\triangle\mathcal{I}$ is the collection of all the sets of the form $(A-B)\cup C$ where $A\in\mathcal{S}, B, C\in\mathcal{I}$; moreover, $\mathcal{S}\triangle\mathcal{I}=\mathcal{S}$ if and only if $\mathcal{I}\subset\mathcal{S}$.

Without difficulty we can check that, for every σ-field $\mathcal{S}$ and any σ-ideal $\mathcal{I}$, the set $\mathcal{L}^*(\mathcal{S},\mathcal{I})$ of all $\mathcal{S}\triangle\mathcal{I}$-measurable real functions defined on X, equipped with convergence with respect to $\mathcal{I}$, is an $\mathcal{L}^*$-space (cf.[1; Problem 1. 7. 18. p 90]). Hence it is posible to define the closure operator on $\mathcal{L}^*(\mathcal{S},\mathcal{I})$ by letting $f\in\bar{A}$ if and only if A contains a sequence converging with respect to $\mathcal{I}$ to the function f. Then $\bar{\phi}=\phi$, $A\subset\bar{A}$, $\overline{A\cup B}=\bar{A}\cup\bar{B}$ for any sets $A, B\subset\mathcal{L}^*(\mathcal{S},\mathcal{I})$ however, $\bar{\bar{A}}=\bar{A}$ holds for each $A\subset\mathcal{L}^*(\mathcal{S},\mathcal{I})$ if and only if the following condition, usually labelled by (L4), is satisfied:

(L4) If $f_j \xrightarrow[j\to\infty]{\mathcal{I}} f$ and $f_{j,n} \xrightarrow[n\to\infty]{\mathcal{I}} f_j$ for each $j\in N$, then there exist sequences $\{j_p\}_{p\in N'}$ $\{n_p\}_{p\in N}$ of positive integers such that $f_{j_p,n_p} \xrightarrow[j\to\infty]{} f$.

If (L4) is fulfilled, then topology introduced in $\mathcal{L}^*(\mathcal{S},\mathcal{I})$ in the sense described above is called the Fréchet topology.

The problem of topologizing the space of measurable functions was studied by Wagner (see [5]). There are both some examples of σ-ideals convergence with respect to them yields the Fréchet topology or not. Our considerations lead to the examples of measurable spaces for which it is not possible to introduce the Fréchet topology.

Let $(X,\mathcal{S})$ be a measurable space and $\mathcal{I}$ be an arbitrary σ-ideal of subsets of X.

The following theorem was proved by Wagner (see th. 1 in [5]):

Theorem A. *Suppose that* $\mathcal{I}\subset\mathcal{S}$ *and every family of disjoint sets in* $\mathcal{S}$ - $\mathcal{I}$ *is at most countable. Then convergence with respect to* $\mathcal{I}$ *yields the Fréchet topology in the space* $\mathcal{L}^*(\mathcal{S},\mathcal{I})$ *if and only if the pair* $(S, \mathcal{I})$ *fulfils condition (E).*

Condition (E) is equivalent to the condition (E') (see ([6]). So let us recall this definition.

Definition 3. *We shall say that a pair* $(\mathcal{S},\mathcal{I})$ *fulfils condition (E') if and only if for each set* $D \in \mathcal{S}-\mathcal{I}$ *and for each double sequence* $\{B_{j,n}\}_{j,n\in N}$ *of* $\mathcal{S}$*-measurable sets such that*

$$1^o \quad B_{j,n} \subset B_{j,n+1} \quad \text{for} \quad \text{any } j,n \in N$$
$$2^o \quad \bigcup_{n=1}^{\infty} B_{j,n} = D \quad \text{for} \quad \text{each } j \in N,$$

there exists a sequence $\{n_j\}_{j\in N}$ *of natural numbers such that* $\bigcap_{j=1}^{\infty} B_{j,n_j} \notin \mathcal{I}$.

Further, we are going to demonstrate the σ-ideals and σ-field fulfilling condition (E), but convergence with respect to them does not yield the Fréchet topology.

At the beginning let us pay attention to the following property:

Proposition 1. *The pair* $(\mathcal{S},\mathcal{I})$ *fulfils condition (E') if and only if the pair* $(\mathcal{S}\triangle\mathcal{I},\mathcal{I})$ *fulfils the condition (E').*

Proof. Necessity. Let us consider any set $D \in (\mathcal{S}\triangle\mathcal{I})-\mathcal{I}$ and $B_{j,n} \in \mathcal{S}\triangle\mathcal{I}$ $(j,n \in N)$ which satisfy conditions (1^o) and (2^o) of Definition 3. There exist sets $A_{j,n} \in \mathcal{S}$, $C_{j,n} \in \mathcal{I}$ and $E_{j,n} \in \mathcal{I}$ such that

$$B_{j,n} = (A_{j,n} - C_{j,n}) \cup E_{j,n}, \text{ where } j,n \in N.$$

Denote

$$D^* = \bigcap_{j=1}^{\infty} \bigcup_{n=1}^{\infty} A_{j,n}, B^*_{j,n} = \bigcup_{k=1}^{n} A_{j,k} \cap D^*$$

and

$$C = \bigcup_{j=1,n=1}^{\infty} C_{j,n} \cup \bigcup_{j=1,n=1}^{\infty} E_{j,n} \text{ for } j,n \in N.$$

Of course, $D^* \in \mathcal{S}$, $C \in \mathcal{I}$ and $B^*_{j,n} \in \mathcal{S}$ for $j,n \in N$. Moreover, $B^*_{j,n} \subset B^*_{j,n+1}$ and $D^* = \bigcup_{n=1}^{\infty} B^*_{j,n}$, whenever $j \in N$. If the pair $(\mathcal{S},\mathcal{I})$ fulfils (E'), then there exists a sequence $\{n_j\}_{j\in N}$ of natural numbers such that $\bigcap_{j=1}^{\infty} B^*_{j,n_j} \notin \mathcal{I}$. Simultaneously,

$$B^*_{j,n} \cup C \subset \bigcup_{k=1}^{n} (A_{j,k} \cup C) \subset \bigcup_{k=1}^{n} B_{j,k} \cup C = B_{j,n} \cup C$$

for any $j, n \in N$. Hence,

$$\bigcap_{j=1}^{\infty} B^{*}_{j,n_j} - \bigcap_{j=i}^{\infty} B_{j,n_j} \in \mathcal{I}$$

so that

$$\bigcap_{j=1}^{\infty} B_{j,n_j} \notin \mathcal{I}.$$

Sufficiency is obvious, because $\mathcal{S} \subset \mathcal{S}\triangle\mathcal{I}$.

Now, we shall define the totally imperfect σ-ideals.

Definition 4. *A σ-ideal $\mathcal{I}$ of subsets of topological space X is totally imperfect if and only if it does not contain nonempty perfect sets in X.*

It is worth mentioning that, there are many natural examples of totally imperfect σ-ideals (see [3]).

Theorem 1. *Let $\mathcal{S}$ be a σ-field of subsets of a perfect Polish space X containing the family of perfect sets in X and $\mathcal{I}$ be an arbitrary totally imperfect σ-ideal of subsets of X. If for each set $A \in \mathcal{S} - \mathcal{I}$ there exists a nonempty perfect set $A^{*} \subset A$, then the pair $(\mathcal{S}\triangle\mathcal{I}, \mathcal{I})$ fulfils condition (E).*

Our proof will be based on the following lemma which is easy to prove.

Lemma 1. *Let $B \subset X$ be an uncountable Borel set. Then, for each positive number ε, there exist nonempty perfect sets C and D, such that $C \cup D \subset B$, $C \cap D = \emptyset$ and $d(C) < \varepsilon$, $d(D) < \varepsilon$, where $d(C), d(D)$ denote, respectively, the diameters of the sets C, D.*

Proof of theorem 1: According to proposition 1 we shall proof that the pair $(\mathcal{S}, \mathcal{I})$ fulfils condition (E'). Let an arbitrary set $D \in \mathcal{S} - \mathcal{I}$ and an arbitrary double sequence $\{B_{j,n}\}_{j,n \in N}$ of $\mathcal{S}$-measurable sets fulfil (1^{o}) and (2^{o}) of condition (E').

We define by induction a decreasing sequence $\{D_j\}_{j \in N}$ of perfect sets and a sequence $\{n_j\}_{j \in N}$ of natural numbers such that $\bigcap_{j=1}^{\infty} D_j$ is a nonempty perfect set and, for each $j \in N$, $D_j \subset B_{j,n_j}$.

Consequently, we shall obtain that $\bigcap_{j=1}^{\infty} B_{j,n_j} \notin \mathcal{I}$.
The sequence $\{D_j\}_{j\in N}$, will be defined in such a way that the following conditions hold:

(a) for each $j \in N, D_j = D_{j,1} \cup D_{j,2} \cup \ldots \cup D_{j,2^j}$,
where $D_{j,k} \cap D_{j,s} = \emptyset$ for $k \neq s, 1 \leq k, s \leq 2^j$
and $D_{j,k}$ is perfect, $d(D_{j,k} < \frac{1}{2^j}$ for each $1 \leq k \leq 2^j$,

(b) for any $j \in N$ and $1 \leq t \leq 2^{j-1}$ there exist
$1 \leq k, s \leq 2^j, k \neq s$ such that $D_{j,k} \cup D_{j,s} \subset D_{j-1,t}$

Since $D = \bigcup_{n=1}^{\infty} B_{1,n}$, there exists a positive integer n_1 such that the set $B_{1,n_1} \in \mathcal{S} - \mathcal{I}$. By Lemma 1 there exist two nonempty perfect sets $D_{1,1}$, $D_{1,2}$ such that $D_{1,1} \subset B_{1,n_1}$, $D_{1,2} \subset B_{1,n_1}$, $d(D_{1,1}) \leq \frac{1}{2}$, $d(D_{1,1}) < \frac{1}{2}$ and $D_{1,1} \cap D_{1,2} = \emptyset$.

Let $D_1 = D_{1,1} \cup D_{1,2}$. Suppose we have defined the sets $D_1, D_2, \ldots$ $\ldots D_{j-1}$ fulfilling the above conditions (a) and (b). Now, we shall define a set D_j. Since

$$D \supset D_{j-1} = D_{j-1,1} \cup D_{j-2,2} \cup \ldots \cup D_{j-1,2^{j-1}},$$

thus,for each $1 \leq k \leq 2^{j-1}$, we have

$$D_{j-1}, k = \bigcup_{n=1}^{\infty} (B_{j,n} \cap D_{j-1,k}),$$

so there exists a natural number n_k such that

$$B_{j,n_k} \cap D_{j-1,k} \notin \mathcal{I}.$$

Let $n_j = \max(n_1, n_2, \ldots n_{2^{j-1}})$. Since $B_{j,n_j} \supset B_{j,n_k}$ for each $1 \leq k \leq 2^{j-1}$, therefore $B_{j,n_j} \cap D_{j-1,k}, \in \mathcal{S} - \mathcal{I}$ for each $1 \leq k \leq 2^{j-1}$. Putting $\varepsilon = 2^{-j}$, by Lemma 1, for each set $B_{j,n_j} \cap D_{j-1,k}$, where $1 \leq k \leq 2^{j-1}$, there exist two nonempty perfect sets $D_{j,2k-1}, D_{j,2k}$ such that $D_{j,2k-1} \cup D_{j,2k} \subset B_{j,n_j} \cap D_{j-1,k}$, $D_{j,2k-1} \cap D_{j,2k} = \emptyset$ and $d(D_{j,2k-1}) < 2^{-j}$, $d(D_{j,2k}) < 2^{-j}$. Let

$$D_j = D_{j,1} \cup D_{j,2} \cup \ldots \cup D_{j,2^j-1} \cup D_{j,2^j}.$$

By virtue of this construction and the properties of the sets D_{j-1}, it is easy to see that the set D_j fulfils conditions (a) and (b). Applying the Cantor theorem, we have that $\bigcap_{j=1}^{\infty} D_j \neq \phi$. Simultaneously, this closed set is perfect (see the fusion lemma in [2]).

Applying the above theorem we conclude.

Corollary 1. *In a perfect Polish space* X *the pair* $(\mathcal{B}\triangle\mathcal{I},\mathcal{I})$ *fulfils condition (E), where* $\mathcal{B}$ *denotes the family of Borel sets in* X *and* $\mathcal{I}$ *an arbitrary totally imperfect* σ*-ideal of subsets of* X.

We can also apply Theorem 1 to the totally imperfect σ-ideal $\mathcal{I}_M$ of sets having the property (s_o) introduced by Marczewski in a Polish space (see [4]). This σ-ideal is included in the σ-field $\mathcal{S}_M$ of sets having the property (s),

Corollary 2. *In a perfect Polish space* X *the pair* $(\mathcal{S}_M,\mathcal{I}_M)$ *fulfils condition (E).*

The following theorem proves that condition (E) is not sufficient to introduce the Fréchet topology.

Theorem 2. *Let* $\mathcal{S}$ *be a* σ*-field of subsets of a perfect Polish space* X *containing the family of perfect sets in* X. *Then convergence with respect to an arbitrary totally imperfect* σ*-ideal* $\mathcal{I}$ *of subsets of* X *does not yield the Fréchet topology in the space* $\mathcal{L}^*(\mathcal{S},\mathcal{I})$.

Proof. We shall prove that condition (L4) is not fulfilled. Let a sequence $\{r_n\}_{n\in N}$ be dense in X and let $\{\varepsilon_n\}_{n\in N}$ be a decreasing sequence of natural numbers converging to 0. Let $f_{j,n} = \chi_{A_{j,n}} + \frac{1}{j}$, where $A_{j,n} = \bigcup_{i=1}^{j} K(r_i,\varepsilon_n)$ for any $j,n \in N$. The functions $f_{j,n}$ are $\mathcal{S}\triangle\mathcal{I}$-measurable for any $j,n \in N$ and $\lim_{n\to\infty} f_{j,n}(x) = \frac{1}{j}$ if and only if

$$x \notin \limsup_{n\in N} A_{j,n} = \bigcap_{j=1}^{\infty}\bigcup_{n=k}^{\infty}\bigcup_{i=1}^{j} K(r_i,\varepsilon_n) = \{r_1,\dots r_j\}.$$

Hence, $f_{j,n} \xrightarrow[n\to\infty]{} f_j = \frac{1}{j}$ and $f_j \xrightarrow[j\to\infty]{} f \equiv 0$. Let $\{j_p\}_{p\in N}$ be an arbitrary increasing sequence of natural numbers and $\{n_p\}_{p\in N}$ an arbitrary sequence of natural numbers. Then

$$\{x :\sim (\lim_{p\to\infty} f_{j_p,n_p}(x) = 0\} \supset \{x : \limsup_{p\to\infty} f_{j_p,n_p}(x) \geq 1\} \supset$$

$$\supset \bigcap_{k=1}^{\infty} \bigcup_{p=k+1}^{\infty} \{x : f_{j_p,n_p}(x) \geq 1\} \supset \bigcap_{k=1}^{\infty} \bigcup_{p=k+1}^{\infty} A_{j_p,n_p}.$$

The set $\bigcup_{p=k+1}^{\infty} A_{j_p,n_p} = \bigcup_{p=k+1}^{\infty} \bigcup_{i=1}^{j} K(x_i, \varepsilon_{n_p})$ is open and dense, so the set $\bigcap_{k=1}^{\infty} \bigcup_{p=k+1}^{\infty} A_{j_p,n_p}$ is Borel and residual. Hence, it contains a nonempty perfect set which implies that $\{x :\sim (\lim_{p\to\infty} f_{j_p,n_p}(x) = 0\} \notin \mathcal{I}$.

Corollary 3. *Convergence with respect to $\mathcal{I}$ does not yield the Fréchet topology in the space $\mathcal{L}^*(\mathcal{B}, \mathcal{I})$.*

Corollary 4. *Convergence with respect to $\mathcal{I}_M$ does not yield the Fréchet topology in the space $\mathcal{L}^*(\mathcal{S}_M, \mathcal{I}_M)$.*

By Theorem A we have

Corollary 5. *There exist, respectively, uncountable families of pairwise, disjoint sets in $\mathcal{B} - \mathcal{I}$ and $\mathcal{S}_M - \mathcal{I}_M$.*

References

[1] Engelking, R.: General topology, PWN, Warszawa, 1977.

[2] Jech, T.: Set theory, Academic Press, New York,1978.

[3] Miller, A.W.: Special subsets of the real line, in Handbook of set-theoretic topology, North-Holland, Amsterdam, 1984.

[4] Szpilrajn-Marczewski, E.: Sur une classe de fonctions de W. Sierpiński et la classe correspondante d'ensembles, Fund. Math. 24 (1935), pp. 17-24

[5] Wagner, E.: Sequences of measurable functions, Fund. Math. 112 (1981) pp. 89-102.

[6] Wagner, E., Wilczyński, W.: Spaces of measurable functions, Rendiconti del Circolo Matematico di Palermo, Serie II, 20 (1981), pp. 97-110.

REZIME

KONVERGENCIJA U ODNOSU NA NEKE σ - IDEALE

U ovom radu se razmatra problem topologiziranja prostora merljivih realnih funkcija pomoću konvergencije u odnosu na totalno imperfektan σ - ideal skupova perfektnog Polish prostora.

Received by Editors May 20, 1988.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 165-172

Review of Research
Faculty of Science
Mathematics Series

OSCILLATIONS AND THE ASYMPTOTIC BEHAVIOUR OF CERTAIN SECOND ORDER NEUTRAL DIFFERENCE EQUATIONS

Mirko Budinčević
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

The paper deals with the oscillatory and asymptotic behaviour of solutions of second order neutral difference equations with variable coefficients and some generalizations of such equations.

AMS Mathematics Subject Classification (1980): 34K15
Key words and phrases: Oscillation, neutral difference equation, asymptotic behaviour.

1. Introduction

This paper deals with the oscillatory and asymptotic behaviour of solutions of the second order neutral difference equation of the form

$$\Delta(a(n)\Delta(x(n)+px(n-n_0)))+q(n)f(x(n-m_0))=0, \tag{1}$$

where $\{a(n)\}$ and $\{q(n)\}$ are positive sequences and Δ is the forward difference operator defined by $\Delta y(n)=y(n+1)-y(n)$. p is a constant and the function f is considered subject to the condition

(0) f is nondecreasing and $uf(u)>0$ for $u\neq 0$.

By a solution of (1) we mean a real sequence $\{y(n)\}$ satisfying (1). Throughout this paper, we usually refer to a solution $\{y(n)\}$ simply as a solution y and consider only nontrivial solutions.

A real sequence $\{r(n)\}$ *eventually* has some property if there exists $N \geq 0$ such that $r(n)$ has this property for $n = N, N+1, \ldots$.

A nontrivial solution y of (1) is said to be *oscillatory* if $y(n)$ changes its sing infinitely many times. Otherwise, y is said to be **nonoscillatory** an equation is called **oscillatory** if all its solutions are oscillatory. Otherwise, it is called **nonoscillatory**.

Similarly, as in the theory of differential equations, in a neutral difference equation the highest difference of the unknown function appears with the argument n (the present state of the system) and with one or more retarded arguments (the past state of the system). Investigations of such systems, beside their theoretical interest, have some importance for application (see [1] and [2]).

There is much current interest in the oscillation theory of differential equations of the neutral type see ([3],[4],[5] and [6]). As far as the author is aware, not to much has been done on the theory of difference equations. This, and the fact that the results for adequate difference equations could be quite different, are motivations for this paper.

2. Preliminaries

In what follows we shall use the following lemmas which give useful information about the bounds for nonoscillatory solutions of the following equation:

$$(2) \qquad \Delta(a(n)\Delta z(n)) + q(n)f(z(n)) = 0, \; n = 0, 1, \ldots, \cdots$$

Lemma 1. *([7]) Consider (2) subject to conditions (0),*
$(C_1) q(n) \geq 0$ for $n = 0, 1, \ldots$, and $q(n)$ is not eventually zero,
$a(n) > 0$ and
(C_2) $\sum_{n=0}^{\infty} \frac{1}{a(n)} < \infty$.

Then, every nonoscillatory solution y of (2) satisfies eventually the following estimate

$$A\rho(n) \leq | y(n) | \leq B$$

for some positive constants A and B (depending on y), where

$$\rho(n) = \sum_{i=n}^{\infty} \frac{1}{a(i)}.$$

Lemma 2. *([8]) Consider (2) subject to conditions*$(0), (C_1), a(n) > 0$ *and* $(C_3)\ \sum_{n=0}^{\infty} \frac{1}{a(n)} = \infty$.

Then, every nonoscillatory solution y of (2) satisfies eventually the following estimate

$$C \leq | y(n) | \leq DR(n)$$

for some positive constants C and D (depending on y), where

$$R(n) = \sum_{i=0}^{n} \frac{1}{a(i)}.$$

Near this *apriori* estimate we next need

Lemma 3. *Suppose that* $x(n) > 0$ *eventually and define*

$$z(n) = \sum_{i=0}^{k} p_i x(n - m_i), \ p_i > 0. \tag{3}$$

If $p_0 > \sum_{i=1}^{k} p_i \equiv P$, *then* $z(n) \to C \geq 0$ *if and only if* $x(n) \to \frac{C}{P+p_i}, n \to \infty$.

Proof. Suppose that $z(n) \to C$. Let $\overline{\lim}_{s\to\infty} x(n) = \lim_{\ell\to\infty} x(n_\ell) = \frac{C+q_1}{P+p_o}$ and

$$\lim_{n\to\infty} x(n) = \lim_{s\to\infty} x(n_s) = \frac{C - q_2}{P + p_0}.$$

We shall prove that $q_1 = q_2 = 0$.
a) suppose that $q_1 \geq q_2 \geq 0$ and $q_1 > 0$. Taking $n = n_\ell + m_0$, (3) implies that

$$C = \frac{C + q_1}{P + p_0} p_0 + \sum_{i=1}^{k} \frac{C + \delta_i}{P + p_0} p_i,$$

where $-q_2 - \epsilon < \delta_i < q_1 + \epsilon$ eventually for every $\epsilon > 0$. The above equation implies

$$p_0 q_1 = -\sum_{i=1}^{k} p_i \delta_i < \sum_{i=1}^{k} p_i (q_2 + \epsilon).$$

Choosing $\epsilon < \frac{p_0 - P}{P} q_1$ we get $q_1 < q_2$, a contradiction.

b) Suppose that $q_2 \geq q_1 \geq 0$ and $q_2 > 0$. Taking $n = n_s + m_0$, (3) implies that

$$C = \frac{C - q_2}{P + p_0} p_0 + \sum_{i=1}^{k} \frac{C + \delta_i}{P + p_0} p_i,$$

where $-q_2 - \epsilon < \delta_i < q_1 + \epsilon$ eventually for every $\epsilon > 0$. The above equation implies

$$p_0 q_2 = \sum_{i=1}^{k} p_i \delta_i < \sum_{i=1}^{k} p_i (q_1 + \epsilon).$$

Choosing $\epsilon < \frac{p_0 - P}{P} q_2$ we get $q_2 < q_1$, a contradiction.

As the convergence of $x(n)$ implies the convergence of $x(n - m_i)$ to the same limit, the proof of the second part of the theorem is obvious.

Remark 1. If $p_0 = P$, $x(n) = 2 + (-1)^n$ for appropriate m_i could be a counter-example.

3. Oscillations and asymptotic behaviour

Consider the second order neutral difference equation

$$\Delta(a(n)\Delta(x(n) + px(n - n_0))) + q(n) f(x(n - m_0)) = 0, \; n = 0, 1, \ldots$$

where $a(n), q(n) > 0$; $0 \leq p < 1$: $n_0, m_0 \in N$ and f satisfies (0).

Theorem 1. *If*

$$\sum_{n=k_0}^{\infty} q(n) = \infty \quad \text{and} \quad \sum_{n=k_0}^{\infty} \frac{1}{a(n)} \sum_{k=k_0}^{n} q(k) = \infty$$

then every solution x of (1) is either oscillatory or else $x(n) \to 0$ as $n \to \infty$.

Proof. Let x be a nonoscillatiry solution of (1). Suppose, without loss of generality, that $x(n) > 0$ eventually. This implies that $x(n - n_0) > 0$ and $x(n - m_0) > 0$ eventually. Set

$$z(n) = x(n) + px(n - n_0), \tag{4}$$

then $z(n) > x(n) > 0$ eventually.
According to (1) we have $\Delta(a(n)\Delta z(n)) < 0$ eventually. Thus, either $\Delta z(n) > 0$ or $\Delta z(n) < 0$ eventually.

a) Assume that $\Delta z(n) > 0$ eventually. It follows that $\Delta z(n - m_0) > 0$ eventually and by (4)

$$x(n - m_0) = z(n - m_0) - px(n - n_0 - m_0) \geq$$

$$\geq z(n - m_0) - pz(n - n_0 - m_0) \geq z(n - m_0) - pz(n - m_0),$$

which implies

$$x(n - m_0) \geq (1 - p)z(n - m_0) = p_1 z(n - m_0).$$

Define a positive sequence w such that

$$w(n) = \frac{a(n)\Delta z(n)}{f(p_1 z(n - m_0))},$$

then

$$\Delta w(n) = \frac{\Delta(a(n)\Delta z(n))}{f(p_1 z(n - m_0))} - \frac{\Delta f(p_1 z(n - m_0))a(n+1)\Delta z(n+1)}{f(p_1 z(n - m_0))f(p_1 z(n - m_0 + 1))} \leq -q(n).$$

Summing the above inequality from N to n we get

$$w(n) \leq w(N) - \sum_{k=n}^{N} q(k) \to -\infty, \ n \to \infty,$$

which is a contradiction.

b) Assume that $\Delta z(n) < 0$ eventually. Then $\lim_{n\to\infty} z(n) = c$ and suppose that $c > 0$. According to Lemma 3 $\lim_{n\to\infty} x(n - m_0) = \frac{C}{1+p}$ which implies that $x(n - m_0) \geq \frac{C}{2(1+p)}$ eventually. Thus

$$\Delta(a(n)\Delta z(n)) \leq -f(\frac{c}{2(1 + p)})q(n)$$

eventually. Summing the above inequality from N to n we get

$$a(n)\Delta z(n) \leq a(N)\Delta z(N) - f(\frac{c}{2(1 + p)}) \sum_{k=N}^{n} q(k).$$

Using the assumptions of the theorem it yields to

$$a(n)\Delta z(n) \le -\frac{1}{2} f(\frac{c}{2(1+p)}) \sum_{k=N}^{n} q(k)$$

eventually. Dividing by $a(n)$ and summing from N to n we get

$$z(n) \le z(N) - \frac{1}{2} f(\frac{c}{2(1+p)}) \sum_{k=N}^{n} \frac{1}{a(s)} \sum_{k=N}^{s} q(k) \to \infty,$$

while $n \to \infty$, and this is a contradiction to the fact that $z(n) \to c > 0$. Thus $c = 0$ and the proof is complete.

The next theorem guarantees that all the solutions of (1) are oscillatory.

Theorem 2. *If* $\sum_{n=k_0}^{\infty} q(n) = \infty$ *and* $\sum_{n=k_0}^{\infty} \frac{1}{a(n)} = \infty$, *then equation (1) is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1) and let $x(n)$ be of a positive sign eventually. As in the proof of Theorem 1, we define $z(n)$. The second condition of the theorem, as it was shown in [8], implies that $\Delta z(n) > 0$ eventually . As the proof follows the same line as in case a) of Theorem 1, it will be omitted.

Remark 2. Accorrding to Lemma 3, we can generalize the assertions of Theorem 1 and the Theorem 2 to the case

$$(5)\Delta(a(n)\Delta(x(n) + \sum_{i=1}^{k} p_i x(n-m_i))) + q(n)f(x(n-m_0)) = 0,\ n = 0, 1,$$

while $\sum_{i=1}^{k} p_i < 1$.

A natural question is what happens when $\sum^{\infty} \frac{1}{a(n)}$ converges. The answer gives the following

Theorem 3. *Consider equation (1) where* $a(n), q(n) > 0;\ 0 \le p < \infty$, $p \ne 1$; *f satisfies (0);*

$$\sum_{n=k_0}^{\infty} \frac{1}{a(n)} < \infty \text{ and } \sum_{n=k_0}^{\infty} \frac{1}{a(n)} \sum_{k=k_0}^{n} q(k) = \infty.$$

Then, every solution x of (1) is either oscillatory or else $x(n) \to 0$ *since* $n \to \infty$.

Proof. As in the proof of Theorem 1 we introduce $z(n)$ and differentiate two cases:

a) Assume that $\Delta z(n) > 0$ eventually. According to Lemma 1 we have that $\lim_{n\to\infty} z(n) = c > 0$, which by Lemma 3 gives that $\lim_{n\to\infty} x(n) = \frac{C}{1+p}$ and the estimate $x(n-m_0) > \frac{z(n-m_0)}{2(1+p)}$ eventually. Conditions $\sum_{n=k_0}^{\infty} \frac{1}{a(n)} < \infty$ and $\sum_{n=k_0}^{\infty} \frac{1}{a(n)} \sum_{n=k}^{n} q(k) = \infty$ imply that $\sum_{n=k_0}^{\infty} q(n) = \infty$ and we can proceed as in the proof of case a) of Theorem 1.

b) Assume that $\Delta z(n) < 0$ eventually. According to the observations given in case a) the proof follows the same line as in the proof of case b) of Theorem 1.

Remark 3. In the light of Lemma 3 we are able to generalize the assertions of Theorem 3 on the difference equation (5) if $0 \le p_i$, $\sum_{i=1}^{k} p_i < 1$ or

$$p_j > 1 + \sum_{\substack{i=1 \\ i \neq j}}^{k} p_i \, .$$

for some fixed j.

References

[1] Brayton, R.K., Willoughby, R.A.: On the numerical integration of a symmetric system of difference-differential equations of neutral type. J. Math. Anal. Appl. 18(1967), 182- 189.

[2] Hale, J.: Theory of functional differential equations. Springer-Verlag, New York, 1977.

[3] Grammatikopoulos, M.K., Grove, E.A., Ladas, G.: Oscillation and asymptotic behaviour of second order neutral differential equations with deviating arguments, Canadian Math. Soc., Conference Proc., Vol. 8 (1981).

[4] Ladas, G., Partheniadis, E.C., Sficas, Y.G.: Oscillations of second order neutral equations, Can. J. Math., Vol. XL, No 6, 1988, 1301-1310.

[5] Arino, O., Györi, I.: Necessaary and sufficient condition for oscillation of a neutral differential system with several delays, J. Diff. Eq. 81(1989), 98-105.

[6] Grace, S.R., Lalli, B.S.: Oscillation and asymptotic behaviour of certain second order neutral differential equations, Radovi mat., Vol. 5(1989), 121-126.

[7] Budinčević, M., Kulenović, M.R.S.: Asymptotic analysis of a nonlinear second order difference equation II, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., 12(1982), 73-92.

[8] Kulenović, M.R.S., Budinčević, M.: Asymptotic analysis of a nonlinear second order difference equation, An. Stiin. Univ. Al. I. Cuza. , Vol. XXX (1984), 39-52.

REZIME

OSCILACIJE I ASIMPTOTSKO PONAŠANJE NEKIH NEUTRALNIH DIFERENCNIH JEDNAČINA DRUGOG REDA

U radu se posmatra neutralna diferencna jednačina drugog reda

$$\Delta(a(n)\Delta(x(n) + px(n - n_0))) + q(n)f(x(n - m_0)) = 0, \ n = 0, 1, ...$$

$a(n), q(n) > 0$; $p \geq 0$ a f je neopadajuća funkcija takva da je $uf(u) > 0$ za $u \neq 0$.
Daju se dovoljni uslovi da bi sva rešenja bila oscilatorna ili da bi još eventualno težila nuli.

Received by the editors April 24, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 173-191

Review of Research
Faculty of Science
Mathematics Series

ON PERIODIC SOLUTIONS OF DIFFERENTIAL INCLUSIONS WITH UNBOUNDED OPERATORS IN BANACH SPACES

M.I.Kamenskii
Voronezh State University, Faculty of Mathematics
394693 Voronezh, USSR
V.V.Obukhovskii
Voronezh State Pedagogical Institute
Faculty of Physics and Mathematics,
394611 Voronezh, USSR

Abstract

In the paper an operator method is suggested using the fixed points index theory of condensing multivalued maps for the investigation of the existence of mild periodic solutions of the differential inclusion

$$x'(t) \in Ax(t) + F(t, x(t)),$$

where A is a linear operator and F is a multivalued map in a separable Banach space. Some applications to the existence of optimal periodic solutions of control systems in a Banach space are considered.

AMS Mathematics Subject Classification (1980): 34C25, 34A60, 34G20, 47H09, 49A27.
Key words and phrases: Differential inclusion, periodic solution, measure of noncompactness, condensing map, multivalued map, fixed points index, optimal periodic control.

1. Introduction

Let Y be a topological space; 2^Y denote a collection of all the subsets of Y. Then

$$P(Y) = \{D \in 2^Y : D \neq \emptyset\},$$

$$Pb(Y) = \{D \in P(Y) : \ D \text{ is bounded } \},$$

$$C(Y) = \{D \in P(Y) : \ D \text{ is closed } \},$$

$$K(Y) = \{D \in P(Y) : \ D \text{ is compact } \}.$$

If Y is a subspace of a topological vector space, then $Cv(Y)[Kv(Y)]$ denote a collection of all the nonempty closed convex [compact convex] subsets of Y.

Let $\mathcal{E}$ be an infinite-dimensional Banach space. A function $\varphi : 2^{\mathcal{E}} \to \mathbf{R}_+$ is said to be a measure of noncompactness in $\mathcal{E}$ if

$$\varphi(\overline{co}\Omega) = \varphi(\Omega)$$

for every $\Omega \in 2^{\mathcal{E}}$ (see, for example, [1],[2]). One of the well-known examples in the Hausdorff measure of noncompactness:

$$\chi(\Omega) = \inf\{\epsilon > 0 : \Omega \text{ has a finite } \epsilon\text{-net } \} .$$

We shall later use its following properties.

$\chi 1)$ If $\Omega_1 \subset \Omega_2$, then $\chi(\Omega_1) \leq \chi(\Omega_2)$.

$\chi 2)$ $\chi(\Omega_1 + \Omega_2) \leq \chi(\Omega_1) + \chi(\Omega_2)$ for all $\Omega_1, \Omega_2 \in 2^{\mathcal{E}}$.

$\chi 3)$ $\chi(Q) = \chi(S) = 1$, where $Q = \{x \in \mathcal{E} : ||x|| \leq 1\}$, $S = \partial Q$.

The symbol E in the sequel will denote a separable Banach space. Then the following properties should be noted. Let the interval $[0, T]$ be equipped with the Lebesgue measure.

$\chi 4)$ **The Kisielewicz lemma.** ([10]) Let $\{g_n\}_{n=1}^{\infty}$ be an integrally bounded sequence of mesurable functions from $[0, T]$ into E. Then $\omega : [0, T] \to \mathbf{R}_+$,

$$\omega(t) = \chi(\{g_n(t)\}_{n=1}^{\infty})$$

is a summable function and

$$\chi(\{\int_{\mathcal{T}} g_n(s)ds\}_{n=1}^{\infty}) \le \int_{\mathcal{T}} \omega(s)ds$$

for every measurable set $\mathcal{T} \subset [0,T]$.

In the sequel, the space $C([0,T];E)$ will be denoted as $\mathcal{C}$.

$\chi 5)$ If $\Omega \subset \mathcal{C}$ is bounded and equicontinuous, then

$$\psi(\Omega) := \sup_{t\in[0,T]} \chi(\Omega(t)) = \chi_{\mathcal{C}}(\Omega),$$

where $\Omega(t) = \{y(t); y \in \Omega\}$ and $\chi_{\mathcal{C}}$ is the Hausdorff measure of noncompactness in $\mathcal{E}$.

Let $L(E)$ be a space of all the bounded linear operators in E. The χ-norm of $B \in L(E)$ is defined as

$$\|B\|^{(\chi)} := \chi(BS),$$

where S is the unit sphere in E. We shall use the following properties of χ-norm.

Lemma 1.1. ([2]) *If $B \in L(E)$, then*

$$\chi(BD) \le \|B\|^{(\chi)}\chi(D)$$

for every $D \in Pb(E)$.

Lemma 1.2. *Let $B \in L(E); \|B\|^{(\chi)} < 1$ and $I - B$ is invertible. Then*

$$\|(I-B)^{-1}\|^{(\chi)} \le (1 - \|B\|^{(\chi)})^{-1}.$$

Proof. Let $R = (I-B)^{-1}S$ and $\{r_n\}_{n=1}^{\infty}$ be a countable dense subset of R. Then the sequence $\{s_n\}_{n=1}^{\infty}$, $s_n = (I-B)r_n$ is dense in S, and using the properties of χ we have

$$1 = \chi(S) = \chi(\{s_n\}_{n=1}^{\infty}) = \chi(\{r_n - Br_n\}_{n=1}^{\infty}) \ge$$
$$\ge \chi(\{r_n\}_{n=1}^{\infty}) - \chi(\{Br_n\}_{n=1}^{\infty}) \ge \chi(R) - \|B\|^{(\chi)}\chi(R) =$$
$$= (1 - \|B\|^{(\chi)})\chi(R).$$

This implies the required inequality □.

A multivalued function (multifunction) $G : [0,T] \to K(E)$ is said to be measurable if it satisfies any of the following two equivalent conditions:

(i) the set $G^{-1}(V) = \{t \in [0,T] : G(t) \subset V\}$ is measurable for every open $V \subset E$;

(ii) there exists the sequence $\{g_n\}_{n=1}^{\infty}$ of measurable functions $g_n : [0,T] \to E$ such that $G(t) = \overline{\{g_n(t)\}_{n=1}^{\infty}}$ for all $t \in [0,T]$ (see, for example [6],[5]).

By the symbol S_G^1 we shall denote the set of all Bochner integrable selectors of the multifunction $G : [0,T] \to P(E)$, i.e.

$$S_G^1 = \{g \in L([0,T]; E) : g(t) \in G(t) \ \text{ a.e.}\}.$$

If $S_G^1 \neq \emptyset$, then the multifunction G is called integrable and

$$\int_{\mathcal{T}} G(s)ds := \{\int_{\mathcal{T}} g(s)ds : g \in s_G^1\}$$

for every measurable set $\mathcal{T} \subseteq [0,T]$. Clearly if G is measurable and integrably bounded (i.e. there exists $\alpha \in L_+^1([0,T])$ such that $\|G(t)\| := \max\{\|y\| : y \in G(t)\} \leq \alpha(t)$ *a.e.*), then G is integrable.

Let us now prove the following generalization of the Kisielewicz lemma (see χ4),

Lemma 1.3. *Let the multifunction* $G : [0,T] \to Pb(E)$ *be integrable, inegrably bounded and*

$$\chi(G(t)) \leq \beta(t) \ a.e. \ on \ [0,T],$$

where $\beta \in L_+^1([0,T])$. *Then*

$$\chi(\int_{\mathcal{T}} G(s)ds) \leq \int_{\mathcal{T}} \beta(s)ds$$

for every measurable set $\mathcal{T} \subseteq [0,T]$. *In particular, if* $\chi(G(\cdot)) \in L_+^1([0,T])$, *then*

$$\chi(\int_{\mathcal{T}} G(s)ds) \leq \int_{\mathcal{T}} \chi(G(s))ds.$$

Proof. Since the space E is separable, so is the space $L^1([0,T]; E)$. Hence the set S_G^1 contains the dense countable subset $\{g_n\}_{n=1}^{\infty}$. Then the set $\{\int_{\mathcal{T}} g_n(s)ds\}_{n=1}^{\infty}$

is dense in $\int_T G(s)ds$. From the properties of the measure of noncompactness χ we have

$$\chi(\int_T G(s)ds) = \chi(\{\int_T g_n(s)ds\}_{n=1}^{\infty})$$

and

$$\chi(\{g_n(t)\}_{n=1}^{\infty}) \leq \chi(G(t)) \leq \beta(t) \ a.e..$$

But, then from $\chi4$, it follows that

$$\chi(\{\int_T g_n(s)ds\}_{n=1}^{\infty}) \leq \int_T \chi(g_n(s)\}_{n=1}^{\infty} ds \leq$$

$$\leq \int_T \beta(s)ds \quad \square.$$

Let X, Y be topological spaces; a multivalued map (multimap) $H : X \to C(Y)$ is said to be; (i) closed if its graph

$$Gr_H = \{(x, y) \in X \times Y : y \in H(x)\}$$

is a closed subset in $X \times Y$; (ii) upper semicontinuous if $H^{-1}(V) = \{x \in X : H(x) \subset V\}$ is an open subset of X for every open $V \subset Y$. If multimap $H : X \to K(Y)$ is closed and compace (i.e. $\overline{H(X)}$ is compact) then H is upper semicontinuous (see, for example [5]).

Let X be a closed subset of a Banach space $\mathcal{E}, \varphi$ a measure of noncompactness in $\mathcal{E}$ and $k \geq 0$. A multimap $H : X \to K(\mathcal{E})$ (or a family of multimaps $G : X \times [0,1] \to K(\mathcal{E})$) is called (k,φ)-contraction (or (k,φ)-contractive) if, respectively,

$$\varphi(H(\mathcal{D})) \leq k\varphi(\mathcal{D}) \ \ or$$

$$\varphi(G(\mathcal{D} \times [0,1])) \leq k\varphi(\mathcal{D})$$

for every $\mathcal{D} \subseteq X$.

Now, let K be a convex closed subset of $\mathcal{E}, \Omega \subset K$ is open in relative topology, $\bar{\Omega}_K$ and $\partial\Omega_K$ denote the closure and the boundary of Ω relative to K. Let $\Gamma : \bar{\Omega}_K \to Kv(K)$ be a closed $(k, \chi_{\mathcal{E}})$ -contraction, where $0 \leq k < 1$ and $\chi_{\mathcal{E}}$ is the Hausdorff measure of noncompactness in $\mathcal{E}$. Assume that the fixed point set of $\Gamma|_\Omega$

$$Fix(\Gamma|_\Omega) = \{x \in \Omega : x \in \Gamma(x)\}$$

is compact. (It is sufficient to suppose that $Fix\Gamma \cap \partial\Omega_K = \emptyset$). Then the topological characteristic: the relative fixed points index $ind_K(\Gamma, \Omega)$ is defined (see [13]). It has the following fundamental properties.

1^0 If $\Gamma(x) \equiv y_0 \in K$ then

$$ind_K(\Gamma, \Omega) = \begin{cases} 1 & \text{if } y_0 \in \Omega \\ 0 & \text{if } y_0 \neq \Omega. \end{cases}$$

2^0 Let Ω_i $(i = 1, 2, \ldots)$ be open in K and mutually disjoint subsets of Ω and $Fix\Gamma \cap (\Omega \setminus \cup_{i=1}^{\infty} \Omega_i) = \emptyset$. Then, the indices $ind_K(\Gamma, \Omega_i)$ are defined, only a finite number of them does not vanish and

$$ind_K(\Gamma, \Omega) = \sum_{i=1}^{\infty} ind_K(\Gamma, \Omega_i).$$

3^0 If closed (k, χ_E)-contractions $\Gamma_0, \Gamma_1 : \bar{\Omega}_K \to Kv(K)$ are homotopic, i.e. there exists the closed (k, χ_E)-contractive family $G : \bar{\Omega}_K \times [0,1] \to Kv(K)$, such that $\cup_{\lambda \in [0,1]} Fix(G(.,\lambda) \mid_{\Omega})$ is compact and $G(.,0) \equiv \Gamma_0, G(.,1) \equiv \Gamma_1$, then $ind_K(\Gamma_0, \Omega) = ind_K(\Gamma_1, \Omega)$.

4^0 If $ind_K(\Gamma, \Omega) \neq 0$, then Γ has a fixed point in Ω.

In this paper we study the existence of mild periodic solutions of the differential inclusion

$$x''(t) \in Ax(t) + F(t, x(t)) \tag{1}$$

in a separable Banach space E. Here A is a closed linear not necessarily bounded operator in E and F is a multimap from $\mathbf{R} \times E$ into E. Let F be T-periodic in the first variable $(T > 0)$, i.e. $F(t + T, .) \equiv F(T), .)$ for all $t \in \mathbf{R}$. In the sequal we shall consider the restriction $F\mid_{[0,T] \times E}$ denoting it by the same symbol F.

We shall assume that A generates an analytic semigroup e^{At} satisfying the estimation

$$\|e^{At}\|^{(x)} \leq e^{-\delta t} \tag{2}$$

for all $t \in [0, T]$, where $\delta > 0$. It should be noted that condition (2) is satisfied if the resolvent of A is completely continuous at a certain point. It will be supposed also that 1 does not belong to the spectrum σe^{AT} of the operator e^{AT}; hence the operator $[I - e^{AT}]^{-1}$ is well defined.

Now we shall assume that the multimap $F : [0, T] \times E \to Kv(E)$ satisfies the following conditions.

F1) for every $e \in E$ the multifunction $F(.,e) : [0,T] \to Kv(E)$ admits a measurable selector;

F2) for almost all $t \in [0,T]$ the multimap $F(t,.) : E \to Kv(E)$ is upper semicontinuous;

F3) for every bounded $D \subset E$ there exists a function $a \in L^p_+([0,T])$, $p > 1$ such that

$$\|F(t,e)\| := \sup\{\|y\| : y \in F(t,e)\} \le a(t)$$

for all $e \in D$ and almost all $t \in [0,T]$;

F4) for every bounded $D \subset E$ we have

$$\chi(F(\{t\} \times D)) \le k\delta\chi(D)$$

for almost all $t \in [0,T]$, where $0 \le k < 1$ and δ is the constant from estimation (2).

It is clear that condition F1 is satisfied if the multifunction $F(.,e)$ is measurable for all $e \in E$. From conditions F1- F3 it follows that $S^1_{F(.,x(.))} \ne \emptyset$ for every $x \in \mathcal{C}$ (see, for example [17]).

In the sequel we shall consider as solutions of the inclusion (1) the mild solutions, i.e. the functions $x \in \mathcal{C}$ of the form

$$x(t) = e^{AT}x(0) + \int_0^t e^{A(t-s)}f(s)ds,$$

where $f \in S^1_{F(.,x(.))}$. it should be noted that the existence of mild solutions for differential inclusions of form (1) is considered for example in the works of N.S. Papageorgiou [14], [16] and V.V.Obukhovskii [12].

It is clear that the question on T-periodic solutions of inclusion (1) is reduced to the existence of solutions $x \in \mathcal{C}$ satisfying the boundary condition of periodicity

$$x(0) = x(T). \tag{3}$$

In the sequal the solution $x \in \mathcal{C}$ satisfying condition (3) will be called the T-periodic solution of (1).

Boundary value problems for differential inclusions with linear operators in a Banach space were considered earlier in the works of P.Zecca-P.Zezza [18] and N.Papageorgiou [15]. In this paper we study the periodic problem under other assumptions, using a method which is new (even in the case of differential equations) and is based on the fixed points index theory for condensing multimaps.

2. Integral multioperator and its properties

In order to investigate T-periodic solutions we shall consider the multivalued integral operator Γ in the space $\mathcal{C}$ defined in the following way

$$\Gamma(x) = \{y \in \mathcal{C} : y(t) = e^{At}[I - e^{AT}]^{-1} \int_0^T e^{A(T-s)} f(s)ds +$$

$$+ \int_0^t e^{A(t-s)} f(s)ds,\ f \in S^1_{F(.,x(.))}\}.$$

Theorem 2.1. *A function $\chi \in \mathcal{C}$ is a T-periodic solution of (1) iff it is a fixed point of Γ.*

Proof. Let $x \in \mathcal{C}$ be a solution of (1) satisfying periodicity condition (3). Then

$$x(0) = e^{AT}x(0) + \int_0^T e^{A(T-s)} f(s)ds,$$

where $f \in S^1_{F(.,x(.))}$. Hence,

$$x(0) = [I - e^{AT}]^{-1} \int_0^T e^{A(T-s)} f(s)ds$$

and

$$x(t) = e^{At}[I - e^{AT}]^{-1} \int_0^T e^{A(T-s)} f(s)ds +$$

$$+ \int_0^t e^{A(t-s)} f(s)ds,$$

i.e. $x \in \Gamma(x)$.

The validity of condition (3) for the function $x \in Fix\Gamma$ can be verified directly. □

Let us now describe the main properties of the multioperator Γ. It follows immediately from the properties of the multimap F that the multioperator Γ has nonempty convex values.

Theorem 2.2. *The multioperator Γ is closed.*

Proof. Let

$$\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subset \mathcal{C};\ y_n \in \Gamma(x_n)(n=1,2,...),\ x_n \to x,\ y_n \to y.$$

We shall show that $y \in \Gamma(x)$.

By assumption we have for every $n \in \mathbf{N}$

$$y_n(t) = e^{At}[I - e^{AT}]^{-1} \int_0^T e^{A(T-s)} f_n(s)ds + \\ + \int_0^t e^{A(t-s)} f_n(s)ds,$$

where $f_n \in S^1_{f(.,x_n(.))}$. From condition F4), it follows that

$$\chi(\{f_n(t)\}_{n=1}^{\infty}) \leq k\delta\chi(\{x_n(t)\}_{n=1}^{\infty}) = 0$$

for almost every $t \in [0,T]$, i.e. $\overline{\{f_n(t)\}_{n=1}^{\infty}}$ is compact in E a.e. $t \in [0,T]$. From the Diestel criterion (see [7], [17]) it then follows that the sequence $\{f_n\}_{n=1}^{\infty}$ is relatively weak compact in the space $L_1([0,T];E)$. Therefore, we can assume that

$$f_n \underset{w}{\rightarrow} f \in L_1([0,T];E).$$

We claim that $f \in S^1_{f(.,x(.))}$.
Indeed, according to the Mazur lemma (see, for example, [8]) the weak convergence $f_n \underset{w}{\rightarrow} f$ implies the existence of the double sequence of nonnegative numbers $\{\lambda_{ik}\}_{k=1}^{\infty}{}_{i=1}^{\infty}$, such that :
1) $\sum_{k=i}^{\infty} \lambda_{ik} = 1$ for all $i = 1,2,...$;
2) $\lambda_{ik} = 0$ for all $k \geq k_0(i)$;
3) the sequence $\{\tilde{f}_i\}_{i=1}^{\infty}$, $\tilde{f}_i(t) = \sum_{k=i}^{\infty} \lambda_{ik} f_k(t)$ converges to f with respect to the norm of $L^1([0,T]); E$. Passing if necessary to a subsequence, we can assume that $\{\tilde{f}_i\}_{i=1}^{\infty}$ converges to f almost everywhere on $[0,T]$.

From condition F2), it follows that for almost all $t \in [0,T]$ for a given $\epsilon > 0$ there exists the integer $i_0 = i_0(\epsilon, t)$, such that

$$F(t, x_i(t)) \subset V_\epsilon(F(t, x(t)))$$

for all $i \geq i_0$, where V_ϵ denotes the ϵ-neighbourhood of a set. But, then $f_i(t) \in V_\epsilon(F(t,x(t)))$ for $i \geq i_0$ and hence

$$\tilde{f}_i(t) \in V_\epsilon(F(t,x(t))),$$

by virtue of the convexity of $V_\epsilon(F(t,x(t)))$. Therefore,

$$f(t) \in F(t,x(t))$$

i.e. $f \in S^1_{F(.,x(.))}$.

For each $t \in [0,T]$ the map $g \mapsto \int_0^t e^{A(t-s)}g(s)ds$ is a continuous linear operator from $L^1([0,T];E)$ into E. It remains continuous if these spaces are endowed with weak topologies. Therefore, for each $t \in [0,T]$ the sequence $y_n(t)$ converges weakly to $e^{At}[I-e^{AT}]^{-1}\int_0^T e^{A(T-s)}f(s)ds + \int_0^t e^{A(t-s)}f(s)ds$. Since by assumption $y_n(t) \to y(t)$, we have $y \in \Gamma(x)$. □

Theorem 2.3. *For every bounded set $\Omega \subset C$ the set $\Gamma(\Omega)$ is bounded and equicontinuous.*

Proof. From condition F3), it follows that there exists a function $a \in L^p_+([0,T])$, $p > 1$, such that for every $x \in \Omega$ and $f \in S^1_{F(.,x(.))}$ we have

$$\|f(t)\| \leq a(t) \ a.e.\ t \in [0,T].$$

By virtue of the analyticity of the semigroup e^{At}, there exist constants C and γ such that

$$\|e^{At}\| \leq Ce^{\gamma t}. \tag{4}$$

Since by assumption $1 \notin \sigma e^{AT}$, we have

$$\|(I - e^{AT})^{-1}\| \leq K, \tag{5}$$

where $K > 0$.

Therefore, if $y \in \Gamma(\Omega)$, then

$$\|y(t)\| \leq KC^2e^{\gamma t}\int_0^T e^{\gamma(T-s)}a(s)ds +$$
$$+C\int_0^t e^{\gamma(T-s)}a(s)ds \leq \tilde{C}\|a\|_{L^p([0,T])},$$

where $\tilde{C}$ is a certain constant. Therefore, $\Gamma(\Omega)$ is bounded.
We now clain that the set $\Gamma(\Omega)$ is equicontinuous. To show this, we may remark that, according to [11], the negative fractional powers $A^{-\alpha}(0 < \alpha < 1)$ of A are defined and the following estimations hold

$$\|A^\alpha e^{At}\| \leq \frac{M(\alpha)}{t^\alpha} \tag{6}$$

$$(7)\qquad \|A^{-\alpha}(e^{At}-I)\| \le \frac{M(1-\alpha)}{\alpha}t^{\alpha},$$

where $M(\alpha)$ is a certain constant depending on α. Choose $\alpha \in (0,1)$ such that $\alpha q < 1$, where $1/p + 1/q = 1$. Then,

$$(8)\int_0^t \frac{a(s)}{\mid t-s\mid^{\alpha}}ds \le (\int_0^t (a(s))^p ds)^{1/p}(\int_0^t \frac{ds}{(t-s)^{\alpha q}})^{1/q} \le \|a\|_{L^p}\frac{T^{(1-\alpha q)/q}}{1-\alpha q}.$$

Now, if $y \in \Gamma(x)$, $x \in \Omega$ and $t_1, t_2 \in [0,T]$, $t_2 > t_1$, then

$$\|y(t_2)-y(t_1)\| \le \|A^{-\alpha}(e^{A(t_2-t_1)}-I)e^{At_1}\| \times$$

$$\|(I-e^{AT})^{-1}\int_0^T A^{\alpha}e^{A(T-s)}f(s)ds\| +$$

$$+\|\int_0^{t_1} A^{-\alpha}(e^{A(t_2-t_1)}-I)A^{\alpha}E^{A(t_1-s)}f(s)ds\| +$$

$$+\|\int_{t_1}^{t_2} e^{A(t_2-s)}f(s)ds\| = J_1 + J_2 + J_3,$$

where $f \in S^1_{F(.,x(.))}$. Using (4)-(8) we have the following estimations:

$$(9)\qquad J_1 \le \frac{M(1-\alpha)}{\alpha}(t_2-t_1)^{\alpha}Ce^{\gamma t_1}K\int_0^Y \frac{a(s)}{(T-s)^{\alpha}}ds \le$$

$$C_1\frac{M(1-\alpha)}{\alpha}\|a\|_{L_p}\frac{T^{(1-\alpha q)/q}}{1-\alpha q}(t_2-t_1)^{\alpha}$$

$$(10)\qquad J_2 \le C_2\frac{M(1-\alpha)}{\alpha}(t_2-t_1)^{\alpha}\int_0^{t_1}\frac{a(s)}{(t_1-s)^{\alpha}}ds$$

$$\le C_2\frac{M(1-\alpha)}{\alpha}\frac{T^{(1-\alpha q)/q}}{1-\alpha q}\|a\|_{L_p}(t_2-t_1)^{\alpha}$$

$$(11)\,J_3 \le \int_{t_1}^{t_2} Ce^{\gamma(t_2-s)}a(s)ds \le (\int_{t_1}^{t_2}(a(s))^p ds)^{1/p}(C\int_{t_1}^{t_2} e^{q\gamma(t_2-s)}ds)^{1/q}$$

$$\le C_3\|a\|_{L_p}(t_2-t_1)^{1/q}.$$

These estimations show that

$$\|y(t_2)-y(t_1)\| \le \tilde{C}_1(t_2-t_1)^{\alpha} + \tilde{C}_2(t_2-t_1)^{1/q}$$

estabilishing the demanded equicontinuity of $\Gamma(\Omega)$.□

Now, consider the measure of noncompactness $\psi : 2^{\mathcal{C}} \to \mathbf{R}_+$

$$\psi(\Omega) = \sup_{t\in[0,T]} \chi(\Omega(t))$$

(see $\chi 5$)).

Theorem 2.4. *The multioperator* Γ *is a* (k,ψ)*-contraction.*

Proof. If $\Omega \subset \mathcal{C}$ is unbounded, then $\psi(\Omega) = +\infty$ and the inequality $\psi(\Gamma(\Omega)) \le k\psi(\Omega)$ evidently holds. Let $\Omega \subset \mathcal{C}$ be nonempty and bounded. Since for every $x \in \Omega$ we have $S^1_{F(.,x(.))} \subset S^1_{F(.,\Omega(.))}$ on each interval $[0,t] \subseteq [0,T]$, the integral

$$\int_0^t e^{A(t-s)} F(s,\Omega(s))ds$$

is defined for all $t \in [0,T]$ and

$$\{\int_0^t e^{A(t-s)} f(s)ds : f \in S^1_{F(.,x(.))},\ x \in \Omega\}$$

$$\subset \int_0^t e^{A(t-s)} F(s,\Omega(s))ds.$$

By virtue of condition F3) the multifunction $S \mapsto e^{A(t-s)}F(s\Omega(s))$, $0 \le s \le t$, is integrally bounded and using Lemma 1.1 and condition F4) we have

$$\chi(e^{A(t-s)}F(s,\Omega(s)) \le \|e^{A(t-s)}\|^{(x)}\chi(F(s,\Omega(s))) \le$$

$$\le k\delta e^{-\delta(t-s)}\chi(\Omega(s)) \le k\delta e^{-\delta(t-s)}\psi(\Omega)$$

for almost all $s \in [0,t]$.

For each $t \in [0,T]$ we have

$$\Gamma(\Omega)(t) \subset e^{At}[I - e^{AT}]^{-1} \int_0^T e^{A(T-s)} F(s,\Omega(s))ds+$$

$$+ \int_0^t e^{A(t-s)} F(s,\Omega(s))ds,$$

and now using the properties $\chi 1), \chi 2)$ and Lemmas 1.1, 1.2 and 1.3, we obtain the following estimation:

$$\chi(\Gamma(\Omega)(t)) \le \|e^{At}\|^{(x)}\|(I - e^{AT})^{-1}\|^{(x)}\times$$

$$\chi\Big(\int_0^T e^{A(t-s)}F(s,\Omega(s))ds\Big) + \chi\Big(\int_0^t e^{A(t-s)}F(s,\Omega(s))ds\Big) \le$$

$$\le e^{-\delta t}(1-e^{-\delta T})^{-1}k\delta\psi(\Omega)\int_0^T e^{-\delta(T-s)}ds +$$

$$+ k\delta\psi(\Omega)\int_0^t e^{-\delta(t-s)}ds =$$

$$k\delta\psi(\Omega)[e^{-\delta t}(1-e^{-\delta T})^{-1}\frac{1}{\delta}(1-e^{-\delta T}) + \frac{1}{\delta}(1-e^{-\delta t})] = k\psi(\Omega).$$

Therefore,

$$\psi(\Gamma(\Omega)) = \sup_{t\in[0,T]} \chi(\Gamma(\Omega)(t)) \le k\psi(\Omega),$$

proving the theorem . □

Corollary 2.5. *For every bounded set $\Omega \subset \mathcal{E}$, the multioperator Γ is a $(k, \chi_{\mathcal{E}})$-contraction on the set $\Omega \cap \overline{co}\Gamma(\Omega)$ where $\chi_{\mathcal{E}}$ is the Hausdorff measure of noncompactness in $\mathcal{E}$.*

Proof. It follows immediately from the property $\chi 5)$ and Theorem 2.3. □

3. The existence of periodic solutions

The properties described above of the integral multioperator allow us to use the theory of the fixed points index to search for periodic solutions of problem (1). Indeed, the following general principle is valid.

Theorem 3.1. *Let $\Omega \subset C$ be an open bounded set; $K = \overline{co}\Gamma(\Omega)$; $x \notin \Gamma(x)$ for all $x \in \partial\Omega_K$ and $ind_K(\Gamma, \Omega_K) \neq 0$ where $\Omega_K = \Omega \cap K$. Then inclusion (1) has at least one T-periodic solution in Ω_K.*

To ensure the conditions of this theorem, we may use a priori bounds of T-periodic solutions of (1). Consider, for example, the following statement.

Theorem 3.2. *Let the multimap F satisfy conditions $F1), F2), F4)$ and $F'3)$, there exists a function $a \in L^p_+([0,T]), p > 1$ such that*

$$\|F(t,e)\| \le a(t)$$

for all $e \in E$ and almost all $t \in [0,T]$. Then, inclusion (1) has at least one $T-$ periodic solution.

Proof. From condition $F'3)$ it follows that the set $\Gamma(\mathcal{C})$ is bounded. Let $\Omega \subset \mathcal{C}$ be an open ball containing the set $\Gamma(\mathcal{C})$. Then, from the properties of the fixed points index we have that $ind_K(\Gamma, \Omega_K) = 1$, where $K = \overline{co}\Gamma(\mathcal{C})$. □

The homotopy invariance property of the fixed points index allows us to apply the general principle to families of differential inclusions continuously depending on a parameter. Let the multimap $F : [0,T] \times E \times [0,1] \to Kv(E)$ satisfy the following properties: s continuously

1Fλ) $F(\cdot, e, \lambda) : [0,T] \to Kv(E)$ admits a measurable selector for all $(e,\lambda) \in E \times [0,1]$;

2Fλ) $F(t, \cdot, \cdot) : E \times [0,1] \to Kv(E)$ is upper semicontinuous for almost all $t \in [0,T]$;

3Fλ) for every bounded $D \subset E$ there exists a function $a \in L^p_+([0,T])$, $p > 1$ such that $\|F(t,e,\lambda)\| \leq a(t)$ for all $(e,\lambda) \in E \times [0,1]$ and almost all $t \in [0,T]$;

4Fλ) for every bounded $D \subset E$ we have $\chi(F(\{t\} \times D \times [0,1])) \leq k\delta\chi(D), 0 \leq k < 1$ for a.e. $t \in [0,T]$.

As an example of such a family we may regard the multimap $\lambda F, \lambda \in [0,1]$, where F satisfies the conditions $F1) - F4)$.

Consider the family of differential inclusions

(1λ) $x'(t) \in Ax(t) + F(t, x(t), \lambda)$, where the multimap F satisfies the conditions $1F\lambda) - 4F\lambda)$.

Modifying our previous reasonings, we may show that the family of integral multioperators

$$\Gamma : \mathcal{C} \times [0,1] \to Cv(\mathcal{C}),$$

$$\Gamma(x,\lambda) = \{y \in \mathcal{C} : y(t) = e^{At}[I - e^{At}]^{-1} \int_0^T e^{A(T-s)} f(s)ds +$$

$$\int_0^t e^{A(t-s)} f(s)de, \ f \in S^1_{F(\cdot, x(\cdot), \lambda)}\}$$

generated by the family (1λ) has the following properties.

1$\Gamma\lambda$) The multimap $\Gamma(x,\lambda)$ is closed.

$2\Gamma\lambda$) For every bounded $\Omega \subset C$, the set $\Gamma(\Omega \times [0,1])$ is bounded and equicontinuous.

$3\Gamma\lambda$) The family $\Gamma(x,\lambda)$ is $(k,\psi)-$ contractive. It is $(k,\chi_C)-$ contractive on every set of the form $\Omega \cap \overline{co}\Gamma(\Omega \times [0,1])$, where $\Omega \subset C$ is bounded.

These properties allow us to justify the following statement as a direct sequence of the property of homotopic invariance of the fixed points index of the multimap $\Gamma(\cdot,\lambda)$.

Theorem 3.3. *Let $\Omega \subset C$ be an open bounded set whose boundary does not contain T-periodic solutions of the family (1λ), $\lambda \in [0,1]$. Let $ind_K(\Gamma(\cdot,0),\Omega_K) \neq 0$, where $K = \overline{co}\Gamma(\Omega \times [0,1])$. Then, the differential inclusion*

$$x'(t) \in Ax(t) + F(t,x(t)) \tag{12}$$

has as least one T-periodic solution in Ω_K. □

The following application of the antipodal theorem is the concretization of this principle.

Theorem 3.4. *Let $\Omega \subset C$ be a symmetric open - bounded neighbourhood of the origin whose boundary does not contain T- periodic solutions of the family (1λ). If*

$$F(t,-e,0) = -F(t,e,0)$$

for all $e \in E$ and almost all $t \in [0,T]$, then the differential inclusion (12) has at least one T-periodic solution in Ω.

Proof . The set $K_1 = \overline{co}(K \cup (-K))$, where $K = \overline{co}\Gamma(\bar{\Omega} \times [0,1])$ is symmetric with respect to the origin and equicontinuous. It is easy to see that the integral multioperator $\Gamma(\cdot,0)$ is odd, and hence $ind_{K_1}(\Gamma(\cdot,0),\Omega_{K_1}) \equiv 1(mod\,2)$([14]). But, now, using the homotopy invariance property we have $ind_{K_1}(\Gamma(\cdot,1),\Omega_{K_1}) \equiv 1(mod\,1)$. □

Corollary 2.5. *Let $\Omega \subset C$ be a symmetric open-bounded neighbourhood of the origin whose boundary does not contain T- periodic solutions of the family*

$$x'(t) \in Ax(t) + \lambda F(t,x(t)),$$

where $\lambda \in [0,1]$ and F satisfies the properties $F1)-F4)$. Then inclusion (1) has at least one T-periodic solution in Ω.

Note, in conclusion, that the operator method described above can be used in the problem of the existence of optimal periodic solutions for control systems in a Banach space. Describe briefly the scheme of this application.

Let E, E_1 be Banach spaces; a map $f : [0,T] \times E \times E_1 \to E$ is such that:

1) $f(\cdot, e, e_1) : [0,T] \to E$ is measurable for all $(e, e_1) \in E \times E_1$;

2) $f(t, \cdot, \cdot) : E \times E_1 \to E$ is continuous for almost all $t \in [0,T]$.

Let a multimap $U : [0,T] \times E \to K(E_1)$ characterize the varying admissible controls domain and have the following peoperties: $U(\cdot, e)$ is measurable for all $e \in E$ and $U(t, \cdot)$ is upper semicontinuous for almost all $t \in [0,T]$.

Consider a nonlinear control system with feedback

$$x'(t) = Ax(t) + f(t, x(t)), u(t)) \tag{13}$$

$$u(t) \in U(t, x(t)) \tag{14}$$

Its solution is the pair $(x(\cdot), u(t))$ consisting of the trajectory $x(\cdot) \in \mathcal{C}$, where $x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)} f(s, x(s), u(s))ds$ is a mild solution of equation (13), and the control $u : [0,T] \to E$, where u is a measurable function satisfying inclusion (14) for almost all $t \in [0,T]$.

The multimap $F : [0,T] \times E \to K(E), F(t,e) = f(t, e, U(t,e))$ satisfies the conditions $F1$) and $F2$) (see, for example [5]). Assume also that the multimap F has convex values and satisfies the conditions $F3$) and $F4$). It should be noted that condition $F4$) is satisfied, if we assume that the multimap $U(t, \cdot)$ is completely continuous for almost all $t \in [0,T]$, i.e. the set $U(\{t\} \times D)$ is relatively compact for every bounded $D \subset E$ and the map $f(t, \cdot, e_1) : E \to E$ is a $k\delta-$ Lipschitzian for all $e_1 \in E_1$ and almost all $t \in [0,T]$ (see [9]),

Therefore the methods described can be applied to the differential inclusion

$$x'(t) \in Ax(t) + f(t, x(t), U(t, x(t))). \tag{15}$$

The existence of the control realizing the $T-$ periodic solution of (15) as the trajectory of the control systems (13), (14) follows from the Filippov implicit functions lemma (see, for example [5]).

The topological properties of the integral multioperator allow us to justify the following optimal control principle.

Let the set of all $T-$periodic solutions of the control system (13), (14) be nonempty and bounded. Then, there exists the control u_* such that the corresponding trajectory x_* minimize the given lower semicontinuous functional $j : \mathcal{C} \to \mathbf{R}$.

It follows directly from the compactness of a fixed points set of a $(k, \chi_C)-$contractive integral multioperator Γ.

References

[1] Akhmerov, R.R., Kamenskiĭ, M.I., Potapov, A.S., Sadovskiĭ, B.N.: Condensing operators (in Russian), Itogi Nauki Tehniki, Ser. Math. Anal. 18 (1980), 185-250.

[2] Akhmerov, R.R., Kamenskiĭ, M.I., Potapov, A.S., Rodkina, A.E., Sadovskiĭ, B.N.: Measures of noncompactness and condensing operators (in Russian). Nauka, Novosibirsk, 1986.

[3] Borisovich, Yu.G., Gel'man, B.D., Myshkis, A.D., Obukhovskiĭ, V.V.: Topological methods in the fixed point theory of multivalued maps (in Russian), Uspehi Mat. Nauk 35, 1(1980), 59- 126.

[4] Borisovich, Yu.G., Gel'man, B.D., Myshkis, A.D., Obukhovskiĭ, V.V.: Multivalued maps (in Russian), Itogi Nauki Tehniki, Ser.Mat. Anal. 19(1982), 127-230.

[5] Borisovich, Yu.G., Gel'man, B.D., Myshkis, A.D., Obukhovskiĭ, V.V.: Introduction to the theory of multivalued maps (in Russian), Izdat. Voronezh Gos. Univ., Voronezh, 1986.

[6] Castaing, C., Valadier, M.: Convex analysis and measurable multifunctions, Lect. Notes in Math., 580. Springer- Verlag, Berlin-Heidelberg-New York, 1977.

[7] Diestel, J.: Remarks on weak compactness in $L_1(\mu, X)$, Glasgow Math. J., 18, 1 (1977), 87-91.

[8] Ekeland, I., Temam, R.: Convex analysis and variational problems, North-Holland Publ. Company, Amsterdam, 1976.

[9] Gel'man, B.D., Obukhovskiĭ, V.V.: Some fixed point theorems for multivalued maps of condensing type (in Russian), to appear in News in global analysis, Izdat. Voronezh Univ., Voronezh.

[10] Kisielewics, M.: Multivalued differential equations in separable Banach spaces. J.Optimizat. Theory and Applic. 37, 2(1982), 231-249.

[11] Krasnosel'skii, M.A., Zabreiko, P.P., Pustyl'nik, E.I., Sobolevskii, P.E.: Integral operators in spaces of summable functions (in Russian), Nauka, Moscow, 1966.

[12] Ovukhovskiĭ, V.: On evolution differential inclusions in a Banach space, Fourth Conf. Diff. Equat. and Appl. , Rousse, 1989, 430.

[13] Obukhovskiĭ, V.V.: On the fixed points index for some classes of noncompact multivalued maps (in Russian) , to appear.

[14] Papageorgiou, N.S.: On multivalued evolution equations and differential inclusions in Banach spaces. Comment. Math. Univ. Sancti Pauli, 36, 1(1987), 21-39.

[15] Papageorgiou, N.S.: Boundary value problems for evolution inclusions, Comment. math. Univ. Carolinae, 29, 2(1988), 355-363.

[16] Papageorgiou, N.S.: On multivalued semilinear evolution equations, Boll. U.M.I., (7) 3-B (1989), 1-16.

[17] Tolstonogov, A.A.: Differential inclusions in a Banach space (in Russian), Nauka, Novosibirsk, 1986.

[18] Zecca, P., Zezza, P.: Nonlinear boundary valued problems in Banach spaces for multivalued differential equations on a non- compact interval, Nonlinear Anal.: Theory, Methods, Appl., 3, 3(1979), 347-352.

REZIME

O PERIODIČNIM REŠENJIMA DEFIRENCIJALNIH INKLUZIJA SA NEOGRANIČENIM OPERATORIMA U BANAHOVIM PROSTORIMA

U ovom radu se razmatra jedna operatorska metoda, korišćenjem teorije indeksa nepokretne tačke kondenzujućeg višeznačnog preslikavanja, za ispitivanje postojanja blago periodičnih rešenja diferencijalne inkluzije

$$x'(t) \in Ax(t) + F(t, x(t)),$$

gde je A linearni operator i F višeznačno preslikavanje u separabilnom Banahovom prostoru. Razmatrane su neke primene na egzistenciju optimalnih periodičnih rešenja kontrolnih sisitema u Banahovim prostorima.

Received by the editors January 7, 1990.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 193-203

Review of Research
Faculty of Science
Mathematics Series

ON COINCIDENCE POINTS IN PROBABILISTIC METRIC SPACES WITH A CONVEX STRUCTURE

Olga Hadžić[1]
Institute of Mathematics, University of Novi Sad,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In this paper a theorem on the existence of a coincidence point in probabilistic metric spaces with a convex structure is proved. The theorem is a generalization of Theorem 3 from [6].

AMS Mathematics Subject Classification (1980): 47H10
Key words and phrases: Coincidence points, probabilistic metric spaces.

1. Introduction

The notion of a probabilistic metric space is introduced by K.Menger in [9] and the first result about the existence of a fixed point of a mapping which is defined on a Menger space $(S, \mathcal{F}, min)$ is obtained by V.Sehgal and A.Barucha-Reid in [1]. Since then, many fixed point theorems for mappings which are defined on a Menger space $(S, \mathcal{F}, t)$ are obtained ([4], [5], [6], [13], [15], [16]). The theory of probabilistic metric spaces is now an important part of the stochastic analysis and in the books [3] and [10] extensive bibliographies on probabilistic metric spaces and their applications are given.

In [6] the notion of a probabilistic metric space with a convex structure W is given and a theorem on coincidence point in such a space is proved. Here, we shall prove a generalization of Theorem 3 from [6].

---

[1] Work supported by Fund for Science of Vojvodina

2. Preliminaries

A triplet $(S, \mathcal{F}, t)$ is a Menger space if and only if S is a nonempty set, $\mathcal{F} : S \times S \to \Delta$, where Δ denotes the set of all distribution functions, and t is a T-norm [10] so that the following conditions are satisfied ($\mathcal{F}(p,q) = F_{p,q}$ for every $p, q \in S$):

1. $F_{p,q}(x) = 1$, for every $x \in \mathbf{R}^+$ if only if $p = q$.
2. $F_{p,q}(0) = 0$, for every $p, q \in S$.
3. $F_{p,q} = F_{q,p}$, for every $p, q \in S$.
4. $F_{p,r}(u+v) \geq t(F_{p,q}(u), F_{q,r}(v))$, for every $p, q, r \in S$ and every $u, v \in \mathbf{R}^+$.

The (ϵ, λ)-topology is introduced by the (ϵ, λ)-neighbourhoods of $v \in S$:

$$U_v(\epsilon, \lambda) = \{u; u \in S, F_{u,v}(\epsilon) > 1 - \lambda\},\ \epsilon > 0,\ \lambda \in (0,1).$$

In [8] some relations between a Menger space and a fuzzy metric space are obtained. It is proved that every Menger space $(S, \mathcal{F}, t)$ is a fuzzy metric space $(S, d, 0, R)$, where

$$R(a,b) = 1 - t(1-a, 1-b)$$

$$d(x,y)(s) = \begin{cases} 0, & s < \sup\{s; F_{x,y}(s) = 0\} \\ 1 - F_{x,y}(s), & s \geq \sup\{s; F_{x,y}(s) = 0\}. \end{cases}$$

Definition 1. *Let $(S, \mathcal{F}, t)$ be a Menger space. A mapping $W : S \times S \times [0,1] \to S$ is said to be a convex structure if for every $(u, x, y, \lambda) \in S \times S \times S \times (0,1)$:*

$$F_{u,W(x,y,\lambda)}(2\epsilon) \geq t(F_{u,x}(\frac{\epsilon}{\lambda}), F_{u,y}(\frac{\epsilon}{1-\lambda})),$$

for every $\epsilon \in \mathbf{R}^+$ and $W(x,y,0) = y$, $W(x,y,1) = x$.

Every random normed spaces [12] is a probabilistic metric space S with a convex structure $W(x,y,\lambda) = \lambda x + (1-\lambda)y,\ x, y \in S,\ \lambda \in (0,1)$.

Example. A nontrivial example of a Menger space with a convex structure is the following. Let (M, d) be a separable metric space with a convex structure W [14] such that for every $\delta \in [0,1]$ the mapping $(x,y) \longmapsto W(x,y,\delta)$ is continuous. Let (Ω, Σ, P) be a probability space. We shall prove that the Menger space $(S, \mathcal{F}, t_m)(t_m(a,b) = \max\{a+b-1, 0\},\ a,b \in [0,1])$ has a convex structure, where S is the space of all measurable mappings from Ω into M (the space of equivalence classes) and for every $X, Y \in S$:

$$F_{X,Y}(u) = P(\{\omega,\ \omega \in \Omega, d(X(\omega), Y(\omega)) < u\}),\ u \in \mathbf{R}^+.$$

Let $\bar{W} : S \times S \times [0,1] \to S$ be define by the relation: $\bar{W}(X,Y,\delta)(\omega) = W[X(\omega), Y(\omega), \delta]$, for every $\omega \in \Omega$, every $X, Y \in S$ and every $\delta \in [0,1]$.

From the measurability of the mappings X and Y and the continuity of the mapping $(x,y) \to W(x,y,\delta)$ it follows that $\bar{W}(X,Y,\delta) \in S$. It is easy to verify that

$$F_{U,\bar{W}(X,Y,\delta)}(2\epsilon) \geq F_{U,X}(\frac{\epsilon}{\delta}) + F_{U,Y}(\frac{\epsilon}{1-\delta}) - 1$$

for every $X, Y, U \in S$; $\delta \in (0,1)$, $\epsilon > 0$. Further, $\bar{W}(X,Y,0) = Y$ and $\bar{W}(X,Y,1) = X$ and so $(S, \mathcal{F}, t_m)$ is a probabilistic metric space with the convex structure $\bar{W}$.

In this paper we shall suppose that a convex structure W on a Menger space $(S, \mathcal{F}, t)$ satisfies the following inequality:

$$F_{W(x,z,\lambda),W(y,z,\lambda)}(\lambda\epsilon) \geq F_{x,y}(\epsilon)$$

for every $(x,y,z,\lambda,\epsilon) \in S \times S \times S \times (0,1) \times \mathbf{R}^+$.

The convex structure $W(x,y,\lambda) = \lambda x + (1-\lambda)y$ in a random normed space satisfies this condition.

In Itoh's paper [7] a similar condition is introduced for metric spaces with a convex structure.

If $(S, \mathcal{F})$ is a probabilistic metric space with a convex structure $W, x_0 \in S$ and $T : S \to S$, then T is said to be (W, x_0)-convex if $W(Tz, x_0, \lambda) = T(W(z, x_0, \lambda))$, for every $\lambda \in (0,1)$ and $z \in S$.

A nonempty set K of S, where $(S, \mathcal{F})$ is a probabilistic metric space with a convex structure W, is said to be starshaped in respect to $x_0 \in K$ if and only if $W(x, x_0, \lambda) \in K$ for every $x \in K$ and every $\lambda \in (0,1)$. Then x_0 is a starcenter of K.

Let $(S, \mathcal{F})$ be a probabilistic metric space and A a nonempty subset of S. The function $D_A(.) : \mathbf{R}^+ \to [0,1]$ defined by [3]:

$$D_A(u) = \sup_{s<u} \inf_{p,q\in A} F_{p,q}(s), \;\; u \in \mathbf{R}^+$$

is called *the probabilistic diameter* of the set A and the set A is *probabilistic bounded* if and only if

$$\sup_{u\in\mathbf{R}^+} D_A(u) = 1.$$

In [15] the function $\{\beta_A(.)\}$, as a generalization of the notion of the measure of noncompactness, is defined for a probabilistic bounded subset $A \subset S$ in the following way:

$$\beta_A(u) = \sup\{\epsilon; \epsilon > 0, \text{ there exists a finite subset}$$

$$A_f \text{ of } S \text{ such that } \tilde{F}_{A,A_f}(u) \geq \epsilon\}$$

where

$$\tilde{F}_{A,B}(u) = \sup_{s<u} \inf_{x\in A} \sup_{y\in B} F_{x,y}(s)$$

for every two probabilistic bounded subsets of S.

The function $\{\beta_A(.)\}$ has the following properties [15] :

1. $\beta_A \in \Delta$.
2. $\beta_A(u) \geq D_A(u)$, for every $u \in \mathbf{R}^+$.
3. $\emptyset \neq A \subset B \subset S \Rightarrow \beta_A(u) \geq \beta_B(u)$, for every $u \in \mathbf{R}^+$.
4. $\beta_{A\cup B}(u) = \min\{\beta_A(u), \beta_B(u)\}$, for every $u \in \mathbf{R}^+$.
5. $\beta_A(u) = \beta_{\bar{A}}(u)$($u \in \mathbf{R}^+$, where $\bar{A}$ is the closure of A).
6. $\beta_A = H \Leftrightarrow A$ is precompact, where

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

Definition 2. *Let $(S, \mathcal{F})$ be a probabilistic metric space, $B(S)$ the family of all probabilistic bounded subset of $S, M \subseteq S$ and A_1 and A_2 mappings from S into S. If for every $K \subseteq M$ such that $A_1(K), A_2(K) \in B(S)$ the implication*

$$\beta_{A_2(K)}(u) \geq \beta_{A_1(K)}(u),\ u \in \mathbf{R}^+ \Rightarrow \bar{K} \text{ is compact}$$

holds, A_1 is said to be a (β, A_2)- densifying mapping on M.

If X is a topological space, $M \subseteq X$ is an attractor for a mapping $F : X \to X$ if for every $x \in X$:

$$M \cap \overline{(\cup_{n\in \mathbf{N}} F^n(x))} \neq \emptyset.$$

3. A theorem on coincidence points

The following theorem is a generalization of Theorem 3 from [6].

Theorem 1. *Let $(S, \mathcal{F}, t)$ be a complete Menger space with a convex structure W and continuous T-norm t, A and B continuous, commutative mappings from S into S such that AS is probabilistic bounded subset of $BS, x_0 \in S$ and B be (W, x_0)-convex so that*

$$F_{Ax,Ay}(\epsilon) \geq F_{Bx,By}(\epsilon),\ \textit{for every}\ \ x, y \in S \ \textit{and every}\ \epsilon \in \mathbf{R}^+.$$

If there exists a nonempty subset M of S such that BM is an attractor for the mapping A and A is (β, B)- densifying on M then there exists $x \in S$ such that $Ax = Bx$.

Proof. Similarly as in [6] it follows that there exists a sequence $\{y_n\}_{n\in\mathbf{N}}$ from M such that

$$(1) \qquad \lim_{n\to\infty} F_{By_n,Ay_n}(\epsilon) = 1,\ \text{for every}\ \ \epsilon > 0$$

but we shall give the proof of (1) because of the completeness.

Let $\{k_n\}_{n\in\mathbf{N}}$ be a sequence from $(0,1)$ such that $\lim_{n\to\infty} k_n = 1$. For every $n \in \mathbf{N}$ and every $x \in S$ let

$$A_n x = W(Ax, x_0, k_n).$$

It is easy to verify that for $A_n, S = B$ and $T = B$ all the conditions of Theorem B from [5] are satisfied. Hence, for every $n \in \mathbf{N}$ there exists $x_n \in S$ such that

$$x_n = A_n x_n = B x_n.$$

Since $(S, \mathcal{F}, t)$ is a probabilistic metric space with a convex structure W we obtain that

$$F_{x_n, Ax_n}(\epsilon) \geq F_{Ax_n, x_0}\left(\frac{\epsilon}{2(1-k_n)}\right) \tag{2}$$

for every $n \in \mathbf{N}$ and every$\epsilon > 0$. From

$$BM \cap (\overline{\cup_{m \in \mathbf{N}} A^m x_n}) \neq \emptyset, \text{ for every } \; n \in \mathbf{N}$$

it follows that there exists, for every $n \in \mathbf{N}, y_n \in M$ such that $By_n \in \overline{\cup_{m \in \mathbf{N}} A^m x_n}$. Further, for every $n \in \mathbf{N}$ and every $k \in \mathbf{N}$

$$F_{A^k x_n, A^{k+1} x_n}(\epsilon) \geq F_{x_n, Ax_n}(\epsilon), \text{ for every } \; \epsilon > 0. \tag{3}$$

Let $\lambda \in (0,1)$. We shall prove that for every $\epsilon > 0$ there exists $n_0(\epsilon, \lambda) \in \mathbf{N}$ such that $F_{By_n, Ay_n}(\epsilon) > 1 - \lambda$ for every $n \geq n_0(\epsilon, \lambda)$.

From (2) and (3) we obtain that for every $\epsilon > 0$ and $k \in \mathbf{N}$

$$\begin{aligned} F_{By_n, Ay_n}(\epsilon) \geq\; & t(F_{By_n, A^k x_n}(\tfrac{\epsilon}{3}), t(F_{Ax_n, x_0}(\tfrac{\epsilon}{6(1-k_n)}), \\ & F_{A^k x_n, By_n}(\tfrac{\epsilon}{3}))). \end{aligned} \tag{4}$$

Since $By_n \in \overline{\cup_{m \in \mathbf{N}} A^m x_n}$ we conclude that for every $n \in \mathbf{N}$ there exists $m_n \in \mathbf{N}$ such that

$$F_{By_n, A^{m_n} x_n}\left(\frac{\epsilon}{3}\right) > \eta, \text{ for every } \; n \in \mathbf{N} \tag{5}$$

where $\eta \in (0,1)$ is such that the following implication holds

$$x, y, z \geq \eta(\lambda) \Rightarrow t(x, t(y, z)) > 1 - \lambda. \tag{6}$$

The existence of such an element η follows from the continuity of the mapping t. Since AS is a probabilistic bounded set it follows that there exists $n_0(\epsilon, \eta(\lambda)) \in \mathbf{N}$ such that

$$F_{Ax_n, x_0}\left(\frac{\epsilon}{6(1-k_n)}\right) > \eta, \text{ for every } \; n \geq n_0(\epsilon, \eta(\lambda)). \tag{7}$$

From (4), (5), (6) and (7) it follows that

$$F_{By_n,Ay_n}(\epsilon) \geq t(\eta, t(\eta,\eta)) > 1-\lambda, \text{ for every } \quad n \geq n_o(\epsilon, \eta(\lambda)).$$

Hence $\lim_{n\to\infty} F_{By_n,Ay_n}(\epsilon) = 1$, for every $\epsilon > 0$.

We shall prove that the set $\overline{\{y_n; n \in \mathrm{N}\}}$ is compact. Using the assumption that A is (β, B) - densifying it is enough to prove that

$$\beta_{B[\{y_n;n\in \mathrm{N}\}]}(u) = \beta_{A[\{y_n;n\in \mathrm{N}\}]}(u) \tag{8}$$

for every $u > 0$.

In order to prove (8) we shall prove (9) and (10) where

$$\beta_{B[\{y_n;n\in \mathrm{N}\}]}(u) \leq \beta_{A[\{y_n;n\in \mathrm{N}\}]}(u) \tag{9}$$

for every $u \in \mathbf{R}^+$ and

$$\beta_{A[\{y_n;n\in \mathrm{N}\}]}(u) \geq \beta_{B[\{y_n;n\in \mathrm{N}\}]}(u) \tag{10}$$

for every $u \in \mathbf{R}^+$.

In order to prove (9) we shall prove that for every $u > 0$ and $s \in (0, u)$

$$\beta_{B[\{y_n;n\in \mathrm{N}\}]}(u-s) \leq \beta_{A[\{y_n;n\in \mathrm{N}\}]}(u). \tag{11}$$

Since β is left continuous (11) implies (9). We can suppose that

$$\beta_{B[\{y_n;n\in \mathrm{N}\}]}(u-s) > 0,$$

since in the oposite case (11) holds. If we prove that

$$0 < r < \beta_{B[\{y_n;n\in \mathrm{N}\}]}(u-s) \Rightarrow r \leq \beta_{A[\{y_n;n\in \mathrm{N}\}]}(u), \tag{12}$$

then (11) holds.

From the inequality $r < \beta_{B[\{y_n;n\in \mathrm{N}\}]}(u-s)$ it follows the existence of a finite subset $A_f \subseteq S$ such that

$$\tilde{F}_{B[\{y_n;n\in \mathrm{N}\}].A_f}(u-s) > r.$$

Hence, for every $n \in \mathbf{N}$ there exists $z_n \in A_f$ such that

$$F_{By_n,z_n}(u-s) > r.$$

Let $\delta_1 \in (0, r)$. From the continuity of t and the relation $t(1, r) = r$ it follows the existence of $\delta_2 \in (0, 1)$ such that $1 \geq h > 1 - \delta_2$ implies $t(h, r) > r - \delta_1$. Let $n(s, \delta_2) \in \mathbf{N}$ be such that $F_{By_n, Ay_n}(\frac{s}{2}) > 1 - \delta_2$, for every $k > n(s, \delta_2)$. Then

$$F_{Ay_n, z_n}(u - \frac{s}{2}) \geq t(F_{Ay_n, By_n}(\frac{s}{2}), F_{By_n, z_n}(u - s)) \geq$$

$$t(F_{Ay_n, By_n}(\frac{s}{2}), r) > r - \delta_1$$

and so $\beta_{A[\{y_n; n \in \mathbf{N}\}]}(u) \geq r$ and (12) is proved. This proves (9). Similarly, we can prove (10) and so we have that for every $u > 0$ (8) holds. Since A is (β, B) densifying we obtain that the set $\overline{\{y_n; n \in \mathbf{N}\}}$ is compact. Let $\lim_{k \to \infty} y_{n_k} = y$. Then from $\lim_{n \to \infty} By_{n_k} - Ay_{n_k} = 0$ and the continuity of B and A it follows that $By = Ay = y$.

Corollary 1. *Let $(S, \mathcal{F}, t)$ be a Menger space with a convex structure W and continuous T-norm t, A and B continuous, commutative mappings from S into S such that AS is probabilistic bounded subset of BS, $x_0 \in S$ and B (W, x_0) - convex so that*

$$F_{Ax, Ay}(\epsilon) \geq F_{Bx, By}(\epsilon), \text{ for every } x, y \in S \text{ and every } \epsilon \in \mathbf{R}^+.$$

If there exists a compact set $M \subseteq S$ such that BM is an attractor for A then there exists $x \in S$ such that $x = Ax = Bx$.

Proof. It is obvious that A is (β, B) - densifying on M since M is a compact set and so all the conditions of the Theorem are satisfied.

References

[1] Banas, J., Goebel, K.: Measures of noncompactness in Banach spaces, Marcel Dekker, New York and Basel, 1980.

[2] Bharucha-Reid, A.T.: Fixed point theorems in probabilistic analysis, Bull.Amer.Math.Soc. 82(1976), 641-657.

[3] Constantin, Gh., Istratescu, I.: Elemente de Analiza Probabilista si Aplicatii, Editura Academiei Republici Socialiste Romania, Bucuresti, 1981.

[4] Hadžić, O.: Some theorems on the fixed points in probabilistic metric and random normed spaces, Boll. Unione Mat.Ital., (6), 1-B(1982), 381-391.

[5] Hadžić, O.: On a common fixed point in Banach and random normed spaces, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 11(1981), 11-18.

[6] Hadžić, O.: On coincidence points in metric and probabilistic metric spaces with a convex structure. Univ. u Novom Sadu Zb. Rad. Prirod.-Mat.Fak. Ser. Mat. 15, 1(1985), 11-22.

[7] Itoh, Sh.: Some fixed point theorems in metric spaces, Fund.Math. C II (1979), 109-117.

[8] Kaleva, O., Seikkala, S.: On fuzzy metric spaces, Fuzzy Sets and Systems 12(1984), 215-229.

[9] Menger, K.: Statistical metric, Proc. Nat. Acad. Sci., USA 28 (1942), 535-537.

[10] Schweizer, B., Sklar, A.: Statistical metric spaces, North-Holland Series in Probability and Applied Mathematics, 5, 1983.

[11] Sehgal, V., Bharucha-Reid, A.: Fixed points of contraction mappings on probabilistic metric spaces, Math. System Theory 6(1972), 97-102.

[12] Sherstnev, A.N.: The notion of random normed spaces, DAN USSR 149(2)(1963), 280-283.

[13] Shih-Sen Chang: On some fixed point theorems in probabilistic metric spaces and its applications, Z.Wahrscheinlichkeitstheorie Verw. Geb. 63(1983), 463-474.

[14] Takahashi, W.: A convexity in metric spaces and nonexpansive mappings I, Kodai Math. Sem. Rep., 22(1970), 142-149.

[15] Tan, D.H.: On probabilistic condensing mappings, Rev. Roum. Math. Pures Appl. 26, 10(1981), 1305-1317.

[16] Xie Ping Ding, Common fixed points for nonexpansive type mappings, in convex and probabilistic convex metric spaces, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat.Fak. Ser. Mat., 16,1(1986), 73-84.

[17] Zadeh, L.A.: Fuzzy sets, Inform. and Control 8(1965), 338-353.

REZIME

O TAČKAMA KOINCIDENCIJE U VEROVATNOSNIM METRIČKIM PROSTORIMA SA KONVEKSNOM STRUKTUROM

U ovom radu je dokazana teorema o postojanju tačke koincidencije u verovatnosnim metričkim prostorima sa konveksnom strukturom. Teorema je uopštenje teoreme 3 iz rada [6].

Received by the editors March 16, 1991.

Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 203-215

Review of Research
Faculty of Science
Mathematics Series

CONTINUOUS DEPENDENCE OF THE FIXED POINTS ON PARAMETERS IN RANDOM NORMED SPACES

Olga Hadžić
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

Continuous dependence of the fixed points on parameters is investigated in many papers ([1], [8], [14], [26]). In this paper we shall give a generalization of Vorel's result from [25] to random normed spaces. As a corollary we shall give a generalization of the well known fixed point theorem of Krasnoselskiĭ in random normed spaces.

AMS Mathematics Subject Classification (1980): 47H10
Key words and phrases: Fixed points, random normed spaces.

1. Introduction

The notion of a probabilistic metric space was introduced by K.Menger [17] and the notion of a random normed space by A.N.Sherstnev in [22]. The first result about the existence of the fixed point of a probabilistic contraction on a probabilistic metric space $(S, F, \min)$ is obtained by V.Sehgal and A. Bharucha - Reid in [21]. Some fixed point theorems for singlevalued and multivalued mappings in probabilistic metric spaces are obtained in [2], [3], [4], [5], [6], [7], [9], [10], [18], [21], [24].

The continuous dependence of the fixed points on parameters for densifying mappings is investigated in [1], [8], [14] and [26]. Using the probabilistic functions α and β ([6], [25]) we shall give in this paper a probabilistic generalization of Z.Vorel's results from [26]. As an application we shall obtain a generalization of the Krasnoselskiĭ fixed point theorem in random normed spaces.

2. Preliminaries

Let Δ denote the set of all distribution functions F such that $F(0) = 0$(F is a nondecreasing, left continuous mapping from the set of real numbers $\mathbf{R}$ into $[0,1]$ so that $\sup_{x\in\mathbf{R}} F(x) = 1$) and $\mathbf{R}^+ = [0,\infty)$.

The ordered pair (S,F) is a probabilistic metric space if S is a nonempty set and $F : S\times S \to \Delta$ ($F(p,q)$ for $p,q\in S$ being denoted by $F_{p,q}$) so that the following conditions are satisfied:

1. $F_{u,v}(x) = 1$, for every $x > 0 \Rightarrow u = v((u,v)\in S\times S)$,
2. F is symmetric,
3. $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1 \Rightarrow F_{u,w}(x+y) = 1((u,v,w)\in S\times S\times S$ and $(x,y)\in\mathbf{R}\times\mathbf{R})$.

A Menger space is a triple (S,F,t), where (S,F) is a probabilistic metric space and t is a $T-$norm so that:

$$F_{u,w}(x+y) \geq t(F_{u,v}(x), F_{v,w}(y)), \quad \text{for every } (u,v,w)\in S\times S\times S$$

and every $(x,y)\in\mathbf{R}\times\mathbf{R}$.

Recall that a mapping $t : [0,1]\times[0,1]\to[0,1]$ is called a $T-$norm if the following conditions are satisfied:

1. For every $a\in[0,1]$, $t(a,1) = a$.
2. For every $(a,b)\in[0,1]\times[0,1] : t(a,b) = t(b,a)$.
3. For every $(a,b,c,d)\in[0,1]\times[0,1]\times[0,1]\times[0,1] : a\geq b,\ c\geq d \Rightarrow t(a,c)\geq t(b,d)$.
4. For every $a,b,c\in[0,1]\times[0,1]\times[0,1]$:

$$t(a,t(b,c)) = t(t(a,b),c).$$

Examples of such functions are:
$t_m(a,b) = \max\{a+b-1,0\}$ and $t_{\min}(a,b) = \min\{a,b\}, (a,b) \in [0,1] \times [0,1]$.

The (ϵ, λ) - topology in a probabilistic metric space is introduced by the family of neighbourhoods given by :

$$U = \{U_v(\epsilon,\lambda)\}_{(v,\epsilon,\lambda)\in S\times \mathbf{R}^+ \times (0,1)}$$

where $U_v(\epsilon,\lambda) = \{u \mid u \in S, F_{u,v}(\epsilon) > 1-\lambda\}$.

A random normed space (S,F,t) is an ordered triple where S is a real or complex vector space, t is a $T-$norm which is stronger then the $T-$norm $t_m (t \geq t_m)$ and the mapping $F : S \to \Delta$ satisfies the following conditions, where

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

(a) $F_p = H \iff p = \mathbf{0}$ ($\mathbf{0}$ is the neutral element of S)

(b) For every $p \in S$, every $u > 0$ and every $r \in K\backslash\{0\}(K$ is the scalar field) $:F_{rp}(u) = F_p(\frac{u}{|r|})$.

(c) For every $(p,q) \in S \times S$ and every $(u,v) \in \mathbf{R}^+ \times \mathbf{R}^+$:

$$F_{p-q}(u+v) \geq t(F_p(u), F_q(v)).$$

Every random normed space is a Menger space, where $F : S \times S \to \Delta$ is defined by $F(p,q) = F_{p-q}$.

If the $T-$norm t is continuous then S is, in the $(\epsilon,\lambda)-$ topology, a topological vector space.

Let (S,F) be a probabilistic metric space. The following two definitions are given in [6].

Definition 1. *Let A be a nonempty subset of S. The function $D_A(\cdot)$, defined on $\mathbf{R}^+$ by*

$$D_A(u) = \sup_{s<u} \inf_{p,q\in A} F_{p,q}(s), \quad u \in \mathbf{R}^+,$$

is called the probabilistic diameter of the set A and the set A is probabilistic bounded if and only if

$$\sup_{u\in\mathbf{R}^+} D_A(u) = 1.$$

Definition 2. *Let A be a probabilistic bounded subset of S. The Kuratowski function $\alpha_A(\cdot)$ is defined by:*

$\alpha_A(u) = \sup\{s \mid s > 0,$ *there is a finite family* $A_j (j \in J)$ *such that* $A = \cup_{j \in J} A_j$ *and* $D_{A_j}(u) \geq s,$ *for every* $j \in J\}$ $(u \in \mathbf{R})$.

The Kuratowski function has the following properties :

1. $\alpha_A \in \Delta$,
2. $\alpha_A(u) \geq D_A(u)$, for every $u \in \mathbf{R}$,
3. $\emptyset \neq A \subset B \subset S \Rightarrow \alpha_A(u) \geq \alpha_B(u)$, for every $u \in \mathbf{R}$,
4. $\alpha_{A \cup B}(u) = \min\{\alpha_A(u), \alpha_B(u)\}$, for every $u \in \mathbf{R}$,
5. $\alpha_A(u) = \alpha_{\bar{A}}(u)$, for every $u \in \mathbf{R}$, where $\overline{A}$ is the closure of A,
6. $\alpha_A = H \Rightarrow A$ is precompact.

In [25] the function $\beta_A(\cdot)$ is defined in the following way: $\beta_A(u) = \sup\{r \mid r > 0$, there exists a finite subset A_f of S such that $\tilde{F}_{A,A_f}(u) \geq r\}$, where for every probabilistic bounded subsets A and B of S

$$\tilde{F}_{A,B}(u) = \sup_{s<u} \inf_{x \in A} \sup_{y \in B} F_{x,y}(s).$$

The function β has properties 1) - 6) for β instead of α.

If the T-norm t is $t_{\min}$ then for every $u \in \mathbf{R}^+$

$$\beta_A(u) \geq \alpha_A(u) \geq \beta_A(\frac{u}{2}).$$

Let (S, F) be a probabilistic metric spaces, K a probabilistic bounded subset of S and T a mapping from K into the family of all nonempty subsets of S. Let for every probabilistic bounded subset A of S, $\gamma_A : \mathbf{R} \to [0, 1]$. If $T(K)$ is a probabilistic bounded subset of S and for every $B \subset K$:

$$\gamma_{T(B)}(u) \leq \gamma_B(u), \quad \text{for every } u > 0 \Rightarrow B \text{ is precompact}$$

then we say that the mapping T is densifying on the set K with respect to the functio γ.

3. Continuous dependence of the fixed points on parameters

In the next theorem we suppose that $\gamma_A(\cdot) = \alpha_A(\cdot)$, for every probabilistic bounded subset A of S or $\gamma_A(\cdot) = \beta_A(\cdot)$ for every probabilistic bounded subset A of S.

Theorem 1 in this paper is a generalization of Z.Vorel's result from [26] to random normed spaces.

Theorem 1. *Let (S, F, t) be a complete random normed space, t a continuous $T-$norm, K a nonempty closed and probabilistic bounded subset of $S, T_k : K \to S(k \in \mathbf{N} \cup \{0\})$, x_k a fixed point of the mapping $T_k(k \in \mathbf{N})$ and x_0 the unique fixed point of the mapping T_0. If the mapping T_0 is continuous and densifying on K with respect to the function γ then the following holds:*

$$\lim_{k\to\infty} x_k = x_0 \Longleftrightarrow \lim_{k\to\infty} (T_k - T_0)x_k = 0.$$

Proof. If $\lim_{k\to\infty} x_k = x_0$ then $T_kx_k - T_0x_k = x_k - x_0 + T_0x_0 - T_0x_k$ converges to 0 since T_0 is a continuous mapping. Suppose that $\lim_{k\to\infty}(T_k - T_0)x_k = 0$. We shall prove that an arbitrary subsequence $\{x_{n_k}\}_{k\in\mathbf{N}}$ has a convergent subsequence y_{k_r} with the limit x_0. Let for every $k \in \mathbf{N}, y_k = x_{n_k}$. Then

$$y_k = T_{n_k}y_k = (T_{n_k} - T_0)y_k + T_0y_k$$

hich implies that $\lim_{k\to\infty} y_k - T_0y_k = 0$. This means that for every $s > 0$, $\lim_{k\to\infty} F_{y_k - T_0y_k}(s) = 1$. We shall prove that:

$$\gamma_{\{y_k|k\in\mathbf{N}\}}(u) = \gamma_{\{T_0y_k|k\in\mathbf{N}\}}(u) \tag{1}$$

for every $u \in \mathbf{R}^+$.

First, we shall prove that:

$$\gamma_{\{y_k|k\in\mathbf{N}\}}(u) \leq \gamma_{\{T_0y_k|k\in\mathbf{N}\}}(u), \quad \text{for every } u \in \mathbf{R}^+. \tag{2}$$

Let $\gamma = \beta$. In order to prove (2) for $\gamma = \beta$ we shall prove that for every $u > 0$ and every $s \in (0, u)$:

$$\beta_{\{y_k|k\in\mathbf{N}\}}(u - s) \leq \beta_{\{T_0y_k|k\in\mathbf{N}\}}(u). \tag{3}$$

Suppose that for some $u > 0$ and $s \in (0, u)$:

$$\beta_{\{y_k|k\in \mathrm{N}\}}(u-s) > 0. \tag{4}$$

If (1) is not satisfied then (3) holds. In order to prove (3) we shall prove the following implication:

$$0 < r < \beta_{\{y_k|k\in \mathrm{N}\}}(u-s) \Rightarrow r \leq \beta_{\{T_0 y_k|k\in \mathrm{N}\}}(u). \tag{5}$$

Let $r < \beta_{\{y_k|k\in N\}}(u-s)$. From the definition of the function $\beta_A(\cdot)$ it follows that there exists a finite set A_f from S such that

$$\tilde{F}_{\{y_k|k\in \mathrm{N}\},A_f}(u-s) > r.$$

Hence, from the definition of the function $\tilde{F}$ it follows that for every $n \in \mathrm{N}$ there exists $z(n) \in A_f$ so that

$$F_{y_n,z(n)}(u-s) > r.$$

Let δ be an arbitrary element from the interval $(0, r)$. We shall prove that

$$\beta_{\{T_0 y_k|k\in \mathrm{N}\}}(u) \geq r - \delta. \tag{6}$$

Since the mapping t is continuous and $t(1, r) = r$ it follows that there exists $\tilde{\delta}$ so that the following implication holds

$$1 \geq h > 1 - \tilde{\delta} \Rightarrow t(h, r) > r - \delta.$$

Let $n_0(s, \tilde{\delta})$ be such a natural number that for every $k > n_0(s, \tilde{\delta})$

$$F_{y_k - T_0 y_k}(\frac{s}{2}) > 1 - \tilde{\delta}.$$

Then for every $k > n_0(s, \tilde{\delta})$ we have that

$$F_{T_0 y_k - z(k)}(u - \frac{s}{2}) \geq t(F_{T_0 y_k - y_k}(\frac{s}{2}), F_{y_k - z(k)}(u-s))$$

$$\geq t(F_{T_0 y_k - y_k}(\frac{s}{2}), r) > r - \delta.$$

From the definition of the function $\beta_A(\cdot)$ and the relation

$$\beta_{\{T_0 y_k|k\in \mathrm{N}\}}(u) = \beta_{\{T_0 y_k|k > n_0(s,\tilde{\delta})\}}(u)$$

it follows that (6) is satisfied. Since δ is an arbitrary element from the interval $(0, r)$ we obtain that the right side in (5) holds.

Hence (3) holds. Since the function $\beta_A(\cdot)$ is left continuous we have that

$$\lim_{s\to 0^+} \beta_{\{y_k|k\in\mathrm{N}\}}(u-s) = \beta_{\{y_k|k\in\mathrm{N}\}}(u) \le \beta_{\{T_0y_k|k\in\mathrm{N}\}}(u)$$

for every $u > 0$.

Similarly we can prove that

$$\beta_{\{y_k|k\in\mathrm{N}\}}(u) > \beta_{\{T_0y_k|k\in\mathrm{N}\}}(u)$$

and so

$$\beta_{\{y_k|k\in\mathrm{N}\}}(u) = \beta_{\{T_0y_k|k\in\mathrm{N}\}}(u).$$

Suppose now that $\gamma = \alpha$ and prove that

$$\alpha_{\{y_k|k\in\mathrm{N}\}}(u) = \alpha_{\{T_0y_k|k\in\mathrm{N}\}}(u)$$

for every $u \in \mathbf{R}^+$.

As in the case $\gamma = \beta$ we shall prove that for every $s \in (0, u)$:

$$\alpha_{\{y_k|k\in\mathrm{N}\}}(u-s) \le \alpha_{\{T_0y_k|k\in\mathrm{N}\}}(u).$$

Let $0 < r < \alpha_{\{y_k|k\in N\}}(u-s)$. Then there exists a finite family $\{Y_1, Y_2, ..., Y_n\}$ $(Y_i \subset S,\ i \in \{1, 2, ..., n\})$ so that $\{y_k \mid k \in \mathrm{N}\} = \cup_{i=1}^{n} Y_i$ and

$$D_{Y_i}(u-s) > r, \quad \text{for every } i \in \{1, 2, ..., n\}. \tag{7}$$

From (7) and the definition of $D_A(\cdot)$ we obtain that for every $x, y \in Y_i$, $F_{x-y}(u-s) > r$. Suppose that δ is an arbitrary element from the interval $(0, r)$. The mapping t is continuous and since $t(1, t(r, 1)) = r$ it follows that there exists $\bar{\delta}$ so that the following implication holds

$$1 \ge v,\ w > 1 - \bar{\delta} \Rightarrow t(v, t(r, w)) > r - \delta.$$

For every $j \in \{1, 2, ..., n\}$ let $Z_j \subset S$ be defined by

$Z_j = \{z \mid z \in S$, there exists some $y \in Y_j$ so that $F_{z-y}(\frac{s}{4}) > 1 - \bar{\delta}\}$.

Further, let $n_1(s, \bar{\delta}) \in \mathrm{N}$ so that for every $k > n_1(s, \bar{\delta})$

$$F_{y_k - T_0y_k}(\frac{s}{4}) > 1 - \bar{\delta}.$$

Then it is easy to see that $\{T_0 y_k \mid k > n_1(s,\tilde{\delta})\} \subset \cup_{j=1}^n Z_j$. In order to prove that

$$r - \delta \le \alpha_{\{T_0 y_k | k > n_1(s,\tilde{\delta})\}}(u)$$

we shall prove that $D_{Z_j}(u) > r - \delta$, for every $j \in \{1, 2, ..., n\}$.

Let $x, y \in Z_j$. Then there exist $\tilde{x}$ and $\tilde{y}$ from Y_j so that

$$F_{x-\tilde{x}}(\frac{s}{4}) > 1 - \tilde{\delta}, \quad F_{y-\tilde{y}}(\frac{s}{4}) > 1 - \tilde{\delta}.$$

Since $F_{\tilde{x}-\tilde{y}}(u - s) > r$ we have that

$$F_{x-y}(u - \frac{s}{2}) \ge t(F_{x-\tilde{x}}(\frac{s}{4}), t(F_{\tilde{x}-\tilde{y}}(u-s), F_{y-\tilde{y}}(\frac{s}{4})))$$

$$\ge t(F_{x-\tilde{x}}(\frac{s}{4}), t(r, F_{y-\tilde{y}}(\frac{s}{4}))) > r - \delta$$

and so

$$D_{Z_j}(u) = \sup_{s<u} \inf_{x,y \in Z_j} F_{x-y}(u) \ge \inf_{x,y\in Z_j} F_{x-y}(u - \frac{s}{2}) > r - \delta.$$

Similarly as in the case $\gamma = \beta$ it can be prove that

$$\alpha_{\{y_k | k \in \mathbf{N}\}}(u) = \alpha_{\{T_0 y_k | k\in \mathbf{N}\}}(u), \quad \text{for every } u \in \mathbf{R}^+.$$

Since T_0 is densifying on K in respect to γ then $\{y_k \mid k \in \mathbf{N}\}$ is precompact. If it is not the case for some $u \in \mathbf{R}^+$ then

$$\gamma_{\{y_k | k\in\mathbf{N}\}}(u) < \gamma_{\{T_0 y_k | k \in \mathbf{N}\}}(u) = \gamma_{\{y_k | k\in \mathbf{N}\}}(u)$$

whish is a contradiction.

Hence, there exists a subsequence $\{y_{k_r}\}_{r\in N}$ such that

$$\lim_{r\to\infty} y_{k_r} = x \in K \quad \text{and } x = \lim_{r\to\infty} y_{k_r} = \lim_{r\to\infty} T_{n_{k_r}} y_{k_r}$$

$$= \lim_{r\to\infty}(T_{n_{k_r}} y_{k_r} - T_0 y_{k_r}) + \lim_{r\to\infty} T_0 y_{k_r} = T_0 x.$$

This means that x is a fixed point of the mapping T_0 and since x_0 is the unique fixed point of the mapping T_0 we have that $x = x_0$. Hence $x_0 = \lim_{r\to\infty} y_{k_r}$ and so $\lim_{k\to\infty} x_k = x_0$.

From Theorem 1 we obtain the following corollary.

Corollary 1. *Let (S, F, t) be a complete random normed space with a continuous $t-$norm t, K a nonempty closed and probabilistic bounded subset of $S, T_0 : K \to K$ a probabilistic $k-$contraction, $T_k : K \to S(k \in \mathbf{N})$ and $x_k = T_k x_k (k \in \mathbf{N} \cup \{0\})$. If*

$$\lim_{k\to\infty} (T_k - T_0)x_k = 0$$

then $\lim_{k\to\infty} x_k = x_0$.

Proof. Since T_0 is a probabilistic $k-$contraction and K is closed and probabilistic bounded from Theorem 1 [12] it follows that there exists one and only element $x_0 \in K$ such that $x_0 = T_0 x_0$. Namely, the existence of x_0 follows from Theorem 1 [12] and the uniqueness follows easily. In Theorem 1 [12] it is proved that T_0 is densifying in respect to the function β. Hence, all the conditions of Theorem 2 are satisfied and so $\lim_{k\to\infty} x_k = x_0$.

In order to obtain a generalization of the Krasnoselskii fixed point theorem [27] we shall give the definition of the notion of an admissible subset in a topological vector space which is given in the paper of V.Klee [26].

Definition 3. *Let X be a topological vector space and K a nonempty subset of X. The subset K is said to be admissible if for every compact subset A of K and every neighbourhood U of zero in X there exists a continuous mapping $h : A \to K$ so that the following conditions are satisfied:*

1. *dim $Linh(A) < \infty$, (Lin- linear hull).*
2. *For every $x \in A$, $x - h(x) \in U$.*

If $K = X$ then X is an admissible space.

The notion of the admissibility is very important in the fixed point theory, because by means of this notion many results from the fixed point theory in locally convex topological vector spaces can be generalized to topological vector spaces which are not necessarily locally convex. The admissibility of a class of random normed spaces is investigated in [20].

Further information about the admissibility can be found in the book [19].

The admissibility of the space $S(0, 1)$ (the space of all classes of Lebesgues measurable real functions defined on the interval $(0, 1)$) is proved by T.Riedrich in [35].

In [21] S.Hahn and K.F.Pötter obtained the following generalization of the well known Tihonov fixed point theorem.

Theorem 2. *Let X be a Hausdorff topological vector space, K a nonempty closed convex and admissible subset of X and $f : K \to K$ a continuous mapping such that $\overline{f(K)}$ is compact. Then there exists $x \in K$ such that $x = fx$.*

This theorem will be used in the proof of the next corollary. Since every nonempty, closed and convex subset of a topological vector space, which is locally convex, is admissible the above fixed point theorem is a generalization of Tihonov's fixed point theorem.

Corollary 2. *Let (S, F, t) be a complete random normed space with a continuous $T-$norm $t, \emptyset \neq K \subset S$ a closed, convex, admissible and probabilistic bounded set, T_1 and T_2 continuous mappings from K into S so that the following conditions are satisfied:*

i) $T_1K + T_2K \subset K$.

ii) T_1 is a probabilistic $k-$contraction.

iii) $\overline{T_2(K)}$ is a compact subset of S.

Then there exists $x \in K$ such that $x = T_1x + T_2x$.

Proof. Since T_1 is a probabilistic $k-$contraction it follows that for every $z \in \overline{T_2K}$ there exists one and only one element $x(z)$ from K such that $x(z) = T_1x(z) + z$. We shall prove that the mapping $z \mapsto x(z)(z \in \overline{T_2K})$ is continuous using Theorems 1 and 2. Suppose that $\{z_k\}_{k \in N}$ is a sequence from $\overline{T_2(K)}$ such that $\lim_{k\to\infty} z_k = z$. We have to prove that $\lim_{k\to\infty} x(z_k) = x(z)$. Let, for every $k \in N : T_kx = Tx + z_k(x \in K)$ and $T_0x = Tx + z$. The mapping T_0 is a probabilistic $k-$contraction and from Theorem 1 [12] it follows that T_0 is densifying on K in respect to the function β. Further, $\lim_{k\to\infty}(T_k - T_0)u = \lim_{k\to\infty} z_k - z = 0$ for every $u \in K$ and hence $\lim_{k\to\infty}(T_k - T_0)x(z_k) = 0$. This implies that all the conditions of Theorem 1 are satisfied and so $\lim_{k\to\infty} x(z_k) = x(z)$. Let $fu = x(T_2u)$, for every $u \in K$. The mapping $f : K \to K$ satisfies the conditions of the Theorem of Hahn and Pötter and so there exists $x \in K$ such that $x = fx$. It is easy to see that $x = T_1x + T_2x$.

References

[1] Artstein, Z.: Continuous dependence of solutions of operator equations, I. Trans. Amer. Math. Soc. 231 (1977), 143-166.

[2] Bocsan, Gh.: Some applications of functions of Kuratowski, Sem. Teor. Funct. si. Mat. Apl. , Timisoara, RS Romania, No 5, 1973.

[3] Bocsan, Gh.: On the Kuratowski function in random normed spaces, Sem. Teor. Funct. si. Mat. Apl., Timisoara, RS Romania, No 8, 1974.

[4] Bocsan, Gh.: On some fixed point theorems in random normed spaces, Sem. Teor. Funct. si. Mat. Apl. , Timisoara, RS Romania, No 13, 1974.

[5] Bocsan, Gh.: On some fixed point theorems in probabilistic metric spaces, Sem. Teor. Funct. si. Mat. Apl. , Timisoara, RS Romania, No 24, 1974.

[6] Bocsan, Gh., Constantin, Gh.: The Kuratowski function and some applications to the probabilistic metric spaces, Sem. Teor. Funct. si. Mat. Apl., Timisoara, RS Romania , No 1, 1973.

[7] Cain, G.L., Kasriel, R.H.: Fixed and periodic points of local contraction mappings on probabilistic metric spaces, Math. Systems Theory, Vol. 9, No. 4 (1976), 289-297.

[8] Cheng, M: On certain condensing operators and the behaviour of their fixed points with respect to parameters, J. Math. Anal. Appl. 64 (1978), 505-517.

[9] Hadžić, O.: Fixed point theorems for multivalued mappings in probabilistic metric spaces, Mat. vesnik 3(16)(31), (1979), 125-133.

[10] Hadžić, O.: Some theorems on the fixed point in probabilistic metric and random normed spaces, Boll. Unione Mat. Ital., (6) , 1-B (1982), 381-391.

[11] Hadžić, O., Nikolić - Despotović, D.: A proof of the admissibility of a class of random normed spaces, Mat. vesnik 3 (16) (31) (1979), 267-271.

[12] Hadžić, O.: Fixed point theorems for multivalued mappings in some classes of fuzzy metric spaces, Fuzzy Sets and Systems 29,1(1989), 115-125.

[13] Hahn, S., Pötter, F.: Über Fixpunkte kompakter Abbildungen in topologischen Vektorräunem, Studia Math. L (1974), 1-16.

[14] Hale, J.: Continuous dependence of fixed points of condensing maps, J. Math. Anal. Appl. 46 (1974), 388-393.

[15] Klee, V.: Leray - Schauder theory without local convexity, Math. Ann. 141 (1960), 286-296.

[16] Krasnoselskiĭ, M.: Two observations about the method of succesive approximations, U. M. N. 10, B. 1 (63) (1955), 123-127.

[17] Menger, K.: Statistical metric, Proc. Nat. Acad. Sci. , USA 28 (1942), 535-537.

[18] Radu, V.: On some fixed point theorems in probabilistic metric spaces, Sem. Teor. Prob. Apl. , Timisoara, RS Romania, No 74, 1984.

[19] Riedrich, T.: Der Raum $S(0,1)$ ist zulässig, Wiss. Z. Technische Universität Dresden , 13 (1964), 1-6.

[20] Schweizer,4B., Sklar, A.: Statistical metric spaces, North - Holland Series in Probability and Applied Mathematics 5, 1983.

[21] Sehgal, V., Bhauicha - Reid, A.: Fixed points of contraction mappings on probabilistic metric spaces, Math. System Theory 6 (1972), 97-102.

[22] Sherstnev, A.N.: The notion of random normed spaces, DAN USSR 149 (2) (1963), 280-283.

[23] Sherwood, H.: Complete probabilistic metric spaaces, Z. Wahrscheinlichkeitstheorie Verw. Geb. 63 (1983), 463-474.

[24] Shih - Sen Chang, On some fixed point theorems in probabilistic metric spaces with applications, Z. Wahrscheinlichkeitstheorie Verw. Geb. 63 (1983), 463-474.

[25] Tan, D.H.: On probabilistic condensing mappings, Rev. Roum. Math. Pures Appl. 26, 10 (1981), 1305-1317.

[26] Vorel, Z.: Necessary and sufficient conditions for continuous dependence of fixed points of $\alpha-$ condensing maps, in the book: Nonlinear Phenomena in Mathematical Statistics, Ed. V.Lakshmikantham, Academic Press, 1982, 987-989.

REZIME

NEPREKIDNA ZAVISNOST NEPOKRETNE TAČKE OD PARAMETARA U SLUČAJNIM NORMIRANIM PROSTORIMA

Neprekidna zavisnost nepokretne tačke od parametara je ispitivana u mnogim radovima ([1], [8], [14], [26]). U ovom radu je dato uopštenje Vorelovog rezultata iz [25] na slučajne normirane prostore. Kao posledica dobijeno je uopštenje poznate teoreme o nepokretnoj tački Krasnoseljskog u slučajnim normiranim prostorima.

Received by the editors March 16, 1989.

151253

Entered in Database

Signature with Date

28-9-07